Sensitivity Analysis (continued)

Recall the general outline:

- Revision of final tableau
- Checking feasibility
- Checking optimality
- Reoptimization if necessary

Case 1: Changing b_i (RHS)

- Considered before
- What if changes in b_i make the current BS infeasible?
  Then reoptimize using dual simplex.

A related scenario:

Adding new constraint

Subcase 1: Current optimal sol-n X^* is feasible for new constraint.

- Then X^* is optimal also for the new problem.

Subcase 2: Current optimal sol-n does not satisfy new constraint.

- Then after introducing the new constraint into tableau (might require elementary operations) can reoptimize using dual simplex.

Example: Final tableau is

	x_1	x_2	x_3	x_4	RHS
z	0	0	$1/3$	$1/6$	4
x_1	1	0	$2/3$	$-1/6$	2
x_2	0	1	$-1/3$	$1/3$	2

$x^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

- Suppose we add a constraint $x_1 \leq 1$. x^* doesn't satisfy it $\Rightarrow$ add the constr. to the tableau and reoptimize.
- $x_1 \leq 1 \quad \rightarrow \quad x_1 + x_5 = 1$, $x_5 \geq 0$

	x_1	x_2	x_3	x_4	x_5	RHS
z	0	0	$1/3$	$1/6$	0	4
x_1	1	0	$2/3$	$-1/6$	0	2
x_2	0	1	$-1/3$	$1/3$	0	2
x_5	1	0	0	0	1	1

This is not in a proper form yet. After row operations:

	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	0	$1/3$	$1/6$	0	4
x_1	1	0	$2/3$	$-1/6$	0	2
x_2	0	1	$-1/3$	$1/3$	0	2
x_5	0	0	$-2/3^{*}$	$1/6$	1	-1

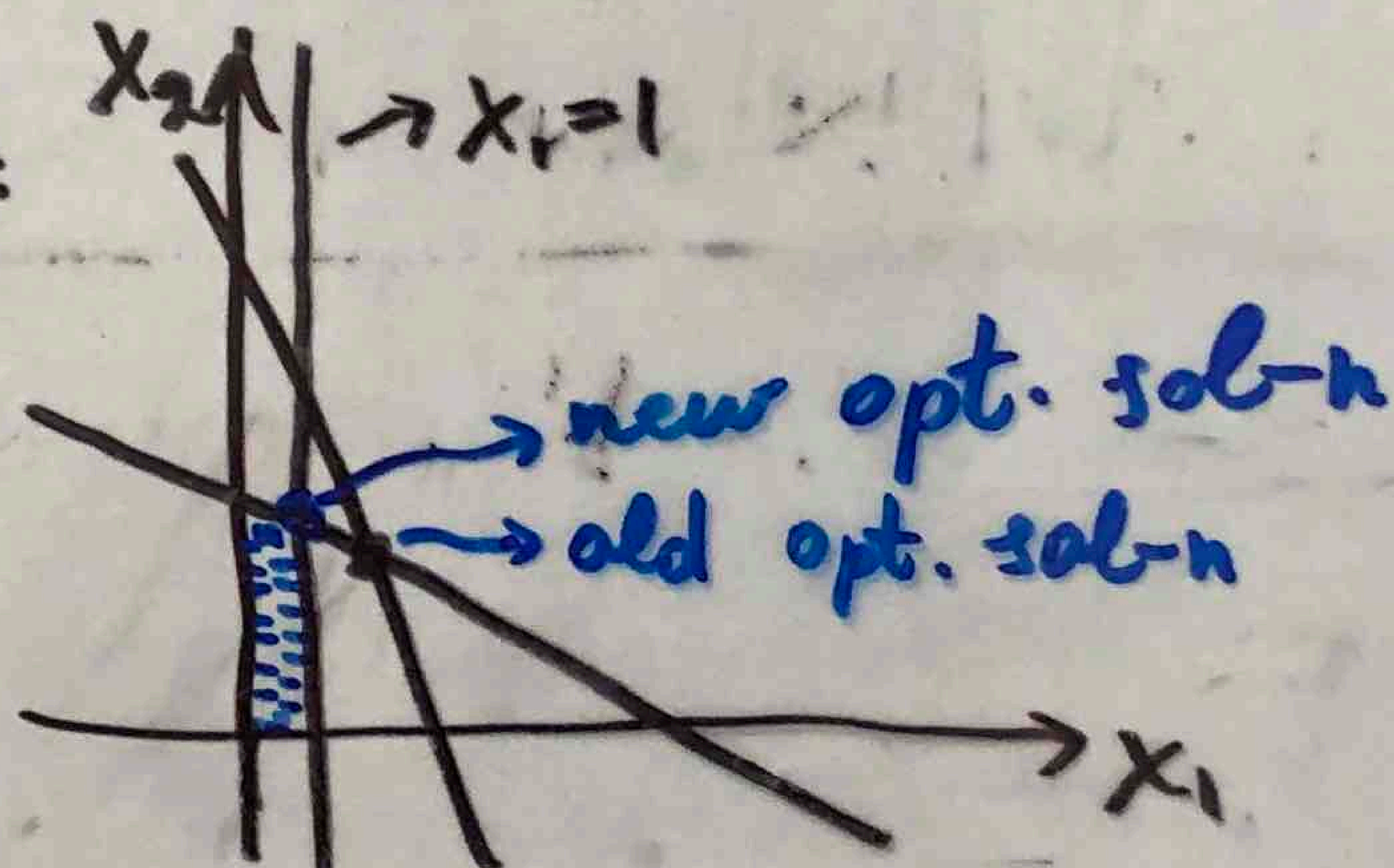
Apply dual simplex:

