Unbounded Problems

$$\max \; x_1 + x_2$$
$$\text{s.t.} \; -x_1 + x_2 \leq 1$$
$$x_1 - x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

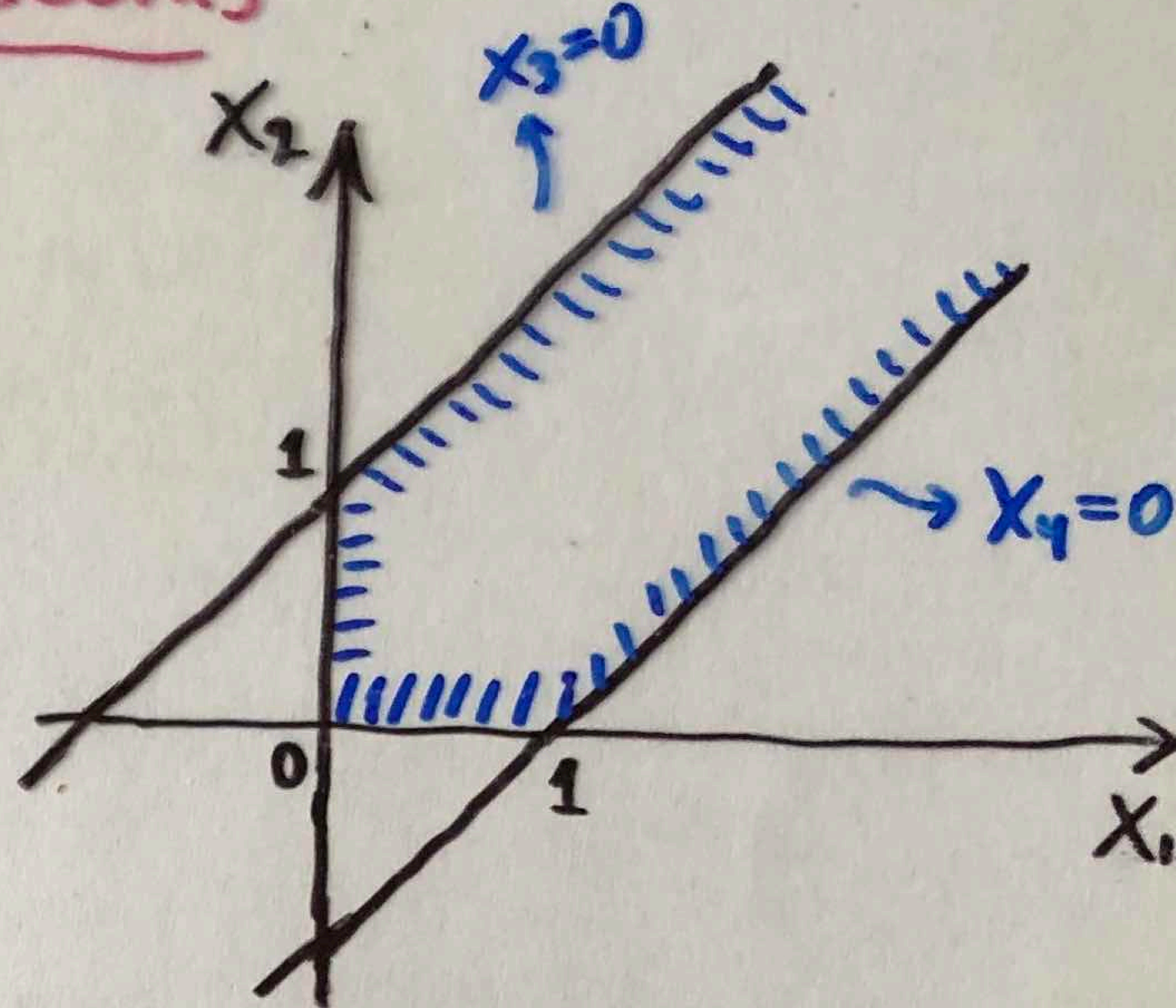


tableau (after introducing slack variables x_3 and x_4)

Basic var.	x_1	x_2	x_3	x_4	RHS
z	-1	-1	0	0	0
x_3	-1	1	1	0	1
x_4	1*	-1	0	1	1

$B = \{3,4\}, N = \{1,2\}$

Basic var.	x_1	x_2	x_3	x_4	RHS
z	0	-2	0	1	1
x_3	0	0	1	1	2
x_1	1	-1	0	1	1

$B = \{3,1\}, N = \{4,2\}$

x_2 is the only candidate to enter the basis; <u>but</u> we can increase x_2 infinitely without violating the constraints.

