Math 4620/5620

Homework 2 Solution

1. The standard form of an LP looks like

$$\begin{aligned} \max c^T x & & \\ \text{subject to } Ax &\leq& b \\ x &\geq& 0 \end{aligned}$$

To add blah

(a) FALSE. An obvious counterexample is given in problem 2 where any solution on the line segment connecting $(5,1)^T$ and $(0,6)^T$ is optimal, but only the endpoints are CPF.

(b) FALSE. This expression is an *upper* bound on the number of basic solutions, so there are at *most* this many CPF solutions (remember that CPF solutions are the same as basic feasible solution).

(c) TRUE. Suppose we have an optimal solution x^* which is not a CPF solution. Recall that any LP problem in standard form that has an optimal solution has a CPF solution which is also optimal. Let x^{**} be an optimal CPF solution to the same problem. Note that any point in the line segment joining x^* and x^{**} will also be optimal.

(d) FALSE. Consider an unbounded LP problem.

2. The initial tableau is the following:

	x_1	x_2	x_3	x_4	x_5	
z	-1	-1	0	0	0	0
x_3	1^*	0	1	0	0	5
x_4	1	1	0	1	0	6
x_5	0	1	0	0	1	4

Entering x_1 and performing the min-ratio test yields that x_3 will leave the basis. And so, after one pivot, we get the following tableau:

	x_1	x_2	x_3	x_4	x_5	
z	0	-1	1	0	0	5
x_1	1	0	1	0	0	5
x_4	0	1^*	-1	1	0	1
x_5	0	1	0	0	1	4

The corresponding solution now is $x = (5,0,0,1,4)^T$, the current value is $z = 5$. After one more pivot step (in which x_2 enters the basis and x_4 leaves), we get the following tableau:

	x_1	x_2	x_3	x_4	x_5	
z	0	0	0	1	0	6
x_1	1	0	1	0	0	5
x_2	0	1	-1	1	0	1
x_5	0	0	1^*	-1	1	3

Since all coefficients in row 0 are nonnegative this tableau is an optimal one and we have found an optimal solution $x^* = (5, 1, 0, 0, 3)$ with optimal value $z^* = 6$. In terms of the original problem variables, the optimal solution is $(x_1, x_2) = (5, 1)$.

Looking at this last tableau a bit more carefully we see that there is a non-basic variable x_3 with an objective function coefficient 0 in row 0. This means that if we were to increase x_3 it would have no effect on the objective value, and we would get another solution with the same value as the current one. The min-ratio test yields that x_5 will leave the basis.

	x_1	x_2	x_3	x_4	x_5	
z	0	0	0	1	0	6
x_1	1	0	0	1	-1	2
x_2	0	1	0	0	1	4
x_3	0	0	1	-1	1	3

Here we obtain another optimal solution $(x_1, x_2) = (2, 4)$ with the same optimal value 6. Then all the points on the line segment connecting $(2, 4)$ and $(5, 1)$ are feasible and their value is also 6 (the optimal value).

3. The starting tableau should look like this:

	x_1	x_2	x_3	x_4	x_5	x_6	
z	-1	-2	1	0	0	0	0
x_4	2	2^*	-2	1	0	0	10
x_5	3	-2	2	0	1	0	5
x_6	1	-4	1	0	0	1	10

The corresponding basis is $\{4, 5, 6\}$, the basic feasible solution $x = (0, 0, 0, 10, 5, 10)$ with objective function value $z = 0$. Now when we pivot, x_2 enters the basis and x_4 leaves:

	x_1	x_2	x_3	x_4	x_5	x_6	
z	1	0	-1	1	0	0	10
x_2	1	1	-1	$\frac{1}{2}$	0	0	5
x_5	5	0	0	1	1	0	15
x_6	5	0	-3	2	0	1	30

From here we see immediately that this problem is unbounded: we can increase x_3 as much as we want as long as we increase x_2 and x_6 correspondingly. More specifically, the ray

$$\left\{ \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 15 \\ 30 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 3 \end{pmatrix} : \alpha \geq 0 \right\},$$

is a feasible ray on which the objective function can increase infinitely. The objective function value on this ray is $10 + \alpha$. To find a solution with value at least 2024:

$$\begin{aligned} 10 + \alpha &\geq 2024 \\ \alpha &\geq 2014 \end{aligned}$$

and one feasible point with objective function value of exactly 2024 is $(x_1, x_2, x_3) = (0, 2019, 2014)$.