	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	0	0	$1/4$	$1/2$	$7/2$
x_1	1	0	0	0	1	1
x_2	0	1	0	$1/4$	$-1/2$	$5/2$
x_3	0	0	1	$-1/4$	$-3/2$	$3/2$

New optimal sol-n:

$$x^* = \begin{pmatrix} 1 \\ 2.5 \\ 1.5 \\ 0 \\ 0 \end{pmatrix}, \quad Z^* = 3.5$$

Graphically:



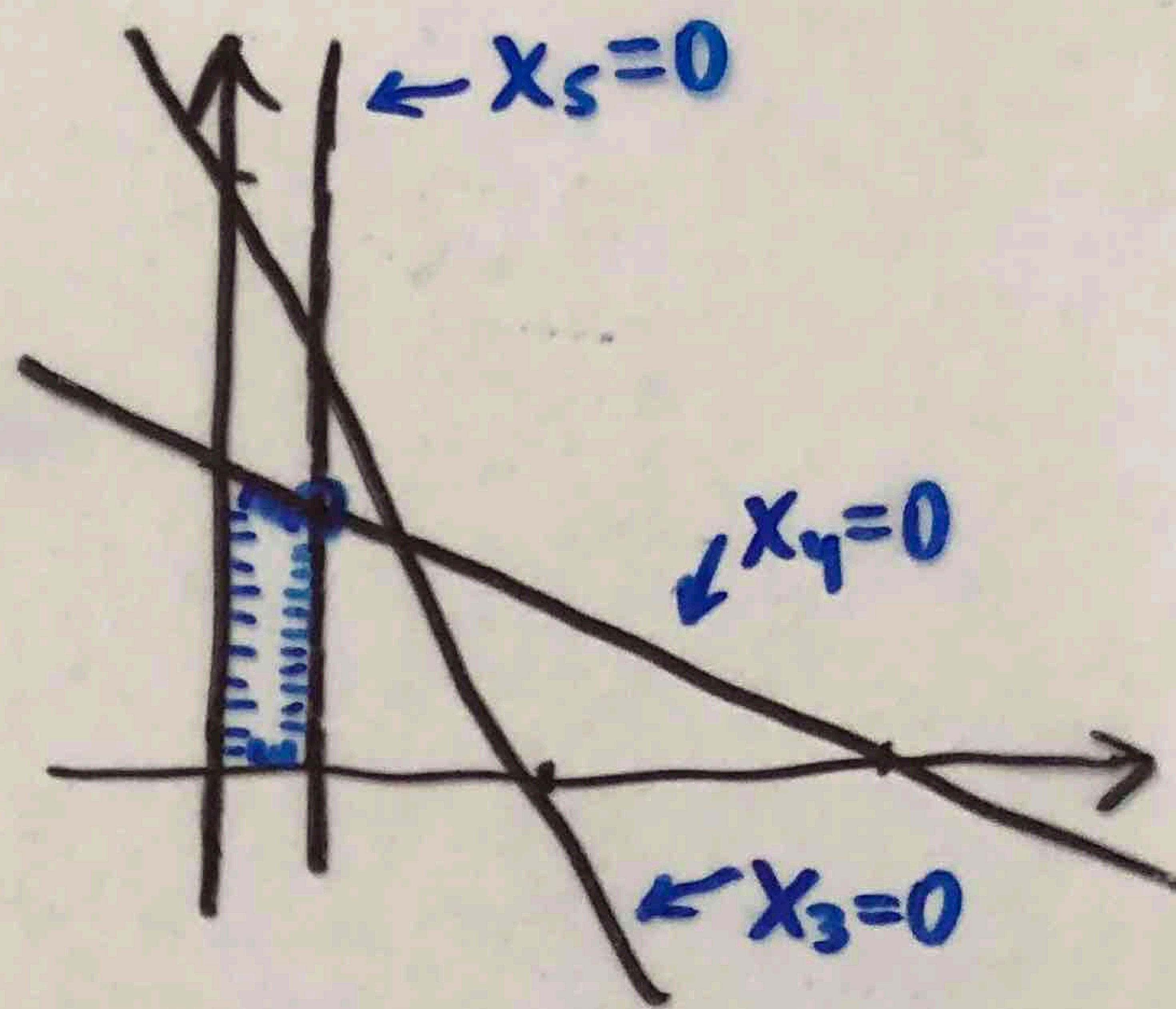
Deleting a constraint (optional)

Subcase 1: The optimal solution satisfies the deleted constraint strictly $\Rightarrow$ the corresponding slack variable is positive $\Rightarrow$ the slack variable is in the basis.

- Then just delete the row and column of the slack variable; the optimal sol-n is the same.

Example:

- Suppose the first constraint: $2X_1 + X_2 \leq 6$ in our problem is deleted.
- This is equivalent to
  - Keeping $2X_1 + X_2 + X_3 = 6$;
  - and changing the sign of X_3 from "≥ 0" to "free".



- But then deleting the row of X_3 will not change the feasible region.

The modified tableau:

	X_1	X_2	X_4	X_5	RHS
Z	0	0	$1/4$	$1/2$	$7/2$
X_1	1	0	0	1	1
X_2	0	1	$1/4$	$-1/2$	$5/2$

optimal sol-n:

$$X^* = \begin{pmatrix} 1 \\ 2.5 \\ 0 \\ 0 \end{pmatrix}, \quad Z^* = 3.5$$

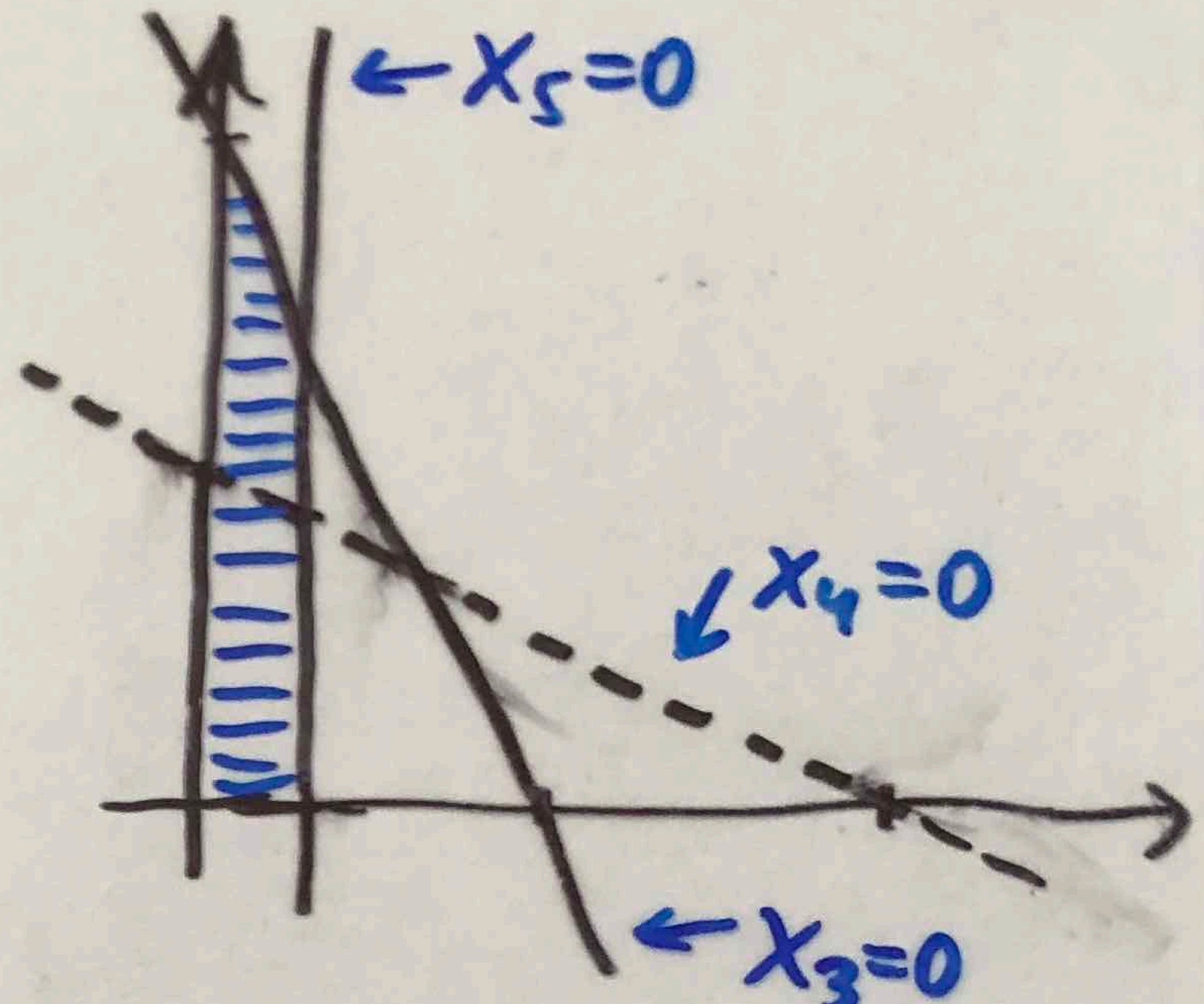
Subcase 2: The slack variable of the deleted constraint is nonbasic.

- Then need to pivot the slack variable into basis, and only after that delete the row. Needs reoptimization afterwards.

Example:

Suppose we delete the second constraint $2x_1+4x_2 \le 12$.
The corresponding slack var. x_4 is nonbasic.

