The Tableau Method

- convenient shorthand for Simplex
- $$\begin{aligned} &\max Z \\ &\text{st } Z - x_1 - x_2 &&= 0 \\ &\quad 2x_1 + x_2 + x_3 &&= 6 \\ &\quad 2x_1 + 4x_2 \quad + x_4 &&= 12 \\ &\quad x_i \geq 0 \ \forall i \end{aligned}$$

Write the same in a "tableau":

↓ enters the basis

Basic var.	x_1	x_2	x_3	x_4	RHS
Z	-1	-1	0	0	0
x_3	2	1	1	0	6
← x_4	2	4^*	0	1	12

leaves the basis

pivot element

min-ratio test: $\min\left(\frac{6}{1}, \frac{12}{4}\right) = 3$

Next tableau (one tableau per iteration)

Basic var.	↓ x_1	x_2	x_3	x_4	RHS
Z	$-\frac{1}{2}$	0	0	$\frac{1}{4}$	3
← x_3	$\frac{3}{2}^*$	0	1	$-\frac{1}{4}$	3
x_2	$\frac{1}{2}$	1	0	$\frac{1}{4}$	3

min ratio test: $\min\left(\frac{3}{\frac{3}{2}}, \frac{3}{\frac{1}{2}}\right) = 2$

Basic var.	x_1	x_2	x_3	x_4	RHS
Z	0	0	$1/3$	$1/6$	4
x_1	1	0	$2/3$	$-\frac{1}{6}$	2
x_2	0	1	$-1/3$	$1/3$	2

optimal tableau

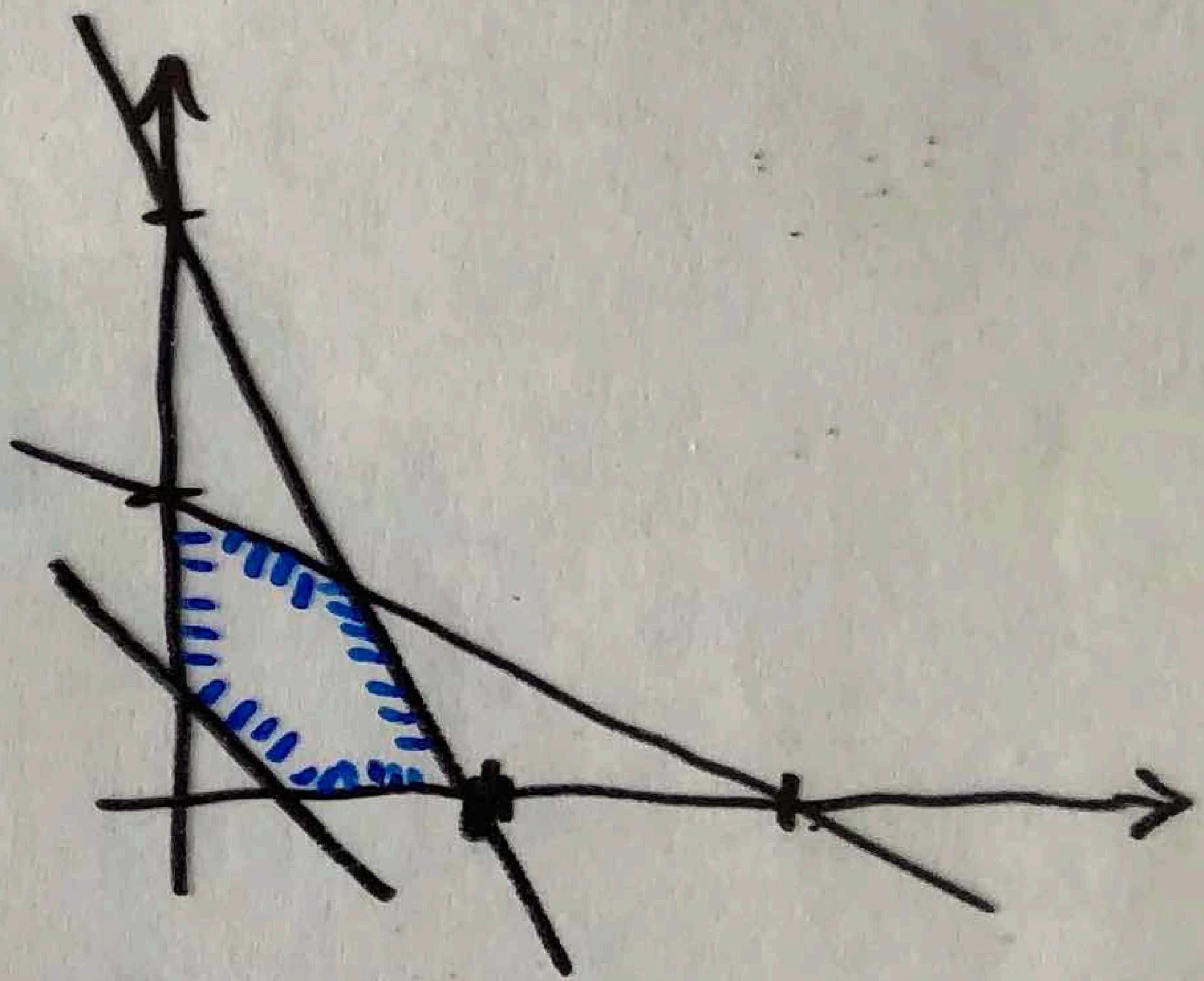
$$x^* = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad Z^* = 4$$

