Math 4620/5620

Homework 3 Solution

1. For the Revised Simplex we first need to fix some representation of the problem (with equality constraints!) which we will use to get the data (A, b and c) in each step.

$$\begin{array}{rrrrrcll} \max & -2x_1 & -x_2 & -3x_3 & & & & \\ \text{s.t.} & 5x_1 & +2x_2 & +7x_3 & & = & 420 & \\ & 3x_1 & +2x_2 & +5x_3 & -x_4 & = & 280 & \\ & & & & x_i & \geq & 0 & \forall i = 1 \ldots 4 \end{array}$$

Normally, we also have to do phase I, but the problem statements says to start with phase II. So we begin with the feasible basis found at the end of phase I: $\mathcal{B} = \{4, 3\}, \mathcal{N} = \{1, 2\}$.

Step 1: Compute $B^{-1}, B^{-1}b, c_B^T B^{-1}b$. B consists of the 4th and the 3rd column of the constraint matrix in the above representation, and so

$$B = \begin{pmatrix} 0 & 7 \\ -1 & 5 \end{pmatrix}$$

Computing B^{-1}, we get

$$B^{-1} = \begin{pmatrix} \frac{5}{7} & -1 \\ \frac{1}{7} & 0 \end{pmatrix}$$

Now

$$x_B = \begin{pmatrix} x_4 \\ x_3 \end{pmatrix} = B^{-1}b = \begin{pmatrix} \frac{5}{7} & -1 \\ \frac{1}{7} & 0 \end{pmatrix} \begin{pmatrix} 420 \\ 280 \end{pmatrix} = \begin{pmatrix} 20 \\ 60 \end{pmatrix}$$

So the current solution is $x = (0, 0, 60, 20)$ with objective function value

$$z = c_B^T B^{-1}b = c_B^T x_B = (c_4, c_3) \begin{pmatrix} 20 \\ 60 \end{pmatrix} = (0, -3) \begin{pmatrix} 20 \\ 60 \end{pmatrix} = -180$$

Step 2: Test optimality, or find entering basic variable. For this we need to compute the row 0 coefficients:

$$\begin{aligned} c_B^T B^{-1} N - c_N^T &= (c_4, c_3) B^{-1} [A_1, A_2] - (c_1, c_2) \\ &= (0, -3) \begin{pmatrix} \frac{5}{7} & -1 \\ \frac{1}{7} & 0 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} - (-2, -1) \\ &= (-\frac{3}{7}, 0) \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} - (-2, -1) \\ &= (-\frac{15}{7}, -\frac{6}{7}) - (-2, -1) \\ &= (-\frac{1}{7}, \frac{1}{7}). \end{aligned}$$

Note that we want to increase x_1 here.

Step 3: To be able to perform the min-ratio test we need to compute the entering column of the tableau:

$$B^{-1} A_1 = \begin{pmatrix} \frac{5}{7} & -1 \\ \frac{1}{7} & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} \\ \frac{5}{7} \end{pmatrix}$$

Step 4: We do the min-ratio test and determine that x_4 leaves the basis.

Step 5: We get that $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$ and we can move on to the next iteration.

Step 1: Compute $B^{-1}, B^{-1}b, c_B^T B^{-1}b$. B now consists of the 1st and the 3rd columns of the constraint matrix in the above representation, and so

$$B = \begin{pmatrix} 5 & 7 \\ 3 & 5 \end{pmatrix}$$

We can either compute B^{-1} from scratch or just perform the usual pivot operation on the previous B^{-1} to get the new B^{-1} (in this case, multiply the first row with $\frac{7}{4}$ and subtract $\frac{5}{4}$ times the first row from the second row). In any case we get that

$$B^{-1} = \begin{pmatrix} \frac{5}{4} & -\frac{7}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix}$$

Now:

$$x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = B^{-1}b = \begin{pmatrix} \frac{5}{4} & -\frac{7}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} 420 \\ 280 \end{pmatrix} = \begin{pmatrix} 35 \\ 35 \end{pmatrix}$$

We get that $x = (35, 0, 35, 0)$ is our current basic feasible solution. The current objective function value is

$$z = c_B^T B^{-1} b = c_B^T x_B = (c_1, c_3) \begin{pmatrix} 35 \\ 35 \end{pmatrix} = (-2, -3) \begin{pmatrix} 35 \\ 35 \end{pmatrix} = -175$$

Step 2: Test optimality, or find entering basic variable. For this we need to compute the row 0 coefficients:

$$\begin{aligned} c_B^T B^{-1} N - c_N^T &= (c_1, c_3) B^{-1} [A_2, A_4] - (c_2, c_4) \\ &= (-2, -3) \begin{pmatrix} \frac{5}{4} & -\frac{7}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix} - (-1, 0) \\ &= (-\frac{1}{4}, -\frac{1}{4}) \begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix} - (-1, 0) \\ &= (-1, \frac{1}{4}) - (-1, 0) \\ &= (0, \frac{1}{4}). \end{aligned}$$

Here we see that our current solution is in fact an optimal one. But since there is a zero non-basic objective function coefficient we can do one more iteration to get an alternative optimal basic feasible solution. To do this, we let x_2 enter the basis.

Step 3: To be able to perform the min-ratio test we need to compute the entering column of the tableau:

$$B^{-1} A_2 = \begin{pmatrix} \frac{5}{4} & -\frac{7}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Step 4: We do the min-ratio test and determine that x_3 leaves the basis.

