Network Simplex

LP formulation for Min-Cost Flow:

$$
(P)\qquad
\begin{aligned}
\min\ & \sum_{i\to j\in E} c_{ij}x_{ij} \\
\text{s.t. } & \sum_{k:(j,k)\in E} x_{j,k} - \sum_{i:(i,j)\in E} x_{ij} = b_j \quad \forall \text{node } j \\
& x_{ij} \geq 0 \quad \forall \text{arc } i\to j
\end{aligned}
$$

- Typical constraint matrix for this problem:

$$
\begin{array}{cc|ccccccccc|c}
 & & & & & & \text{arcs} & & & & & \\
 & & x_{12} & x_{13} & x_{23} & x_{24} & x_{25} & x_{34} & \cdots & & & \\
\hline
 & 1 & 1 & 1 & & & & & & -1 & & \pi_1 \\
 & 2 & -1 & & 1 & 1 & 1 & & & & -1 & \pi_2 \\
\text{nodes} & 3 & & -1 & -1 & & & 1 & 1 & 1 & & \pi_3 \\
 & 4 & & & & -1 & & -1 & & & & \pi_4 \\
 & 5 & & & & & -1 & & -1 & & 1 & \pi_5 \\
\hline
\end{array}
$$

- The dual problem:

$$
(D)\qquad
\begin{aligned}
\max\ & \sum_{i\in V} b_i\pi_i \\
\text{st } & \pi_i - \pi_j \leq c_{ij}, \quad \forall i\to j\in E \\
& \pi_i \text{ free}, \quad \forall i\in V
\end{aligned}
$$

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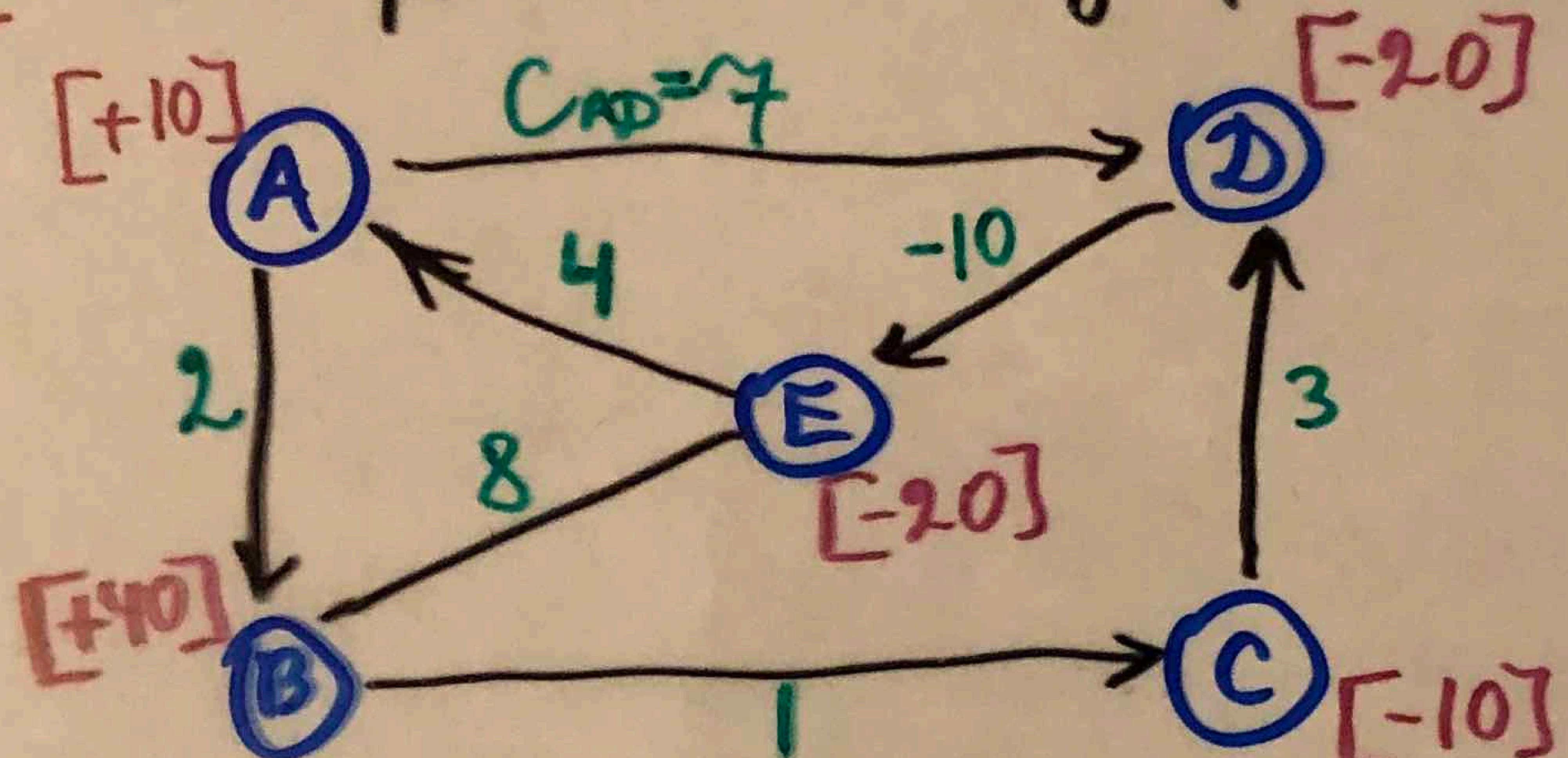
- From last time,