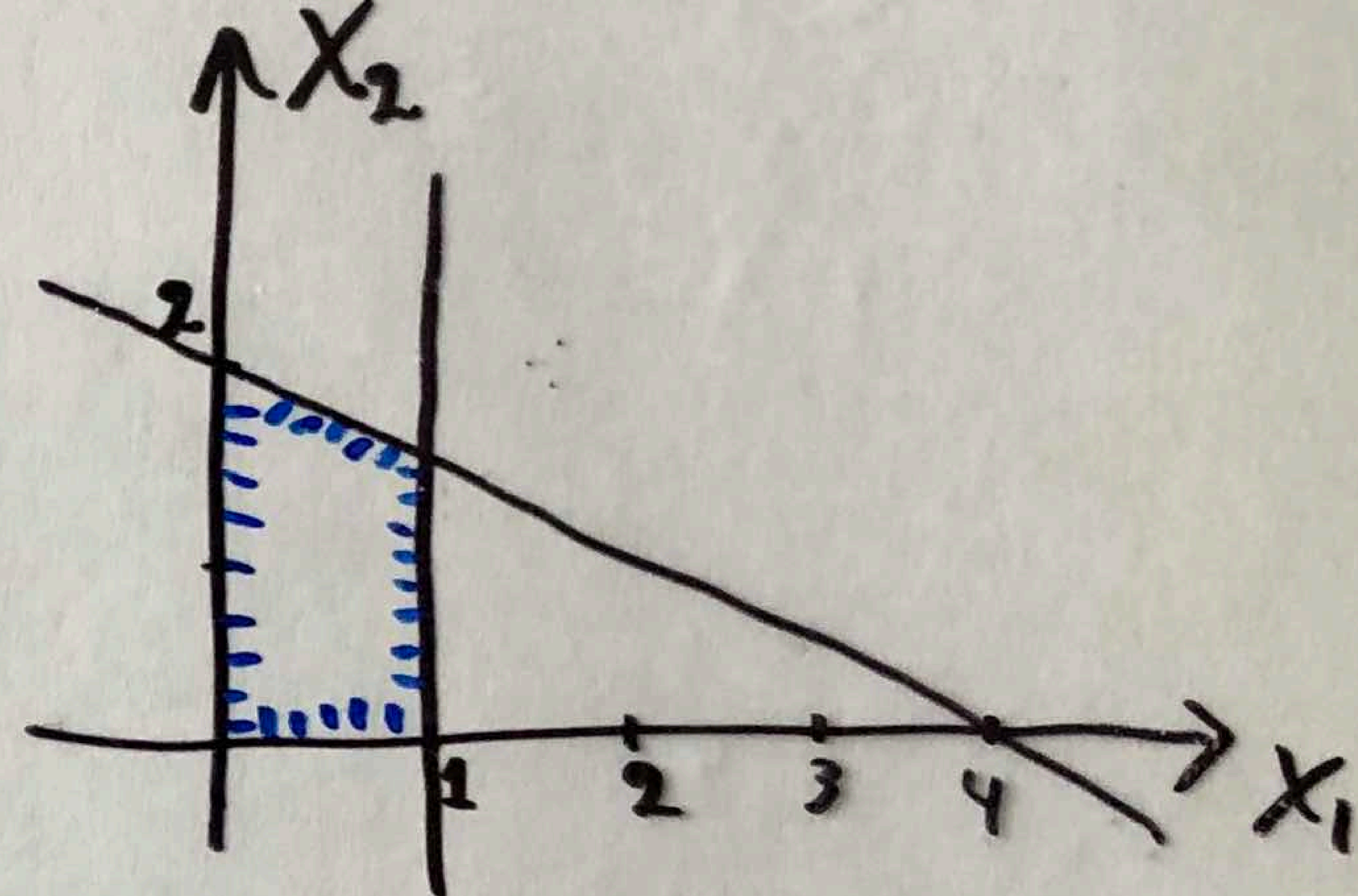
Thus, the problem is unbounded.

$$x(\lambda) = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$ is a feasible ray (feasible for all $\lambda \geq 0$);

and $Z = 1 + 2\lambda$ on that ray.

