Sensitivity Analysis I

$$\max \; c^T x$$
$$\text{st} \quad Ax \leq b$$
$$x \geq 0$$

- Suppose the optimal sol-n to this problem is x^*, z^*.
- Questions:
  - What happens if parameters (A, b, c) are changed slightly (or not so slightly)?
  - Will current sol-n still be feasible? optimal?
  - Will the value z^* change?
  - Which parameters are especially sensitive to those changes?
- Why is sensitivity analysis important?
  - Given data are not certain in reality.
  - Sometimes the sol-n to the original problem might look unreasonable.
  - Some parameters are related to policy decisions which might be changed.

- How to perform sensitivity analysis?
  - One way is to solve the LP with modified data from scratch.
    This way is <u>not</u> effective.
  - Use the optimal sol-n (tableau) of the original problem <u>plus</u> the abstract view of simplex (fundamental insight, ...) to do the analysis.

We will adopt the second way.

Next:

Simple sensitivity analysis

- changing coefficients of obj. f-n
- changing RHS

Sensitivity Analysis

Here: some cases

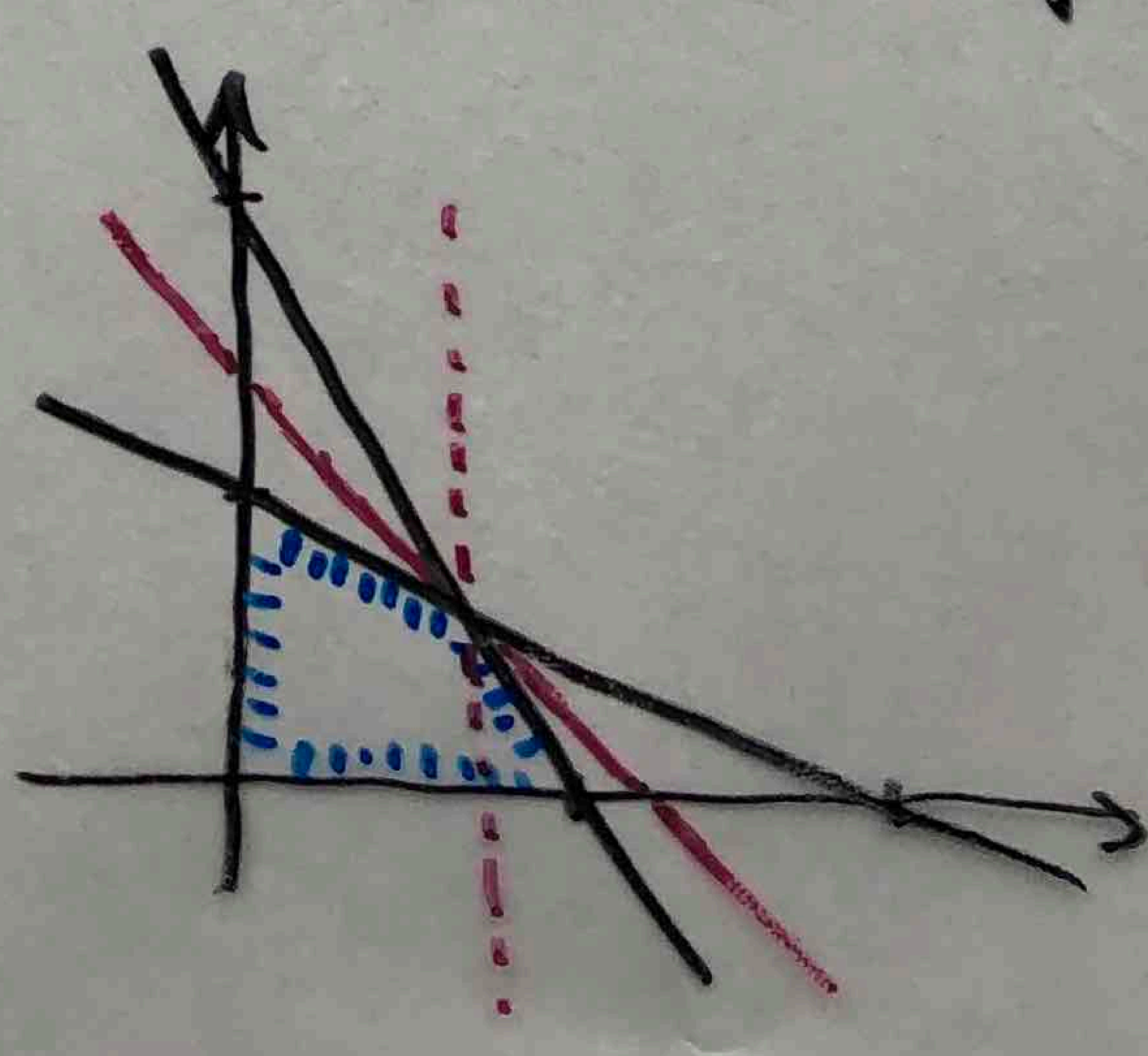
The rest: after introducing Duality Theory

Changing coefficients of objective f-n

$$\begin{aligned} \max \ & c^T x \\ \text{st} \ & Ax \le b \\ & x \ge 0 \end{aligned}$$

- Opt. sol-n : x^*, z^*
- Suppose c_j is changed to $\bar{c}_j$.
- Questions:
  - Is feasibility of x^* changed?
    — No because the feasible region is the same.
  - Is optimality of x^* changed?
    — Can't answer immediately; depends on the magnitude of changes.

