Abstract view of Simplex

Consider LP in augmented form:

$$\begin{aligned} \max \quad & c^T x \\ \text{st} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad \text{(LP)}$$

- For some BF sol-n, let B = set of basic var.
  N = set of nonbasic var.
- Permute columns and rows of the system such that

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \quad c = \begin{pmatrix} c_B \\ c_N \end{pmatrix}, \quad A = [B \,\vdots\, N]$$

- Now can rewrite (LP) as

$$\left.\begin{aligned} \max \quad & \begin{pmatrix} c_B \\ c_N \end{pmatrix}^T \begin{pmatrix} x_B \\ x_N \end{pmatrix} \\ \text{s.t.} \quad & [B \,\vdots\, N] \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \\ & \begin{pmatrix} x_B \\ x_N \end{pmatrix} \geq 0 \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} \max \quad & c_B^T x_B + c_N^T x_N \\ \text{s.t.} \quad & B \cdot x_B + N \cdot x_N = b \\ & x_B, x_N \geq 0 \end{aligned}\right.$$

Question: Can we get the Simplex tableau corresponding to current basis without doing all those Simplex iterations.

- In a simplex tableau, columns of basic variables form an identity matrix, I.

Thus, to get that tableau, we need to premultiply the original constraints $B \cdot X_B + N \cdot X_N = b$ by B^{-1}:

$$B^{-1} \cdot (B \cdot X_B + N \cdot X_N) = B^{-1} b$$

$$X_B + B^{-1} N \cdot X_N = B^{-1} b \qquad (1')$$

- The coefficients computed in this system give the tableau corresponding to current basis (except row 0):

$$\left[I \quad B^{-1}N \;\middle|\; B^{-1}b \right]$$

Ex.:

$$\begin{aligned} 2X_1 + X_2 + X_3 \quad &= 6 \\ 2X_1 + 4X_2 \quad + X_4 &= 12 \end{aligned}$$

Suppose we want to get the simplex tableau corresponding to basis $\{X_1, X_2\}$.

- Then $B = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$.
- Compute B^{-1}, $B^{-1}N$, $B^{-1}b$

Example:

- Compute B^{-1}, $B^{-1}N$, $B^{-1}b$

$$B^{-1}=\begin{bmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{bmatrix}, \quad B^{-1}N=\begin{bmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{bmatrix}$$

$$B^{-1}b=\begin{bmatrix} 2/3 & -\frac{1}{6} \\ -1/3 & 1/3 \end{bmatrix}\cdot\begin{bmatrix} 6 \\ 12 \end{bmatrix}=\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

The tableau we get is the same as the one we got as a result of several simplex iterations. (see handout Simplex 4, page 2)

x_1	1	0	2/3	-1/6	2
x_2	0	1	-1/3	1/3	2

- Can compute also the coefficients of the obj. f-n.
  - Recall eq-n (1'): $x_B + B^{-1}N\cdot x_N = B^{-1}b$.
  - Express x_B in terms of x_N:

$$x_B = B^{-1}b - B^{-1}N\cdot x_N$$

  - Plug these values of x_B in the objective f-n: $z = c_B^T x_B + c_N^T x_N = c_B^T\cdot(B^{-1}b - B^{-1}N\cdot x_N) + c_N^T x_N =$

$$= \underbrace{C_B^T B^{-1} b}_{\substack{\text{current}\\ \text{obj. f-n}\\ \text{value}}} + \underbrace{(C_N^T - C_B^T B^{-1} N)}_{\text{nonbasic coefficients (with opposite signs)}} \cdot X_N$$

Ex. (cont.):

$$C_B^T B^{-1} b = [1 \quad 1]\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 4$$

$$C_N^T - C_B^T B^{-1} N = [0 \quad 0] - [1 \quad 1]\begin{bmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{bmatrix} = [-1/3 \quad -1/6]$$

These are the same coefficients as in the tableau of (Simplex 4, slide 2).

- Summarizing, given a basis the corresponding Simplex tableau can be directly computed from given data:

Basic var.	X_B	X_N	RHS
Z	0	$C_B^T B^{-1} N - C_N^T$	$C_B^T B^{-1} b$
X_B	I	$B^{-1} N$	$B^{-1} b$

(*)

- Based on this observation, a streamlined variation of Simplex (next page)

Revised Simplex Algorithm

- **Motivation**: Suppose we get a problem with 10 constraints and 10,000,000 variables.
  To Keep the whole simplex tableau in storage, need to Keep >100 mil. real numbers in memory.
- **Question**: Do we really need to Keep the whole tableau in storage?