Multiple Optimal Solutions

$$\begin{aligned} \max \quad & X_1 + 2X_2 \\ \text{s.t.} \quad & X_1 + 2X_2 \leq 4 \\ & X_1 \leq 1 \\ & X_1, X_2 \geq 0 \end{aligned}$$



After introducing slack variables:

Basic var.	X_1	X_2	X_3	X_4	RHS
Z	-1	-2	0	0	0
X_3	1	2^*	1	0	4
X_4	1	0	0	1	1

Basic var.	X_1	X_2	X_3	X_4	RHS
Z	0	0	1	0	4
X_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	2
X_4	1	0	0	1	1

$B = \{2, 4\}$, $N = \{1, 3\}$

$X^* = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ is opt.

$Z^* = 4$

- In this tableau, x_1 is a nonbasic variable and its coefficient is 0 in row 0 $\rightarrow$ increasing x_1 will not change the obj. f-n value.
- Do another iteration, entering x_1 into basis. Min-ratio test: $\min\left(\frac{2}{\frac{1}{2}}, \frac{1}{1}\right)=1$

Basic var.	x_1	x_2	x_3	x_4	RHS
z	0	0	1	0	4
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1.5
x_1	1	0	0	1	1

$B=\{1,2\}$, $N=\{3,4\}$

$\tilde{x}^* = \begin{pmatrix} 1 \\ 1.5 \\ 0 \\ 0 \end{pmatrix}$ is another optimal sol-n

In fact, any point on the segment joining $\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1.5 \\ 0 \\ 0 \end{pmatrix}$ is optimal.