- First need to enter x_4 into basis.



	x_1	x_2	x_3	x_4 ↓	x_5	RHS
z	0	0	0	$1/4$	$1/2$	$7/2$
x_1	1	0	0	0	1	1
x_2	0	1	0	$1/4^*$	$-1/2$	$5/2$
x_3	0	0	1	$-1/4$	$-3/2$	$3/2$

	x_1	x_2	x_3	x_4	x_5	RHS
z	0	-1	0	0	1	1
x_1	1	0	0	0	1	1
~~x_4~~	~~0~~	~~4~~	~~0~~	~~1~~	~~-2~~	~~10~~
x_3	0	1	1	0	-2	4

Delete row + column of x_4:

	x_1	x_2 ↓	x_3	x_5	RHS
z	0	-1	0	1	1
x_1	1	0	0	1	1
x_3	0	1	1	-2	4

	x_1	x_2	x_3	x_5 ↓	RHS
z	0	0	1	-1	5
x_1	1	0	0	1	1
x_2	0	1	1	-2	4

	x_1	x_2	x_3	x_5	RHS
z	1	0	1	0	6
x_5	1	0	0	1	1
x_2	2	1	1	0	6

optimal sol-n:

$$x^* = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_5 \end{array} \qquad z^* = 6$$

Case 2.

Changing coefficients of nonbasic variables.

- Recall the current Simplex tableau:

B.v.	x_B	x_N	RHS
z	0	$C_B^T B^{-1} N - C_N^T$	$C_B^T B^{-1} b$
x_B	I	$B^{-1} N$	$B^{-1} b$

(*)

- Suppose we made changes in the column of non-basic var. x_j, i.e., C_j and A_j are changed in original data.
- Then the only changes in (*) are in the column of x_j. Namely, $C_B^T B^{-1} A_j - C_j$ and $B^{-1} A_j$ will be changed.

Recall,

original system →

...	x_j	...
	c_j	
	A_j	

...	x_j	...
	$C_B^T B^{-1} A_j - C_j$	
	$B^{-1} A_j$	

← current tableau

- In the result of changes, the coefficient of x_j in row 0 might get negative.
  In that case, reoptimize using <u>primal simplex</u>.

Related Situation:
Adding a new variable

Suppose a new variable x_j is added.

Idea: Pretend that x_j was in the original model with all coefficients equal to zero.

- So those coefficients are still zero in the final tableau, and x_j is a nonbasic variable in the current BF sol-n.
  - Now consider the real coefficients of x_j as changes from zeros, and proceed as in the previous case.

Example: Suppose x_5 is added to our problem:

$$\begin{array}{ll} \max & x_1 + x_2 + .5 \cdot x_5 \\ \text{st} & 2x_1 + x_2 + x_5 \le 6 \\ & 2x_1 + 4x_2 + 2x_5 \le 12 \\ & x_1, x_2, x_5 \ge 0 \end{array}$$

- Suppose in the final tableau we had a column of x_5 with all zeros:

Basic var.	x_1	x_2	x_3	x_4	x_5	RHS
z	0	0	$\frac{1}{3}$	$\frac{1}{6}$	0	4
x_1	1	0	$2/3$	$-1/6$	0	2
x_2	0	1	$-1/3$	$\frac{1}{3}$	0	2

- Then after making changes, $c_5 = .5$, $A_5 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- So the new entries in the final tableau:

$$C_B^T B^{-1} A_5 - c_5 = \left(\tfrac{1}{3}\ \tfrac{1}{6}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} - .5 = \frac{2}{3} - .5 = \frac{1}{6}$$

$$B^{-1} A_5 = \begin{pmatrix} 2/3 & -1/6 \\ -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

- The modified tableau:

	x_1	x_2	x_3	x_4	x_5	RHS
z	0	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	4
x_1	1	0	$2/3$	$-1/6$	$1/3$	2
x_2	0	1	$-1/3$	$\frac{1}{3}$	$\frac{1}{3}$	2

No need for reoptimization here.

$$x^* = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad z^* = 4$$

Note: Another way to check the optimality is to check the feasibility of the dual constraint corresponding to x_5:

$y_1 + 2y_2 \geq .5$. Really, $\frac{1}{3} + 2 \cdot \frac{1}{6} = \frac{2}{3} \geq .5$