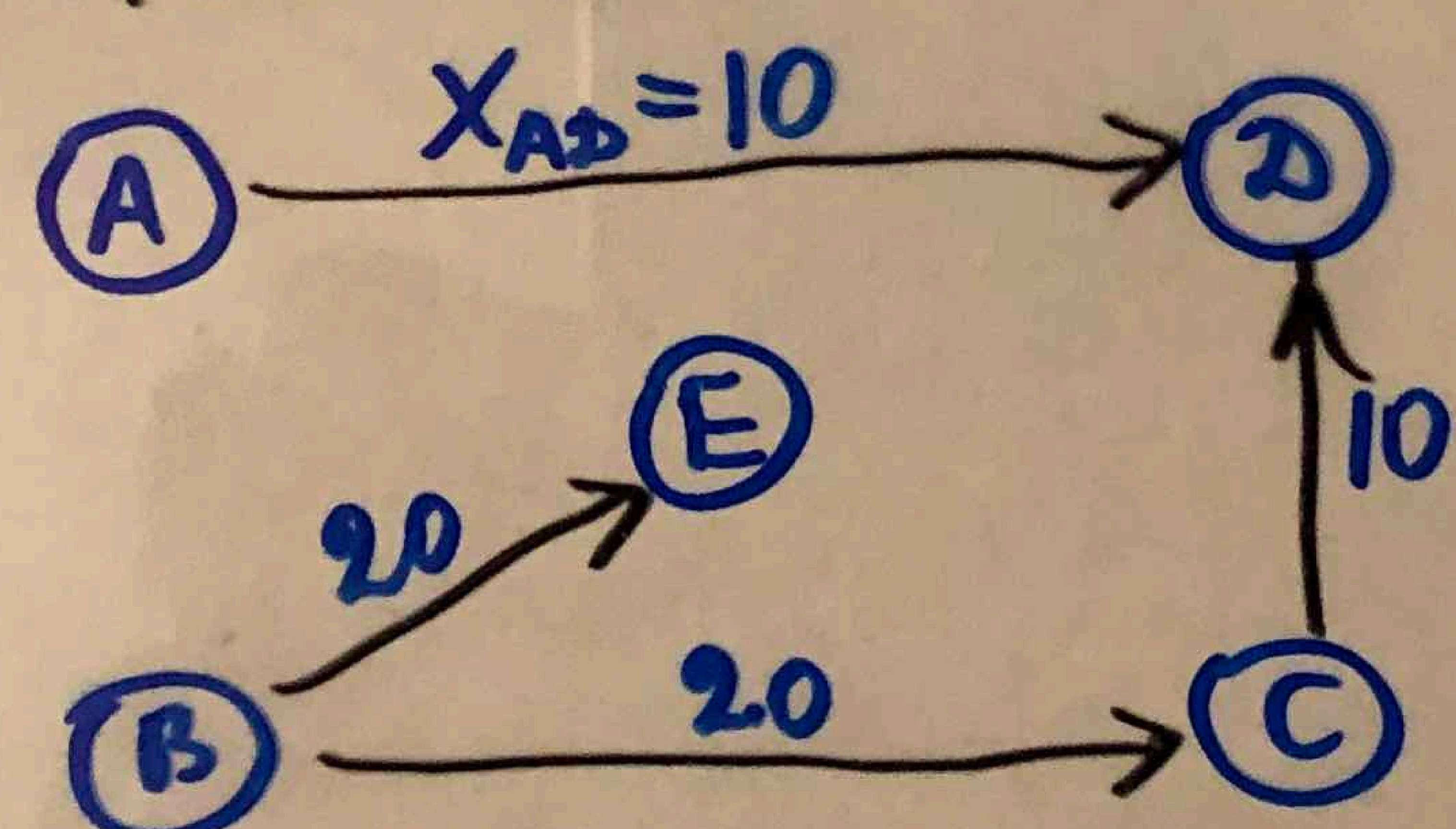
  basic solutions for (P) $\Longleftrightarrow$ spanning tree solutions