Two-Phase Method – finding initial BF sol-n

What if the origin is not a BF sol-n?

Example:

$$\begin{aligned} \max\ & x_1 + x_2 \\ \text{s.t. } & 2x_1 + x_2 \le 6 \\ & 2x_1 + 4x_2 \le 12 \\ & x_1 + x_2 \ge 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$



$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is basic but not feasible. In the system,

$$\begin{aligned} 2x_1 + x_2 + x_3 \quad\quad\quad &= 6 \\ 2x_1 + 4x_2 \quad + x_4 \quad &= 12 \\ -x_1 - x_2 \quad\quad + x_5 &= -1 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

$x = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 12 \\ -1 \end{pmatrix}$ is infeasible basic sol-n

Trick: introduce "artificial variable" X_6 for the 3^{rd} constraint:

$$\begin{aligned} 2X_1 + X_2 + X_3 &= 6 \\ 2X_1 + 4X_2 + X_4 &= 12 \\ X_1 + X_2 - X_5 + X_6 &= 1 \\ X_i \geq 0,\ i=1,..,6 \end{aligned}$$

- $X = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 12 \\ 0 \\ 1 \end{pmatrix}$ is a BF sol-n for this new system.
- Want to drive X_6 to 0 to make the sol-n feasible also for the original system
- Thus, in Phase 1 minimize the sum of artificial variables: $\min \sum(\text{art. var.'s})$
- In our ex., Phase 1 LP is:

$$\begin{aligned} \min\ & X_6 \\ \text{s.t. } & 2X_1 + X_2 + X_3 = 6 \\ & 2X_1 + 4X_2 + X_4 = 12 \\ & X_1 + X_2 - X_5 + X_6 = 1 \\ & X_i \geq 0 \ \forall i \end{aligned}$$

- $\min X_6 \Leftrightarrow \max -X_6 = X_1 + X_2 - X_5 - 1$

(Need this to get a proper row 0 in the initial tableau)

- Thus, the initial tableau in Phase 1:

Basic var.	↓ X_1	X_2	X_3	X_4	X_5	X_6	RHS
$\tilde{Z}$	-1	-1	0	0	1	0	-1
X_3	2	1	1	0	0	0	6
X_4	2	4	0	1	0	0	12
← X_6	1^*	1	0	0	-1	1	1