Breaking Ties, Degeneracy

- Tie for entering variable → break the tie arbitrarily

- Tie in the min-ratio test

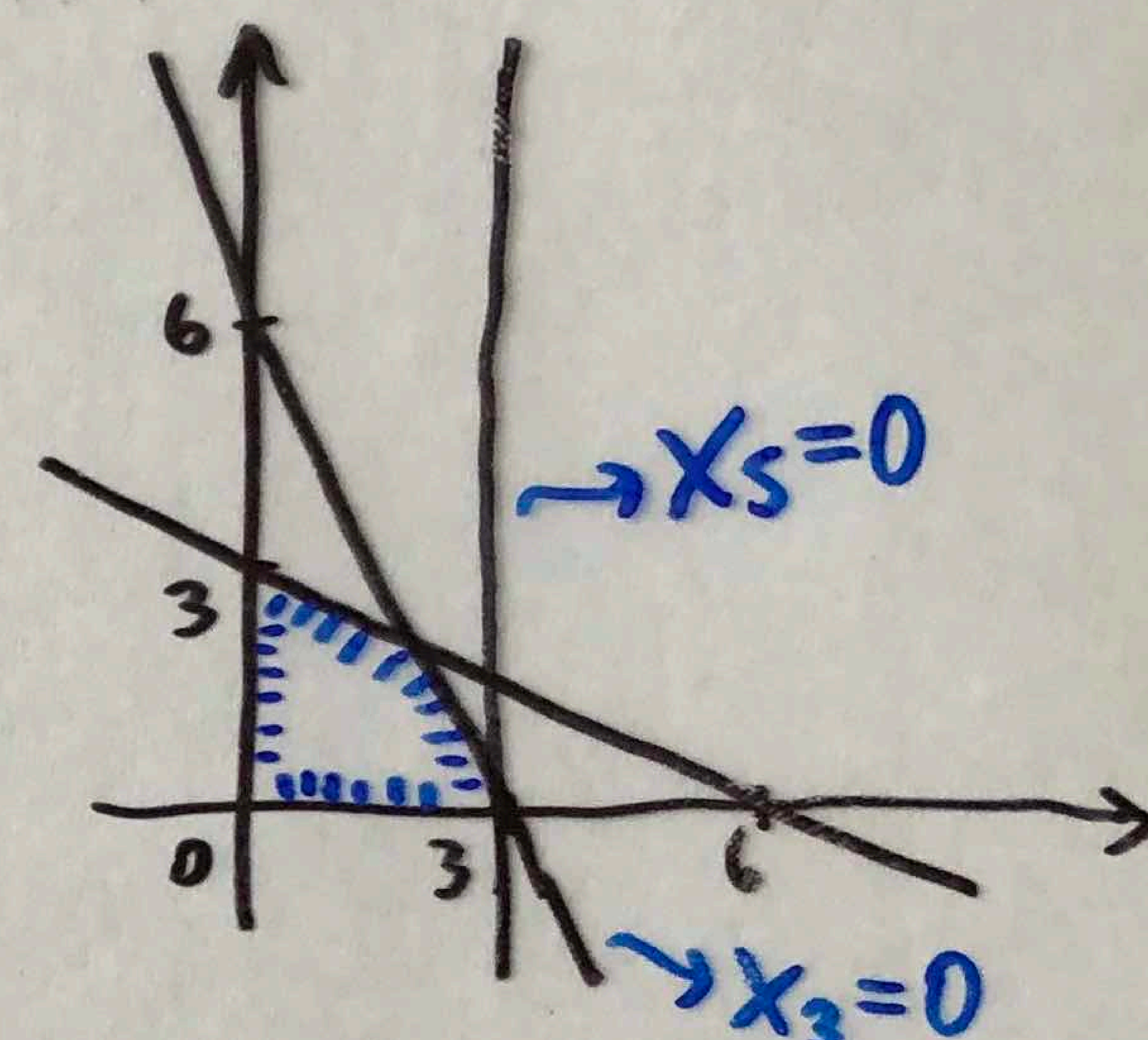
Ex.: $\max \; x_1 + x_2$

s.t. $2x_1 + x_2 \leq 6$

$2x_1 + 4x_2 \leq 12$

$x_1 \leq 3$

$x_1, x_2 \geq 0$



The augmented system:

$$\begin{aligned} 2x_1 + x_2 + x_3 \quad\quad\quad &= 6 \\ 2x_1 + 4x_2 \quad + x_4 \quad &= 12 \\ x_1 \quad\quad\quad + x_5 &= 3 \\ x_i \geq 0 \;\; \forall i \end{aligned}$$

x_2, x_5 nonbasic $\Rightarrow \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$

x_2, x_3 nonbasic $\Rightarrow \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$

x_3, x_5 nonbasic $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$

all three basic solutions: $x = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 6 \\ 0 \end{pmatrix}$

Def-n: A basic sol-n X is called **degenerate** if it has a basic variable that is 0.

Continue with the example.

Basic var.	x_1	x_2	x_3	x_4	x_5	RHS
Z	-1	-1	0	0	0	0
← x_3	2^*	1	1	0	0	6
x_4	2	4	0	1	0	12
x_5	1	0	0	0	1	3

min-ratio test: $\min\left(\frac{6}{2}, \frac{12}{2}, \frac{3}{1}\right) = \min(3, 6, 3) = 3$

tie

2 basic variables x_3 and x_5 drop to 0 simultaneously; but only one of them can become nonbasic.

Thus, tie in min-ratio test → next basic sol-n is degenerate:

Basic var.	x_1	x_2	x_3	x_4	x_5	RHS
Z	0	-.5	.5	0	0	3
x_1	1	.5	.5	0	0	3
x_4	0	3	-1	1	0	6
x_5	0	-.5	-.5	0	1	0

- What problems might degeneracy cause?

In degenerate LP's, Simplex method might cycle: repeat the same sequence of solutions without increasing the obj. f-n. (see Beale's example)

- To prevent cycling, "anti-cycling rules" are invented.
- One of the best-known rules is by Prof. Robert Bland (Cornell OR&IE)

Least-index rule ("Bland's rule"):

i) Among those variables eligible to enter the basis, choose the one with the smallest index (the one which is most to the left in the tableau).

ii) Among all variables eligible to leave the basis, select the one with the smallest index.

E.g.: Suppose x_6, x_2 and x_4 are eligible to leave (ratioes for corresponding rows are the same) $\rightarrow$ choose x_2.