- Suppose we have initial BFS.
  (later we'll see how to get it)

  Ex.: The problem (in graph notation):



  - Initial BFS:



- Need to check optimality of this sol-n
  $\Longleftrightarrow$ check feasibility of complementary dual sol-n.

- How to get the complementary dual solution?

- a primal variable (arc) is basic → corresponding dual constraint is tight
- $n-1$ primal basic variables →

  $\left.\begin{array}{l} n-1 \text{ tight dual constraints} \\ n \text{ dual variables} \end{array}\right\} \rightarrow$ has one degree of freedom

  to solve the system,

  →
  - set one dual variable to 0;
  - solve the rest of the system.

Ex. (cont.)

algebraically:

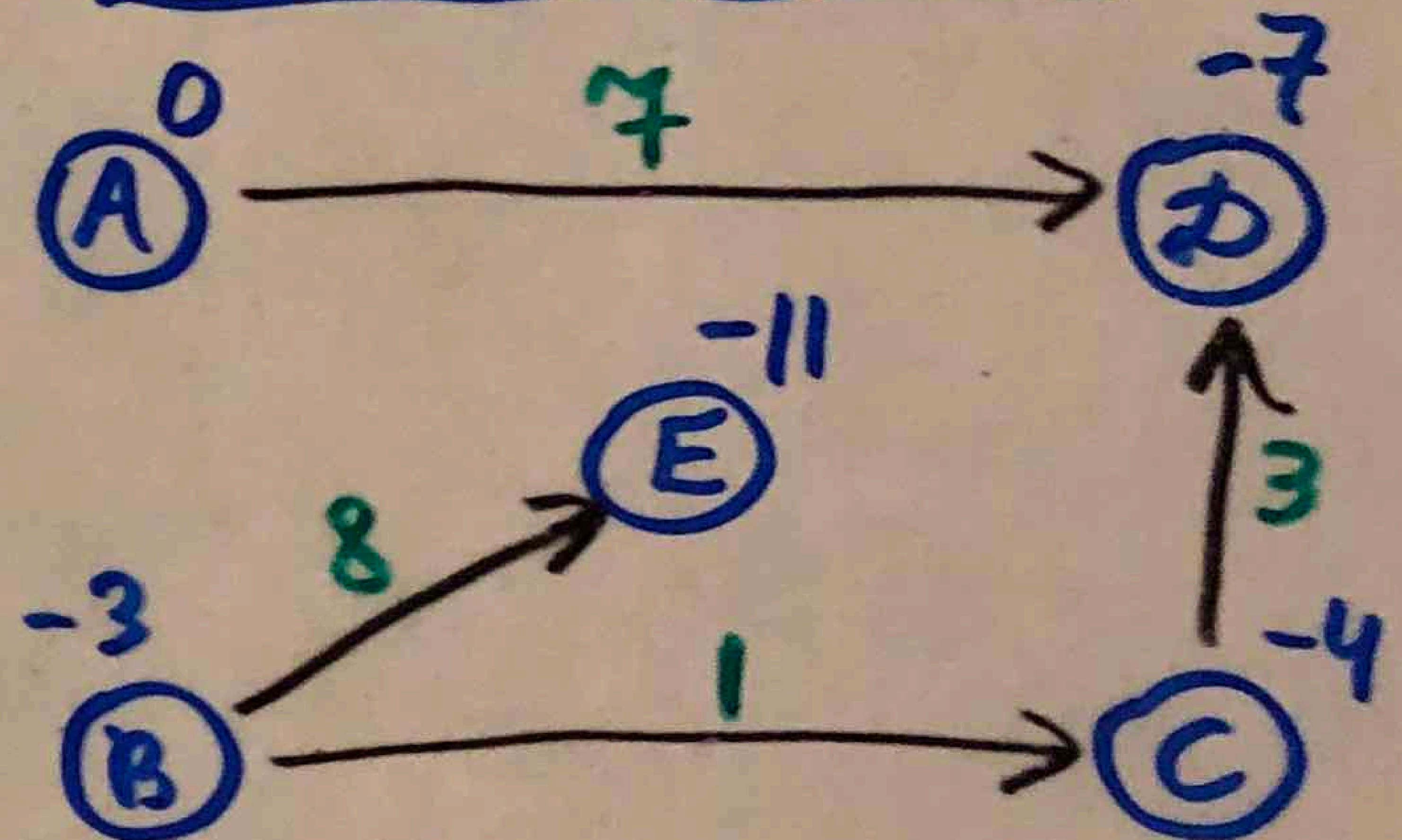
$A \to D$: $\pi_A - \pi_D = 7$

$C \to D$: $\pi_C - \pi_D = 3$

$B \to C$: $\pi_B - \pi_C = 1$

$B \to E$: $\pi_B - \pi_E = 8$

in graph notation:



- set $\pi_A = 0$; call node A root
- get other π_i's (node potentials)

- Check whether dual constraints corresponding to primal nonbasic variables (arcs) are satisfied.

  (checking dual feasibility)

Ex. (cont.):

$i \to j$ Nonbasic variables	Corresponding dual constraint: $c_{ij} - (\pi_i - \pi_j) \geq 0$
$E \to A$	$4 - (-11 - 0) = 15 > 0$
$D \to E$	$-10 - (-7 - (-11)) = -14 < 0$
$A \to B$	$2 - (0 - (-3)) = -1 < 0$

- Dual sol-n is infeasible:
  dual constraints corresponding to $D \to E$, $A \to B$ are negative $\Rightarrow$ $D \to E$ and $A \to B$ are candidates for entering basis
- Choose the arc with most negative coefficient (highest violation of the constraint):
  $D \to E$ is entering variable

