Math 4620/5620

Homework 1 Solution

1. There are many different ways of doing these formulations, and here we present just one of them.

   Decision Variables:
   Let x_{ci}, y_{ci} and z_{ci} represent the number of flights per day made to city c by the i'th plane of B737, AB321 and ATR-70 respectively.

   Constraints:

   1. minimum trip constraints

   $$\begin{aligned}
   \sum_{i=1}^{10} x_{\text{LA},i} + \sum_{i=1}^{15} y_{\text{LA},i} &= 2 \text{ trips} \\
   \sum_{i=1}^{10} x_{\text{BOS},i} + \sum_{i=1}^{15} y_{\text{BOS},i} + \sum_{i=1}^{2} z_{\text{BOS},i} &\geq 4 \text{ trips} \\
   \sum_{i=1}^{10} x_{\text{DC},i} + \sum_{i=1}^{15} y_{\text{DC},i} + \sum_{i=1}^{2} z_{\text{DC},i} &\geq 4 \text{ trips} \\
   \sum_{i=1}^{10} x_{\text{MIA},i} + \sum_{i=1}^{15} y_{\text{MIA},i} &\geq 4 \text{ trips}
   \end{aligned}$$

   And of course, non–negativity is important as well.

   $$x_{ci}, y_{ci}, z_{ci} \geq 0 \text{ for all city } c \text{ and plane } i$$

   2. maximum flying time constraints

   $$\begin{aligned}
   x_{BOS,i} + 2x_{DC,i} + 4x_{MIA,i} + 6x_{LA,i} &\leq 15 \text{ hours for each } i = 1, \cdots, 10 \\
   y_{BOS,i} + 2y_{DC,i} + 5y_{MIA,i} + 8y_{LA,i} &\leq 15 \text{ hours for each } i = 1, \cdots, 15 \\
   2z_{BOS,i} + 4z_{DC,i} &\leq 18 \text{ hours for each } i = 1, 2
   \end{aligned}$$

   Objective Functions:

   1.

   $$\begin{aligned}
   \text{Minimize} \quad & 6000\sum_{i=1}^{10} x_{BOS,i} + 7000\sum_{i=1}^{10} x_{DC,i} + 8000\sum_{i=1}^{10} x_{MIA,i} + 10000\sum_{i=1}^{10} x_{LA,i} + \\
   & 4000\sum_{i=1}^{15} y_{BOS,i} + 3500\sum_{i=1}^{15} y_{DC,i} + 6000\sum_{i=1}^{15} y_{MIA,i} + 10000\sum_{i=1}^{15} y_{LA,i} + \\
   & 1000\sum_{i=1}^{2} z_{BOS,i} + 2000\sum_{i=1}^{2} z_{DC,i}
   \end{aligned}$$

2.

$$\text{Maximize} \quad -1000\sum_{i=1}^{10} x_{BOS,i} + 2000\sum_{i=1}^{10} x_{MIA,i} + 8000\sum_{i=1}^{10} x_{LA,i} +$$
$$-1000\sum_{i=1}^{15} y_{BOS,i} + 2000\sum_{i=1}^{15} y_{DC,i} + 2000\sum_{i=1}^{15} y_{MIA,i} + 4000\sum_{i=1}^{15} y_{LA,i} +$$
$$2000\sum_{i=1}^{2} z_{BOS,i} + 2000\sum_{i=1}^{2} z_{DC,i}$$

3.

$$\text{Minimize} \quad 1\sum_{i=1}^{10} x_{BOS,i} + 2\sum_{i=1}^{10} x_{DC,i} + 4\sum_{i=1}^{10} x_{MIA,i} + 6\sum_{i=1}^{10} x_{LA,i} +$$
$$1\sum_{i=1}^{15} y_{BOS,i} + 2\sum_{i=1}^{15} y_{DC,i} + 5\sum_{i=1}^{15} y_{MIA,i} + 8\sum_{i=1}^{15} y_{LA,i} +$$
$$2\sum_{i=1}^{2} z_{BOS,i} + 4\sum_{i=1}^{2} z_{DC,i}$$

2. Let x_{pi} where $p = A, B, C$ and $i = 1, 2, 3, 4$ be the amount (MG / day) of water treated in plant p using treatment i. Then we want to minimize the total cost

$$\sum_{i=1}^{4} c_i(x_{Ai} + x_{Bi} + x_{Ci})$$

subject to the following set of constraints:

lower bound:

$$\sum_{i=1}^{4} x_{pi} \geq K_p \text{ for each } p = A, B, C$$

upper bound (we cannot treat more water than the flow):

$$\sum_{i=1}^{4} x_{pi} \leq M_p \text{ for each } p = A, B, C$$

where $M_A = M_1$, $M_B = M_2$ and $M_C = M_1 + M_2$

permissible pollutant:

$$\begin{aligned}
\sum_{i=1}^{4} x_{Ai} \cdot P_i &\leq P_S \cdot M_1 \text{ (Phenol at A)} \\
\sum_{i=1}^{4} x_{Ai} \cdot N_i &\leq N_S \cdot M_1 \text{ (Nitrogen at A)} \\
\sum_{i=1}^{4} x_{Bi} \cdot P_i &\leq P_B \cdot M_2 \text{ (Phenol at B)} \\
\sum_{i=1}^{4} x_{Bi} \cdot N_i &\leq N_B \cdot M_2 \text{ (Nitrogen at B)} \\
\sum_{i=1}^{4} (x_{Ai} + x_{Bi} + x_{Ci}) \cdot P_i &\leq P_S \cdot (M_1 + M_2) \text{ (Phenol at C)} \\
\sum_{i=1}^{4} (x_{Ai} + x_{Bi} + x_{Ci}) \cdot N_i &\leq N_S \cdot (M_1 + M_2) \text{ (Nitrogen at C)}
\end{aligned}$$

nonnegativity:

$$x_{pi} \geq 0 \text{ for each } p = A, B, C, \text{ and } i = 1, 2, 3, 4$$

3. Let x, y and z denote the number of kilograms of corn, tankage and alfalfa respectively. Then we have the following LP:

$$\begin{aligned}
\text{Minimize } 84x + 72y + 60z & \qquad \text{(cost in cents)} \\
\text{subject to } 90x + 20y + 40z &\geq 200 \text{ (carbohydrates)} \\
30x + 80y + 60z &\geq 180 \text{ (protein)} \\
10x + 20y + 60z &\geq 150 \text{ (vitamins)} \\
x, y, z &\geq 0
\end{aligned}$$

4. (a) TRUE. Consider the isoprofit line through $(0, 2)$, the isoprofit line through $(3, 3)$, and the isoprofit line through $(6, 3)$. Recall that these lines are all parallel to each other. Since $(3, 3)$ produces a larger value of the objective function than $(0, 2)$ and $(6, 3)$, it follows that the isoprofit line through $(3, 3)$ cannot coincide with the isoprofit line through $(0, 2)$ or $(6, 3)$, and moreover, it can be obtained by shifting (while maintaining the slope) the isoprofit line through $(0, 2)$ or $(6, 3)$ in the direction of increasing z. This means that $(3, 3)$ corresponds to the isoprofit line with maximum z.

   An example of objective function: $Z = 6x_2 - x_1$.

   (b) TRUE. Since $(3, 3)$ is an optimal solution, it is clear from the graph that multiple optimal solutions will only exist if the isoprofit line through $(3, 3)$ coincides with either one of the boundaries (of the feasible region) containing $(3, 3)$. Hence, either $(0, 2)$ or $(6, 3)$ are also optimal.

   Examples of objective functions: $Z = 3x_2 - x_1$ (makes $(0, 2)$ optimal), $Z = x_2$ (makes $(6, 3)$ optimal).

   (c) FALSE. Consider the objective function $z = -x_1 - x_2$.

5. (a) The LP is:

$$
\begin{array}{lrcl}
\max & 16x_1 + 20x_2 & & \text{(revenue)} \\
\text{s. t.} & 3x_1 + 4x_2 & \leq & 12 \text{ (skilled labor)} \\
 & 4x_1 + 2x_2 & \leq & 8 \text{ (unskilled labor)} \\
 & 2x_1 + x_2 & \leq & 16 \text{ (raw material)} \\
 & x_2 & \geq & 2 \text{ (marketing consideration)} \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

(b) The feasible region is the quadrangle with the vertices at $(0,2)$, $(0,3)$, $(1,2)$ and $(0.8, 2.4)$. The optimal solution is $(0.8, 2.4)$ with optimal value 60.8.

6.
Decision variables.
Let
x_t be the number of calculators produced in time period t during regular shift
$t = 1,\ldots,6$
y_t be the number of calculators produced in time period t during overtime hours
$t = 1,\ldots,6$
i_t be the number of calculators stored at the end of period t
$t = 0,\ldots,6$

Objective function:

$$Min \sum_{t=1}^{6}(10x_t + 12y_t + 0.5i_t)$$

Subject to

Production capacity constraints:
$x_t \leq 100$ for $t = 1,\ldots,6$
$y_t \leq 75$ for $t = 1,\ldots,6$

No inventory at the beginning of time period 1:
$i_0 = 0$

Balance constraints:
$i_{t-1} + x_t + y_t = d_t + i_t$ for $t = 1,\ldots,6$

(i.e.,
inventory from previous time period + total number produced in this time period
= demand in current time period + inventory at the end of current time period)

Nonnegativity constraints:
$x_t, y_t, i_t \geq 0$ for $t = 1,\ldots,6$