Step 5: We get that $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$ and we can move on to the next iteration.

Step 1: Compute $B^{-1}, B^{-1}b, c_B^T B^{-1}b$. B now consists of the 1st and the 2nd columns of the constraint matrix in the above representation, and so

$$B = \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix}$$

We compute

$$B^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix}$$

Now:

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B^{-1}b = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} 420 \\ 280 \end{pmatrix} = \begin{pmatrix} 70 \\ 35 \end{pmatrix}$$

We get that $x = (70, 35, 0, 0)$ is our current basic feasible solution. The current objective function value is

$$z = c_B^T B^{-1} b = c_B^T x_B = (c_1, c_2) \begin{pmatrix} 70 \\ 35 \end{pmatrix} = (-2, -1) \begin{pmatrix} 70 \\ 35 \end{pmatrix} = -175$$

Step 2: Test optimality, or find entering basic variable. For this we need to compute the row 0 coefficients:

$$\begin{aligned} c_B^T B^{-1} N - c_N^T &= (c_1, c_2) B^{-1} [A_3, A_4] - (c_3, c_4) \\ &= (-2, -1) \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 5 & -1 \end{pmatrix} - (-3, 0) \\ &= (-\frac{1}{4}, -\frac{1}{4}) \begin{pmatrix} 7 & 0 \\ 5 & -1 \end{pmatrix} - (-3, 0) \\ &= (-3, \frac{1}{4}) - (-3, 0) \\ &= (0, \frac{1}{4}). \end{aligned}$$

Hence, we have another optimal solution. (Note that the last Step 2 may be skipped since we know that row 0 will not change.)

2. From what is left in the final tableau, we have that the basic variables are $\{x_5, x_3, x_1\}$ in that order. We can also get B^{-1} from the final tableau.

$$B^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix}$$

We can now get the constraint coefficients of the final tableau by calculating

$$B^{-1} \cdot A = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & \frac{1}{2} & 1 & 0 & 0 \\ -4 & -2 & -\frac{3}{2} & 0 & 1 & 0 \\ 1 & 2 & \frac{1}{2} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 & 1 & 1 & 2 \\ 0 & 4 & 1 & -2 & 0 & 4 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

The right hand side is

$$B^{-1} \cdot b = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix}$$

In the original problem (and using the optimal basis),

$$c_B = (c_5, c_3, c_1) = (0, 2, 6)$$

So the final objective function coefficient for x_2 will be

$$c_B \cdot B^{-1} \cdot A_2 - c_2 = (0, 2, 6) \cdot \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} - 1 = 7.$$

Finally, the optimal objective function value is

$$c_b \cdot B^{-1} \cdot b = (0,2,6) \cdot \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix} = 6$$

Summarizing, the optimal and final tableau is as follows:

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	7	0	2	0	2	6
x_5	0	4	0	1	1	2	7
x_3	0	4	1	-2	0	4	0
x_1	1	0	0	1	0	-1	1

3. (a) Any change in the r.h.s will change the values of the basic variables and hence change the objective function value, so both b_1 and b_2 are sensitive parameters. If c_1 changes by a small amount, $\bar{c}_1 = c_B^T B^{-1} A_1 - c_1$ will still be positive so x_1 will still be nonbasic. So a change in c_1 will not affect the optimal solution or its value. On the other hand any change in c_2 or c_3 will have an effect on the objective function value.

   (b) We saw earlier that a change in c_1 does not affect the current basis, so we only need to make sure that the change does not make $\bar{c}_1$ negative. So we need

$$\begin{aligned} \bar{c}_1 &= c_B^T B^{-1} A_1 - c_1 \\ &= (2,1) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - c_1 \\ &= (2,1) \begin{pmatrix} 3 \\ 5 \end{pmatrix} - c_1 = 11 - c_1 \geq 0 \end{aligned}$$

   So as long as $c_1 \leq 11$, the current tableau is optimal.
   The analysis for the other c_j's is a bit more complicated. A change in c_2 or c_3 affects all coefficients of nonbasic variables, so we have to make sure that they all are nonnegative:

$$\begin{aligned} \bar{c}_N &= c_B^T B^{-1} N - c_N \\ &= (c_3, c_2) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} - (3,0,0) \\ &= (3c_3 + 5c_2 - 3, c_3 + c_2, c_3 + 2c_2) \geq 0 \end{aligned}$$

   So we need to make sure the following inequalities are fulfilled:

$$\begin{aligned} 3c_3 + 5c_2 &\geq 3 \\ c_3 + c_2 &\geq 0 \\ c_3 + 2c_2 &\geq 0 \end{aligned}$$

   Holding c_2 at 1, we get that $c_3 \geq \max\{(3-5)/3, -1, -2\} = -2/3$. Similarly, keeping c_3 at 2 we get that $c_2 \geq \max\{(3-6)/5, -2, -1\} = -3/5$.

(c) Finally, changes in the right hand side only affect the final right hand side and the objective function value. We require that $\bar{b} \geq 0$ (for feasibility). So we need

$$B^{-1}b \geq 0$$

which is the same as saying

$$\begin{aligned} b_1 + b_2 &\geq 0 \\ b_1 + 2b_2 &\geq 0 \end{aligned}$$

Keeping b_1 at its current value 20, we see that b_2 must be greater than or equal to $\max\{-20, -10\} = -10$ and similarly, with b_2 fixed at 10, b_1 must be greater than or equal to $\max\{-10, -20\} = -10$.