$\min\left(\frac{6}{2}, \frac{12}{2}, \frac{1}{1}\right) = 1$

Basic var.	X_1	X_2	X_3	X_4	X_5	X_6	RHS
$\tilde{Z}$	0	0	0	0	0	1	0
X_3	0	-1	1	0	2	-2	4
X_4	0	2	0	1	2	-2	10
X_1	1	1	0	0	-1	1	1

optimal tableau for phase 1

$\tilde{x} = \begin{pmatrix}1\\0\\4\\10\\0\\0\end{pmatrix}$ is a BF sol-n for Phase 1 LP with $X_6 = 0$ $\Big\} \rightarrow$

$x = \begin{pmatrix}1\\0\\4\\10\\0\end{pmatrix}$ is BF sol-n for the original LP

- Start Phase 2: modify final tableau of Phase 1 the following way:
  - drop the column of X_6
  - modify row 0 to optimize the original obj. f-n

- Need to re-express the original obj. f-n in terms of <u>current nonbasics</u>:

  from the x_1-row, $X_1 = 1 - X_2 + X_5$ →

  max $X_1 + X_2 = 1 + X_5$.

- The initial **Phase 2** tableau:

Basic var.	X_1	X_2	X_3	X_4	↓ X_5	RHS
Z	0	0	0	0	-1	1
← X_3	0	-1	1	0	2^*	4
X_4	0	2	0	1	2	10
X_1	1	1	0	0	-1	1

$\min\left(\frac{4}{2}, \frac{10}{2}\right) = 2$

Basic var.	X_1	↓ X_2	X_3	X_4	X_5	RHS
Z	0	$-1/2$	$1/2$	0	0	3
X_5	0	$-1/2$	$1/2$	0	1	2
← X_4	0	3^*	-1	1	0	6
X_1	1	$1/2$	$1/2$	0	0	3

$\min\left(\frac{6}{3}, \frac{3}{.5}\right) = 2$

Basic var.	X_1	X_2	X_3	X_4	X_5	RHS
Z	0	0	$1/3$	$1/6$	0	4
X_5	0	0	$1/3$	$1/6$	1	3
X_2	0	1	$-1/3$	$1/3$	0	2
X_1	1	0	$2/3$	$-1/6$	0	2

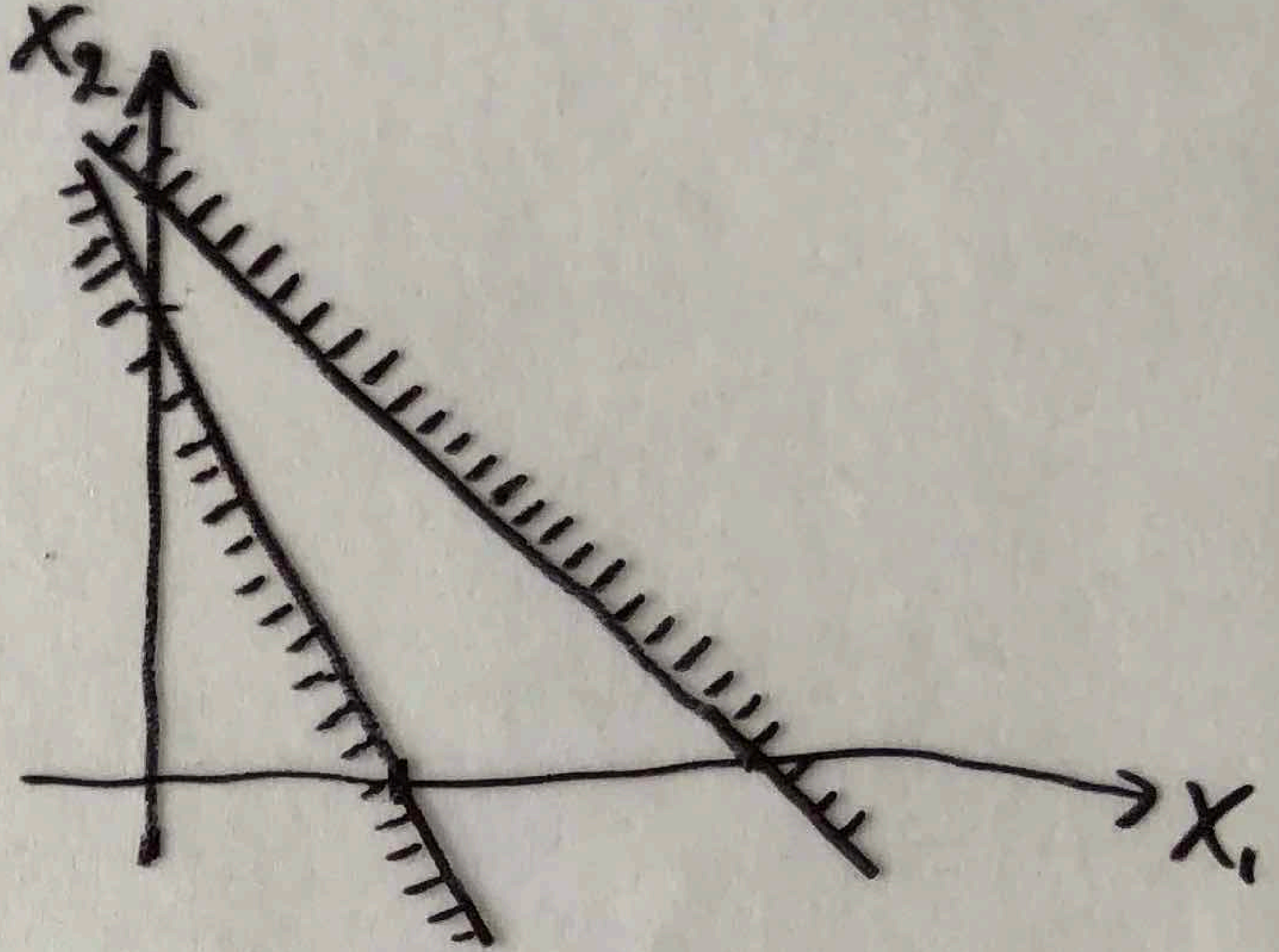
optimal tableau.