Graphically:



depends on the slope of isoprofit lines

Algebraically:

Again use table (*):

Basic var.	X_B	X_N	RHS
Z	0	$C_B^T B^{-1} N - C_N^T$	$C_B^T B^{-1} b$
X_B	I	$B^{-1} N$	$B^{-1} b$

(*)

Example: $\max \ (1,1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

s.t. $\begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 6 \\ 12 \end{pmatrix}$

$(x_1 \ \ x_2) \geq (0 \ \ 0)$

- What if $c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ changes to $\begin{pmatrix} 1+\delta \\ 1 \end{pmatrix}$? Will $x^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ still be optimal?
- Only row (0) in (*) is affected by this change.

  Recall that current basis is $B = \{1, 2\}$.

  New $C_B^T = (1+\delta, 1)$; $C_N^T = (0, 0)$.

  $B^{-1} = \begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{pmatrix}$; $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

- Thus, the new coefficients of non-basic variables in row (0):

$$C_B^T B^{-1} N - C_N^T$$

$$= (1+\delta, 1)\begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - (0, 0)$$

$$= \left(\frac{1}{3} + \frac{2}{3}\delta, \frac{1}{6} - \frac{1}{6}\delta\right)$$

- The coefficients should be nonnegative to maintain optimality:

$$\begin{cases} \frac{1}{3} + \frac{2}{3}\delta \geq 0 \\ \frac{1}{6} - \frac{1}{6}\delta \geq 0 \end{cases} \longleftrightarrow \begin{cases} \delta \geq -\frac{1}{2} \\ \delta \leq 1 \end{cases}$$

- <u>If</u> $\delta \in [-\frac{1}{2}; 1]$ <u>then</u> $X^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is still optimal

<u>otherwise</u>
- it is not optimal;
- get new row (0) and <u>reoptimize</u>.

<u>Note</u>: reoptimize starting from the <u>final tableau</u> of the <u>original</u> problem. By this, we might save a lot of iterations.

- $-\frac{1}{2} \leq \delta \leq 1 \leftrightarrow 1-\frac{1}{2} \leq 1+\delta \leq 1+1 \rightarrow$

$x^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is still optimal if c_1 is in the range $[\frac{1}{2}; 2]$.

This is called <u>allowable range to stay optimal</u> for c_1.

- Even if x^* is still optimal, z^* might be changed.

Compute new z^* by the expression from $(*)$:

$$\text{New } z^* = c_B^T B^{-1} b = (1+\delta, 1)\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 4 + 2\delta$$

<u>More observations:</u>

- <u>If</u> a coefficient of <u>nonbasic</u> variable x_j is changed <u>then</u> need to recompute only x_j's coefficient in the final tableau.
- <u>Can</u> do sensitivity analysis also when several coefficients are changed simultaneously.

General Outline of Sensitivity Analysis

1) Revision of Final tableau
2) Feasibility test
3) Optimality test
4) Reoptimization

Changing RHS

Suppose b is changed to $\hat{b}$

- From (*), only RHS of final tableau will be changed in the result: $B^{-1}b \to B^{-1}\hat{b}$
- Need to check feasibility: $B^{-1}\hat{b} \overset{?}{\geq} 0$.
- If feasibility of current BF sol-n is maintained then it is also optimal.

Example:

$$\max \ (1,1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{st} \ \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$$(x_1, x_2) \geq (0,0)$$

- Suppose b_1 is changed from 6 to $6+\delta$.
- Then the new RHS is

$$B^{-1}\hat{b} = \begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 6+\delta \\ 12 \end{pmatrix} = \begin{pmatrix} 2+\frac{2}{3}\delta \\ 2-\frac{1}{3}\delta \end{pmatrix}$$

- The current BF sol-n stays feasible if

$$\begin{cases} 2+\frac{2}{3}\delta \geq 0 \\ 2-\frac{1}{3}\delta \geq 0 \end{cases} \leftrightarrow \begin{cases} \delta \geq -3 \\ \delta \leq 6 \end{cases} \leftrightarrow$$

$-3 \leq \delta \leq 6 \Rightarrow$ the current BF sol-n stays feasible if $6-3 \leq b_1 \leq 6+6$

$$3 \leq b_1 \leq 12$$

allowable range to stay feasible for b_1

- What if the current basic sol-n is not feasible any more? How to reoptimize? The answer after learning Duality Theory.
- Change in objective f-n? From (*)

$$C_B^T B^{-1}\hat{b} = (1 \;\; 1)\begin{pmatrix} 2+\frac{2}{3}\delta \\ 2-\frac{1}{3}\delta \end{pmatrix} = 4+\frac{1}{3}\delta$$

- Increasing b_1 by 1, increases obj. fn value by $1/3$. If b_1 corresponds to resource 1 (e.g., wood, laborhours, ...) then this value $1/3$ is called shadow price of resource 1.

- Can change b_1 and b_2 simultaneously. Suppose $\begin{pmatrix} 6 \\ 12 \end{pmatrix}$ is changed to $\begin{pmatrix} 6+\delta \\ 12-3\delta \end{pmatrix}$.
  - New RHS:
  $$B^{-1}\hat{b} = \begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 6+\delta \\ 12-3\delta \end{pmatrix} = \begin{pmatrix} 2+7/6\,\delta \\ 2-4/3\,\delta \end{pmatrix}$$
  - To stay feasible, $\begin{cases} 2+7/6\,\delta \geq 0 \\ 2-4/3\,\delta \geq 0 \end{cases} \leftrightarrow \delta \in [-12/7, 3/2]$
  - If stays feasible, new opt. sol-n is
  $$x^* = \begin{pmatrix} 2+7/6\,\delta \\ 2-4/3\,\delta \\ 0 \\ 0 \end{pmatrix}, \quad z^* = c_B^T B^{-1}\hat{b}$$
  $$= (1\ 1)\begin{pmatrix} 2+7/6\,\delta \\ 2-4/3\,\delta \end{pmatrix}$$
  $$= 4 - 1/6\,\delta$$