# Math 4620/5620

## Homework 4 Solution

1. Suppose the primal has a degenerate optimal basic solution. Then, an optimal tableau contains a row whose r.h.s. column entry is 0. We can proceed with the dual simplex method by choosing the basic variable in this row to leave the basis. If there is a negative entry in the row, we can pivot on that entry to obtain another dual optimal basis. Otherwise, we have an optimal ray for the dual problem, and so there are multiple dual solutions.

   Now, if the primal has multiple optimal solutions, then the dual has an optimal degenerate solution. To see this, note that there must be a primal optimal tableau that contains a 0 in a nonbasic column in row 0. This corresponds to a dual basic variable being equal to 0, which implies that the complementary dual solution is degenerate.

2. (a) The dual problem is given by

$$\begin{array}{rrcl} \min & 4y_1 + 7y_2 & & \\ \text{s. t.} & 2y_1 + y_2 & \geq & 5 \\ & 3y_1 + 2y_2 & \geq & 2 \\ & y_1 + 3y_2 & \geq & 5 \\ & y_1, y_2 & \geq & 0 \end{array}$$

   (b) The optimal solution to the dual problem is $y^* = (2, 1)^T$ with objective function value $w^* = 15$.

   (c) Since $y_1 = 2 > 0$ and $y_2 = 1 > 0$ in the dual optimal solution, it follows that the two functional constraints will be tight in the primal optimal solution, i.e.

$$\begin{array}{rcl} 2x_1 + 3x_2 + x_3 & = & 4 \\ x_1 + 2x_2 + 3x_3 & = & 7 \end{array}$$

   Moreover, only the second constraint in the dual is not tight in the dual optimal solution. Hence, we must have $x_2 = 0$ in the primal optimal solution. Solving these three equations, we obtain the primal optimal solution $x^* = (1, 0, 2)^T$ with $z^* = 15$.

3. (a) Let $x_i$ be the number of units of food type $i$ ($i = 1, 2, 3, 4, 5$) included in the diet. Then the LP is:

$$\begin{array}{lrcl} \min & 20x_1 + 20x_2 + 31x_3 + 11x_4 + 12x_5 & & \\ \text{s. t.} & x_1 + \quad\quad x_3 + x_4 + 2x_5 & \geq & 21 \\ & x_2 + 2x_3 + x_4 + x_5 & \geq & 12 \\ & x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

   We transform the problem into standard form, introduce slack variables $x_6$ and $x_7$ and set up the following tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $-z$ | 20 | 20 | 31 | 11 | 12 | 0 | 0 | 0 |
| $x_6$ | $-1$ | 0 | $-1$ | $-1$ | $-2^*$ | 1 | 0 | $-21$ |
| $x_7$ | 0 | $-1$ | $-2$ | $-1$ | $-1$ | 0 | 1 | $-12$ |

We see that the initial basic feasible solution is infeasible, but its complementary dual solution is feasible for the dual problem, so we apply the Dual Simplex Method. $x_6$ leaves the basis, the min-ratio test yields that $x_5$ will enter:

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \hline -z & 14 & 20 & 25 & 5 & 0 & 6 & 0 & -126 \\ \hline x_5 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & \frac{21}{2} \\ x_7 & \frac{1}{2} & -1 & -\frac{3}{2} & -\frac{1}{2}^* & 0 & -\frac{1}{2} & 1 & -\frac{3}{2} \\ \hline \end{array}$$

Now $x_7$ leaves the basis, the min-ratio test yields that $x_4$ should enter:

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \hline -z & 19 & 10 & 10 & 0 & 0 & 1 & 10 & -141 \\ \hline x_5 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & 9 \\ x_4 & -1 & 2 & 3 & 1 & 0 & 1 & -2 & 3 \\ \hline \end{array}$$

This is an optimal tableau and we conclude that the optimal diet consists of 3 units of food type 4 and 9 units of food type 5. We find $B^{-1}$, as usual, under the slack variables in the last tableau:

$$B^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

(b) We can read the complementary dual solutions in row 0 of the respective tableaus: in the first tableau $y = (0, 0)$, in the second one we find $y = (6, 0)$ and finally we get $y^* = (1, 10)$.

(c) The shadow price of the vitamin K constraint is 10 cents. Note that this can be read from the optimal tableau as the coefficient of $x_7$ in row (0). This is the amount of decrease in the objective function (cost) per unit decrease in the rhs of the vitamin K constraint. To be competitive the pharmacist needs to sell the vitamin K pills for *less* than that, so 12 cents is not competitive.