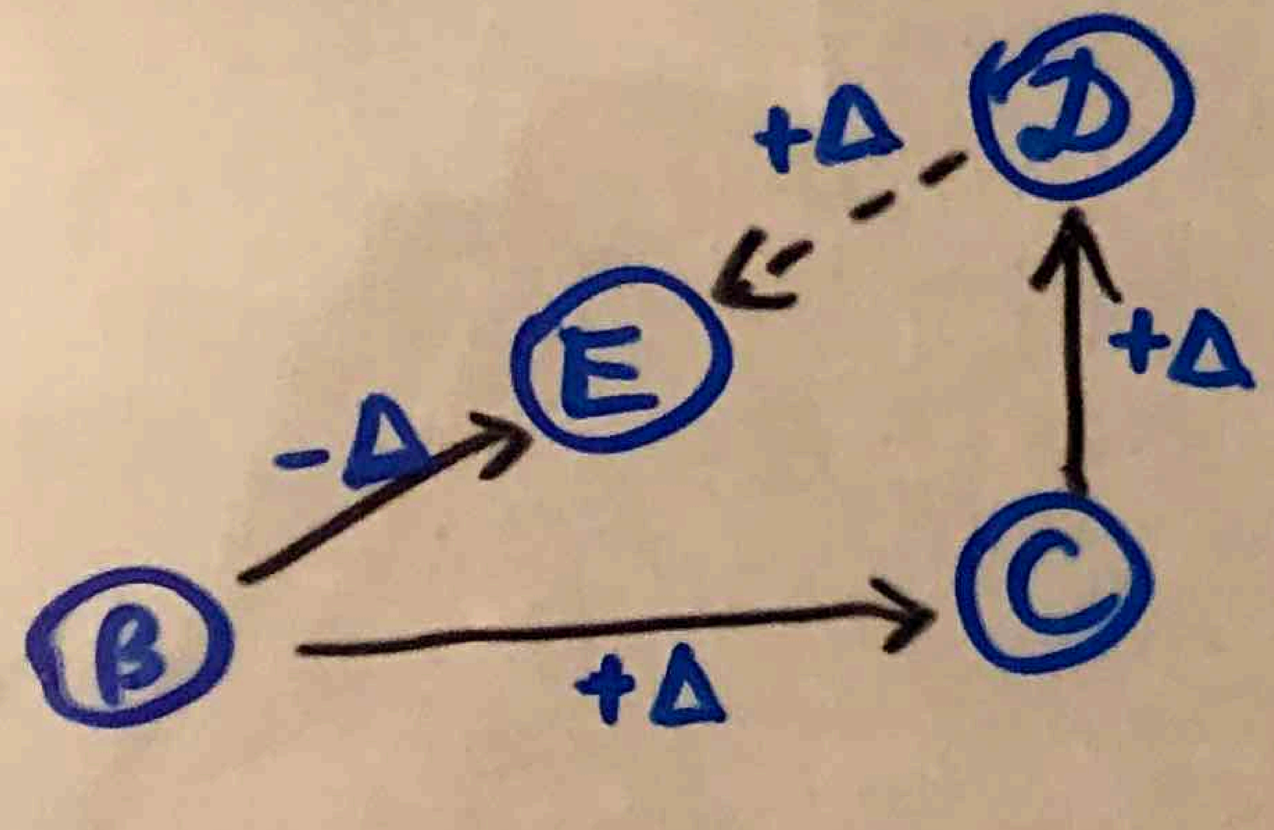
- **What is the leaving variable?**
  - Should get something equivalent to min-ratio test.
  - Adding one arc to a spanning tree creates a unique (undirected) cycle. Then one of the arcs of that cycle should leave the basis to keep the tree structure of basis solutions.

Ex. (cont.)

- Suppose x_{DE} is increased from 0 to Δ. Then the arc $D \to E$ is added to the basis tree. The created undirected cycle is

$$D - E - B - C - D.$$

- To keep cons.-of-flow satisfied, need to change the flow on the other arcs of the cycle:
  - if an arc has the same direction as $D \to E$ then its flow is increased by Δ.
  - if an arc has opposite direction of $D \to E$ then its flow is decreased by Δ.



- Increase Δ until the flow on one of the arcs of the cycle gets zero. (we can't have negative flow since we keep primal feasibility).

- To stay feasible:

$$B \to E:\ 20-\Delta \geq 0 \longrightarrow \Delta \leq 20$$

<u>$B \to E$ leaves the basis</u>

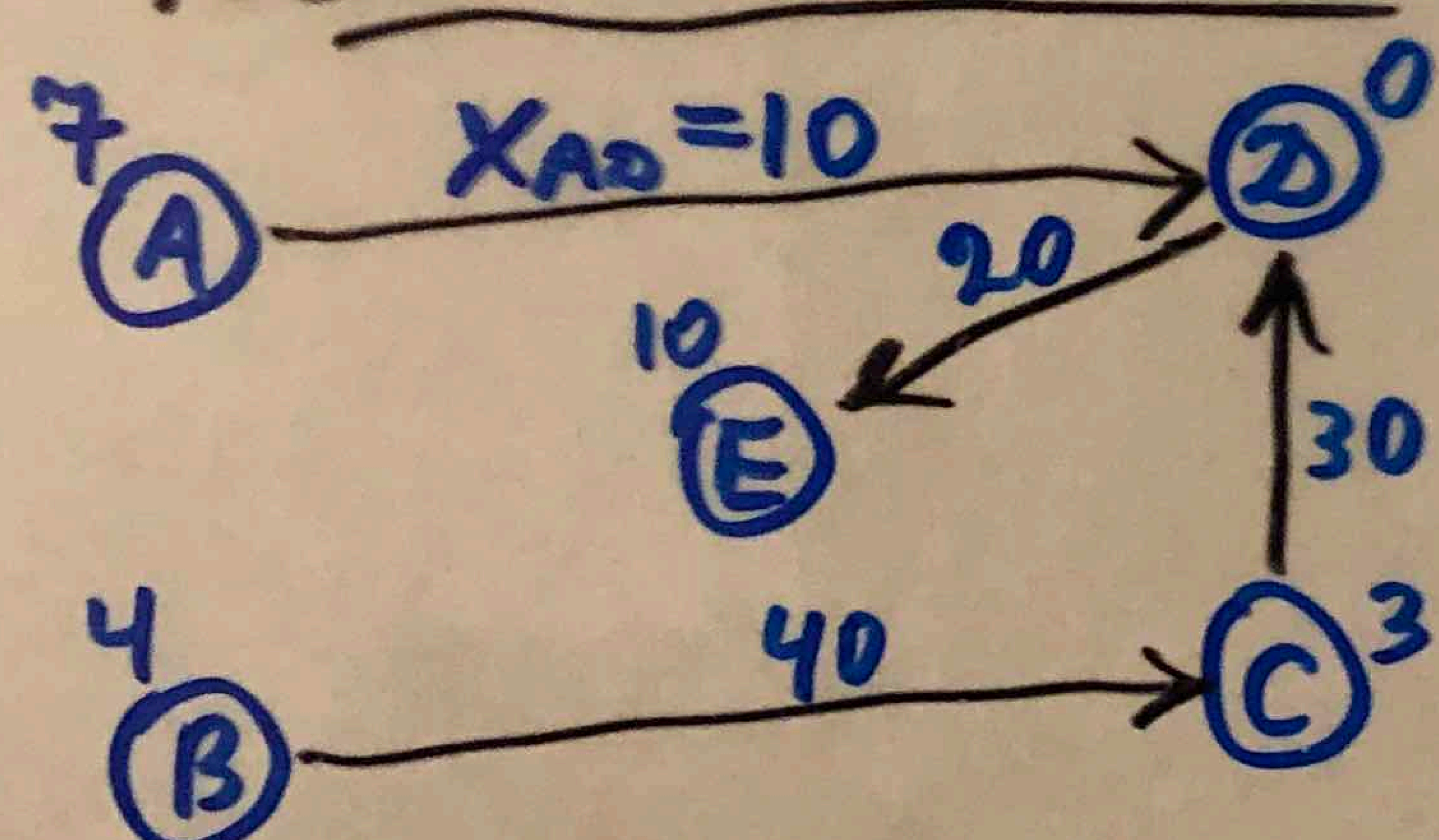
(Generally, might have <u>several</u> candidates for leaving basis. Choose the one with the smallest possible change of Δ).

- max increase of x_{DE} is $\Delta = 20$.