$X^* = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \quad Z^* = 4$

What happens if the problem is infeasible?

Example:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 6 \\ & x_1 + x_2 \geq 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$



- Augmented form:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 = 6 \\ & -x_1 - x_2 + x_4 = -7 \\ & x_i \geq 0 \ \forall i \end{aligned} \qquad (*)$$

- Add artificial variable x_5 and do Phase 1:

$$\begin{aligned} \min \quad & x_5 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 = 6 \\ & x_1 + x_2 - x_4 + x_5 = 7 \\ & x_i \geq 0 \ \forall i \end{aligned} \qquad (**)$$

- $\min x_5 \rightarrow \max -x_5 = x_1 + x_2 - x_4 - 7$

basic var.	x_1	↓ x_2	x_3	x_4	x_5	RHS
$\tilde{z}$	-1	-1	0	1	0	-7
← x_3	2	1^*	1	0	0	6
x_5	1	1	0	-1	1	7

$\min\left(\frac{6}{1}, \frac{7}{1}\right) = 6$

basic var.	x_1	x_2	x_3	x_4	x_5	RHS
$\tilde{z}$	1	0	1	1	0	-1
x_2	2	1	1	0	0	6
x_5	-1	0	-1	-1	1	1

$\tilde{z}^* = -1 \rightarrow$ opt. value of (**) is 1

This is an optimal tableau for Phase 1 but x_5 is still in basis at value 1 $\Rightarrow$ the original problem is infeasible

Proof: Assume the opposite: the original LP is feasible. Then (*) has a feasible sol-n: $\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix}$. But then $\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ 0 \end{pmatrix}$ is a feasible sol-n for (**) with obj. f-n value 0 $\Rightarrow$ optimal value of (**) is 0. This contradicts the final tableau of Phase 1.

Thus, the contrary assumption is wrong, and the original problem is infeasible.

□

Summary of Phase 1

- If the optimal value of Phase 1 LP is 0 then the optimal sol-n of Phase 1 LP is an initial BF sol-n for Phase 2.
- If the optimal value of Phase 1 LP is >0 then the original problem is infeasible.

Next time

- unboundedness
- multiple optimal sol-ns
- degeneracy
- breaking ties