(d) We introduce a new variable $x_8$ and set $c_8 = -28$ and $A_8 = (-3, -2)^T$. The corresponding row 0 entry in the final tableau becomes

$$(y^*)^T A_8 - c_8 = (1, 10) \begin{pmatrix} -3 \\ -2 \end{pmatrix} - (-28) = 5 > 0,$$

so it is *not* optimal to pivot this food into the basis. Remember that introducing a new variable corresponds to introducing a new constraint in the dual problem. What we have just shown is that our old optimal dual solution satisfies this new constraint and hence remains optimal.

(e) $x_5$ is a basic variable in the final tableau, so we need to pivot it out of the basis before we can just drop it. Doing the min-ratio-test (on the negative coefficients in the $x_5$-row) gives us that $x_6$ enters the basis. We pivot on the corresponding entry in the last tableau and get:

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \hline -z & 20 & 9 & 9 & 0 & 1 & 0 & 11 & -132 \\ \hline x_6 & -1 & 1 & 1 & 0 & -1 & 1 & -1 & -9 \\ x_4 & 0 & 1 & 2 & 1 & 1 & 0 & -1 & 12 \\ \hline \end{array}$$

Now that $x_5$ is nonbasic (and thus 0) we can just ignore it and proceed by reoptimizing with the Dual Simplex Method. To reoptimize we want $x_6$ to leave the basis, the min-ratio-test yields that $x_7$ should enter instead:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|
| $-z$ | 9 | 20 | 20 | 0 | 11 | 0 | $-231$ |
| $x_7$ | 1 | $-1$ | $-1$ | 0 | $-1$ | 1 | 9 |
| $x_4$ | 1 | 0 | 1 | 1 | $-1$ | 0 | 21 |

This is an optimal tableau, the new optimal diet consists of just 21 units of food 4. The new complementary dual solution is $y = (11, 0)$.

Again, deleting a primal variable is the same as deleting a constraint in the dual. You should be able to see what happens to the graph above and why the solution moved as it did.

(f) We introduce a new constraint:

$$x_1 + x_2 + x_4 + 2x_5 \leq 20$$

The current optimal diet does not satisfy this constraint, so we need to reoptimize. We add a slack variable $x_8$ and use the optimal tableau to reexpress this constraint in terms of the nonbasic variables. Substituting in for $x_4$ and $x_5$ yields

$$x_2 - x_3 + x_6 + x_8 = -1.$$

Now we can reoptimize using the Dual Simplex Method:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | |
|---|---|---|---|---|---|---|---|---|---|
| $-z$ | 19 | 10 | 10 | 0 | 0 | 1 | 10 | 0 | $-141$ |
| $x_5$ | 1 | $-1$ | $-1$ | 0 | 1 | $-1$ | 1 | 0 | 9 |
| $x_4$ | $-1$ | 2 | 3 | 1 | 0 | 1 | $-2$ | 0 | 3 |
| $x_8$ | 0 | 1 | $-1^*$ | 0 | 0 | 1 | 0 | 1 | $-1$ |

Clearly, $x_8$ leaves the basis and $x_3$ enters, and we get the following optimal tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | |
|---|---|---|---|---|---|---|---|---|---|
| $-z$ | 19 | 20 | 0 | 0 | 0 | 11 | 10 | 10 | $-151$ |
| $x_5$ | 1 | $-2$ | 0 | 0 | 1 | $-2$ | 1 | $-1$ | 10 |
| $x_4$ | $-1$ | 5 | 0 | 1 | 0 | 4 | $-2$ | 3 | 0 |
| $x_3$ | 0 | $-1$ | 1 | 0 | 0 | $-1$ | 0 | $-1$ | 1 |

The optimal solution now becomes $x^* = (0, 0, 1, 0, 10, 0, 0, 0)$, and it's degenerate since we have a basic variable at 0.

4. For the linear programming problem in the standard form, we have the following primal and dual pair.

**Primal:**

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s. t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

**Dual:**

$$\begin{array}{lrcl} \min & b^T y & & \\ \text{s. t.} & A^T y & \geq & c \\ & y & \geq & 0 \end{array}$$

(a) TRUE. Suppose $A$ is an $m$ by $n$ matrix. Then, in the standard form, the number of functional constraints for the primal is $m$; for the dual $n$. The number of variables for the primal is $n$; for the dual, $m$. Therefore for both, the sum is $m + n$.

(b) FALSE. The simplex method simultaneously identifies a basic feasible solution for the primal problem and a basic feasible solution for the dual problem such that their objective values are the same *only at optimality.* By the weak duality theorem, at each primal simplex iteration, the dual basic solution is infeasible unless the current iteration is optimal.

(c) FALSE. If the primal problem has an unbounded objective function, then the dual problem must be *infeasible* by weak duality.