4. The given problem is

$$\begin{array}{llllll} \min & 2x_1 & +x_2 & +3x_3 & & \\ \text{s.t.} & 5x_1 & +2x_2 & +7x_3 & = & 420 \\ & 3x_1 & +2x_2 & +5x_3 & \geq & 280 \\ & & & x_i & \geq & 0 \quad \forall i = 1 \ldots 3 \end{array}$$

First we need to get this problem into equality constrained form, so we add a slack variable to the second constraint (after transforming it into a "$\leq$"-constraint):

$$\begin{array}{lllllll} \min & 2x_1 & +x_2 & +3x_3 & & & \\ \text{s.t.} & 5x_1 & +2x_2 & +7x_3 & & = & 420 \\ & -3x_1 & -2x_2 & -5x_3 & +x_4 & = & -280 \\ & & & & x_i & \geq & 0 \quad \forall i = 1 \ldots 4 \end{array}$$

Now we multiply the second constraint with -1 and add artificial variables $\bar{x}_5$ and $\bar{x}_6$ since we don't have a "natural" basic variable in the first constraint and since the second r.h.s component is negative. Our phase-I problem now becomes

$$\begin{array}{lllllllll} \min & & & & & \bar{x}_5 & +\bar{x}_6 & & \\ \text{s.t.} & 5x_1 & +2x_2 & +7x_3 & & +\bar{x}_5 & & = & 420 \\ & 3x_1 & +2x_2 & +5x_3 & -x_4 & & +\bar{x}_6 & = & 280 \\ & x_1, & x_2, & x_3, & x_4, & \bar{x}_5, & \bar{x}_6 & \geq & 0 \end{array}$$

Using the equality constraints to re-express the objective function in terms of the non-basic variables we get that $-\bar{x}_5 - \bar{x}_6 = -700 + 8x_1 + 4x_2 + 12x_3 - x_4$. Here's the corresponding phase-I tableau:

	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	
z	-8	-4	-12	1	0	0	-700
$\bar{x}_5$	5	2	7	0	1	0	420
$\bar{x}_6$	3	2	5^*	-1	0	1	280

We choose x_3 to enter the basis, the min-ratio test yields that $\bar{x}_6$ should leave the basis:

	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	
z	$-\frac{4}{5}$	$\frac{4}{5}$	0	$-\frac{7}{5}$	0	$\frac{12}{5}$	-28
$\bar{x}_5$	$\frac{4}{5}$	$-\frac{4}{5}$	0	$\frac{7}{5}^*$	1	$-\frac{7}{5}$	28
x_3	$\frac{3}{5}$	$\frac{2}{5}$	1	$-\frac{1}{5}$	0	$\frac{1}{5}$	56

Now x_4 enters and $\bar{x}_5$ leaves the basis:

	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	
z	0	0	0	0	1	1	0
x_4	$\frac{4}{7}$	$-\frac{4}{7}$	0	1	$\frac{5}{7}$	-1	20
x_3	$\frac{5}{7}$	$\frac{2}{7}$	1	0	$\frac{1}{7}$	$\frac{1}{5}$	60

This is an optimal phase-I tableau: the artificial variables are non-basic and hence 0. Now we cross the artificial columns out of the tableau and re-express the original objective function in terms of the current basis: maximize $z = -2x_1 - x_2 - 3x_3 = -2x_1 - x_2 - 3(60 - \frac{5}{7}x_1 - \frac{2}{7}x_2) = \frac{1}{7}x_1 - \frac{1}{7}x_2 - 180$. So the first phase-II tableau becomes:

	x_1	x_2	x_3	x_4	
z	$-\frac{1}{7}$	$\frac{1}{7}$	0	0	-180
x_4	$\frac{4}{7}^*$	$-\frac{4}{7}$	0	1	20
x_3	$\frac{5}{7}$	$\frac{2}{7}$	1	0	60

Now x_1 enters the basis and x_4 leaves:

	x_1	x_2	x_3	x_4	
z	0	0	0	$\frac{1}{4}$	-175
x_1	1	-1	0	$\frac{7}{4}$	35
x_3	0	$\frac{3}{7}$	1	$-\frac{5}{4}$	35

This is an optimal tableau, the corresponding optimal solution is $x^* = (35, 0, 35, 0)$ with value $z^* = -175$ (for the original minimization problem the optimal value is 175).

5. We add slacks, multiply all inequalities with -1 and then add artificial variables to get

$$\begin{array}{ccccccccc} -3x_1 & -2x_2 & +x_3 & -x_4 & & +\bar{x}_6 & & = & 3 \\ x_1 & +x_2 & -2x_3 & & -x_5 & & +\bar{x}_7 & = & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5, & \bar{x}_6, & \bar{x}_7 & \geq & 0 \end{array}$$

The phase I objective is now to $\max -\bar{x}_6 - \bar{x}_7$, which, re-expressed in terms of the non-basics is just $\max -4 - 2x_1 - x_2 - x_3 - x_4 - x_5$. From this we see immediately that the current representation is optimal for phase I and we conclude that the system is infeasible.

6. (a) FALSE. The objective function of the phase I problem is clearly bounded below by 0.
   (b) FALSE. The phase I problem is *constructed* in a way so that we start with a bfs for it.
   (c) TRUE. In fact, all feasible solutions for the original problem are optimal for the phase 1 problem (if you add 0's for the artificial variables).
   (d) TRUE. This is exactly the purpose of the min-ratio rule. See your lecture notes.
   (e) FALSE. The LP $\min x_1$ subject to $x_1, x_2 \geq 0$ is a trivial counterexample.