- Suppose at the beginning of an iteration we Know just the current basis B (no simplex tableau is given).
  - ▼ 10 constraints → the basis matrix is 10×10
    → first compute B^{-1}
  - ▼ To start the iteration, need row 0 to find entering variable x_j.
    Compute it based on (*) of page 2:
    $$C_B^T B^{-1} N - C_N^T$$
  - ▼ Suppose x_j is chosen to enter the basis. To determine the leaving variable, we need the column of x_j and current RHS:

Again from (*),

current column of x_j is $B^{-1}A_j$;

current RHS is $B^{-1}b$.

- After min-ratio test we get a new basis and can iterate.

Statement of the algorithm (Revised Simplex)

Start with a BF sol-n (do Phase 1 if necessary)

1. Compute B^{-1}, $B^{-1}b$, $C_B^T B^{-1} b$
2. Test optimality (or find entering basic variable).
   Compute $C_B^T B^{-1} N - C_N^T$.
   If $C_B^T B^{-1} N - C_N^T \geq 0$ then have an optimal sol-n. Stop.
   Else determine an entering variable x_j
3. Compute the pivot column of tableau: $B^{-1}A_j$.
4. Perform min-ratio test to determine leaving basic variable
   (or to detect unboundedness)
5. Update the basis and go to step 1.

Example: $\max\ x_1 + x_2$

$$\text{s.t. } 2x_1 + x_2 + x_3 = 6$$
$$2x_1 + 4x_2 \quad + x_4 = 12$$
$$x_i \geq 0 \ \forall i$$

$B_1 = \{3,4\}$, $N_1 = \{1,2\}$

Iteration #1:

① $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B^{-1}$

$B^{-1}b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$ → current solution is $x = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 12 \end{pmatrix}$

$C_B^T B^{-1} b = (0\ 0)\begin{pmatrix} 6 \\ 12 \end{pmatrix} = 0$ → current value is 0

② $C_B^T \cdot B^{-1} N - C_N^T = (0\ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix} - (1\ 1) = (-1, -1)$

pick x_2 to enter the basis

③ Compute pivot column:

$B^{-1}A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$RHS = B^{-1}b = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$

④ min-ratio test: $\min\left(\frac{6}{1}, \frac{12}{4}\right) = \min(6,3) = 3$

x_4 leaves the basis

⑤ $B_2 = \{3,2\}$, $N_2 = \{1,4\}$

↑ keep the order

Iteration 2:

① $B = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & -1/4 \\ 0 & 1/4 \end{pmatrix}$

$B^{-1}b = \begin{pmatrix} 1 & -1/4 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \rightarrow \quad x = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix}$

$C_B^T B^{-1} b = (0 \ \ 1) \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \leftarrow$ current value

② $C_B^T B^{-1} N - C_N^T = (0 \ \ 1) \begin{pmatrix} 1 & -1/4 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} - (1, 0) =$

$= (-1/2, \frac{1}{4})$

$\rightarrow$ x_1 enters the basis

③ pivot column: $B^{-1} A_1 = \begin{pmatrix} 1 & -1/4 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$

④ min-ratio test:

...	x_1	...	RHS
	$3/2$	...	3
	$1/2$		3

$\min\left(\frac{3}{\frac{3}{2}}, \frac{3}{\frac{1}{2}}\right) = \min(2, 6) = 2 \quad \rightarrow$ x_3 leaves the basis

⑤ $B_3 = \{1, 2\}$, $N = \{3, 4\}$

Iteration 3:

① $B = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{pmatrix}$

$B^{-1}b = \begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\quad x = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \leftarrow$ current sol-n

$C_B^T B^{-1} b = (1 \; 1)\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 4 \leftarrow$ current obj. f-n value

② $C_B^T B^{-1} N - C_N^T = (1 \; 1)\begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - (0,0)$

$= \left(\frac{1}{3}, \frac{1}{6}\right) > 0 \longrightarrow$

current sol-n is optimal