<u>End of Iteration 1</u>

<u>Iteration 2:</u>

<u>New basic sol-n:</u>



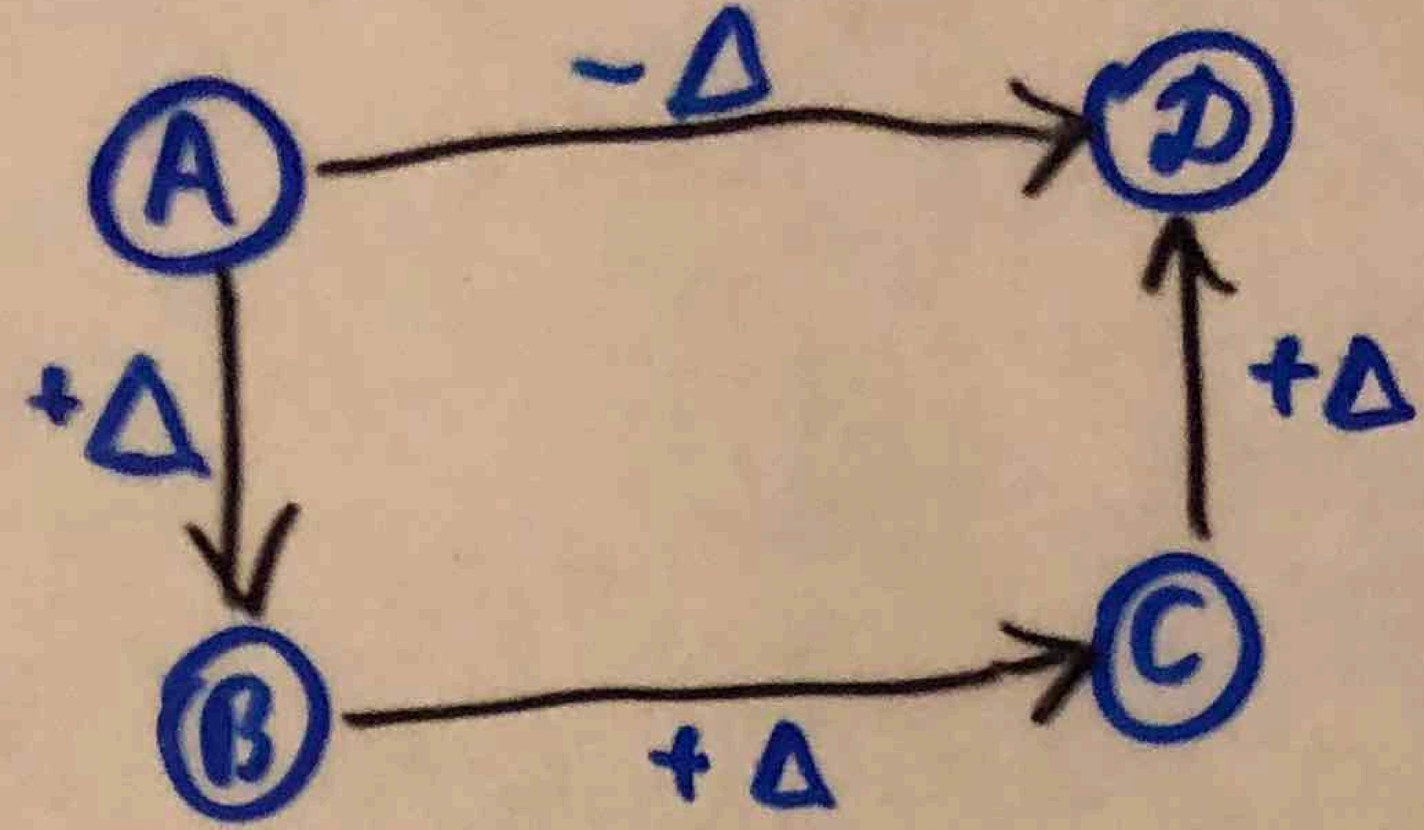
Compute node potentials (can do directly on the graph)

- <u>Feasibility test:</u>

Nonbasic arcs	Dual constr.
$E \to A$	$4-(10-7)=1>0$
$B \to E$	$8-(4-10)=14>0$
enters basis $\leftarrow$ $A \to B$	$2-(7-4)=\underline{-1<0}$

▼ Determine leaving variable.

The cycle:



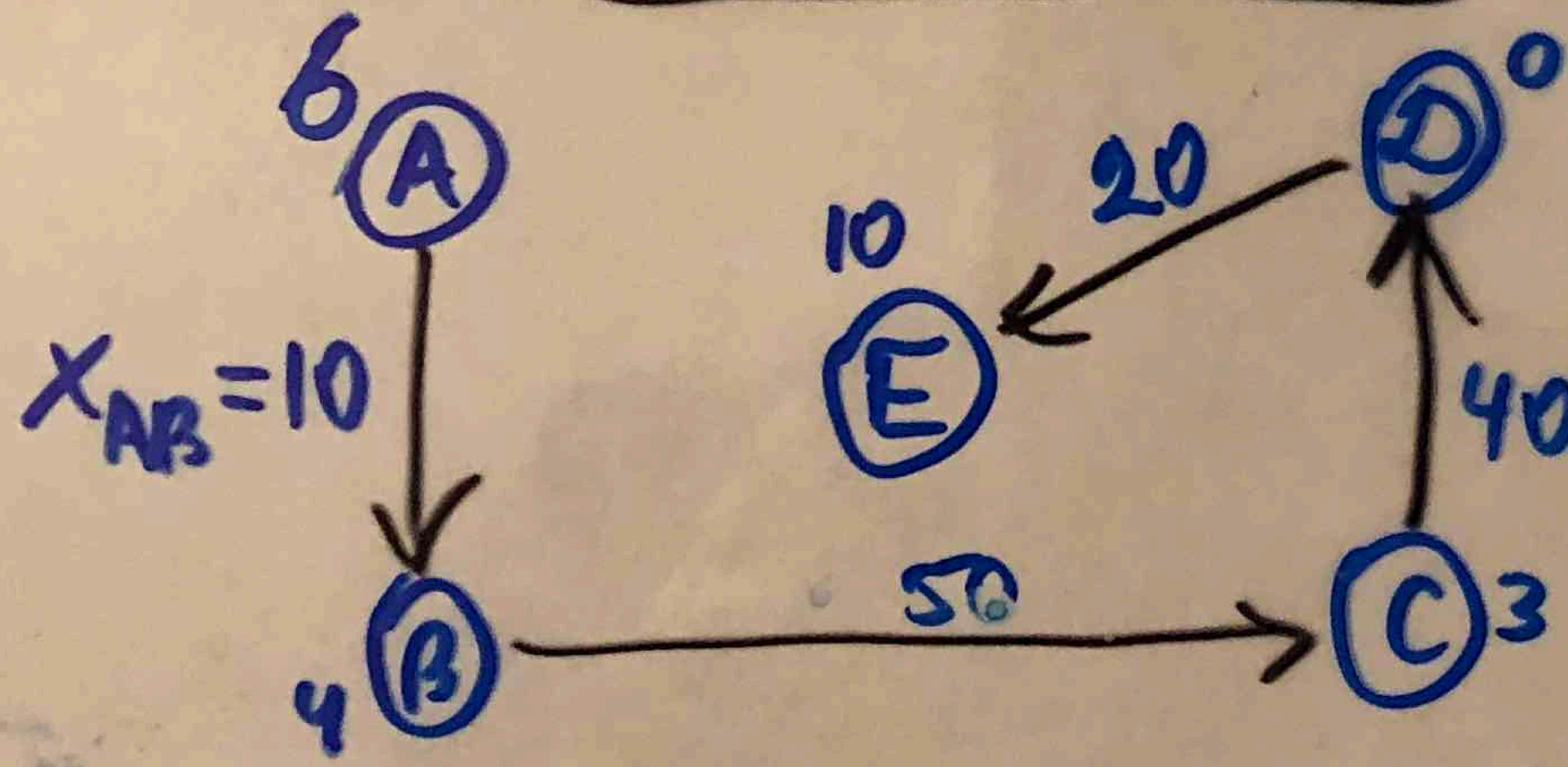
Min-ratio test: $A \to D: \ 10 - \Delta \geq 0 \to \Delta \leq 10$

$A \to D$ leaves basis

- max increase: $\Delta = 10$

• Iteration 3:

New basic sol-n:



▼ Feasibility test:

$A \to D: \ 7 - (6 - 0) = 1 > 0$

$B \to E: \ 8 - (4 - 10) = 14 > 0$

$E \to A: \ 4 - (10 - 6) = 0$

all nonnegative → optimal solution

$X_{AB} = 10, \ X_{BC} = 50, \ X_{CD} = 40, \ X_{DE} = 20, \ X_{ij} = 0$ for rest

total cost: $10 \cdot 2 + 50 \cdot 1 + 40 \cdot 3 - 10 \cdot 20 = -10$

Network Simplex – Algorithm Statement

- Start with initial BFS (= feasible spanning tree)
- Determine values of basic variables.
- Use compementary slackness, to get the complementary dual sol-n (i.e., compute node potentials)
- Check dual feasibility:
  for nonbasic $(i,j) \in E$, check $\pi_i - \pi_j \le c_{ij}$.
  If complem. sol-n is dual feasible
  then the sol-n is optimal.
  else find entering variable
  (corresp. dual constraint violated)
- Add entering variable (arc) to spanning tree, increase flow on it by Δ.
  Find the unique cycle and determine maximum increase of Δ.
  If increase of Δ is limited
  then find a leaving variable and go to
  else problem is unbounded.