4

Solving Linear Programming Problems: The Simplex Method

We now are ready to begin studying the *simplex method,* a general procedure for solving linear programming problems. Developed by George Dantzig in 1947, it has proved to be a remarkably efficient method that is used routinely to solve huge problems on today's computers. Except for its use on tiny problems, this method is always executed on a computer, and sophisticated software packages are widely available. Extensions and variations of the simplex method also are used to perform *postoptimality analysis* (including sensitivity analysis) on the model.

This chapter describes and illustrates the main features of the simplex method. The first section introduces its general nature, including its geometric interpretation. The following three sections then develop the procedure for solving any linear programming model that is in our standard form (maximization, all functional constraints in $\leq$ form, and nonnegativity constraints on all variables) and has only *nonnegative* right-hand sides b_i in the functional constraints. Certain details on resolving ties are deferred to Sec. 4.5, and Sec. 4.6 describes how to adapt the simplex method to other model forms. Next we discuss postoptimality analysis (Sec. 4.7), and describe the computer implementation of the simplex method (Sec. 4.8). Section 4.9 then introduces an alternative to the simplex method (the interior-point approach) for solving large linear programming problems.

4.1 THE ESSENCE OF THE SIMPLEX METHOD

The simplex method is an *algebraic* procedure. However, its underlying concepts are *geometric*. Understanding these geometric concepts provides a strong intuitive feeling for how the simplex method operates and what makes it so efficient. Therefore, before delving into algebraic details, we focus in this section on the big picture from a geometric viewpoint.

To illustrate the general geometric concepts, we shall use the Wyndor Glass Co. example presented in Sec. 3.1. (Sections 4.2 and 4.3 use the *algebra* of the simplex method to solve this same example.) Section 5.1 will elaborate further on these geometric concepts for larger problems.

To refresh your memory, the model and graph for this example are repeated in Fig. 4.1. The five constraint boundaries and their points of intersection are highlighted in this figure because they are the keys to the analysis. Here, each **constraint boundary** is a line that forms the boundary of what is permitted by the corresponding constraint. The points
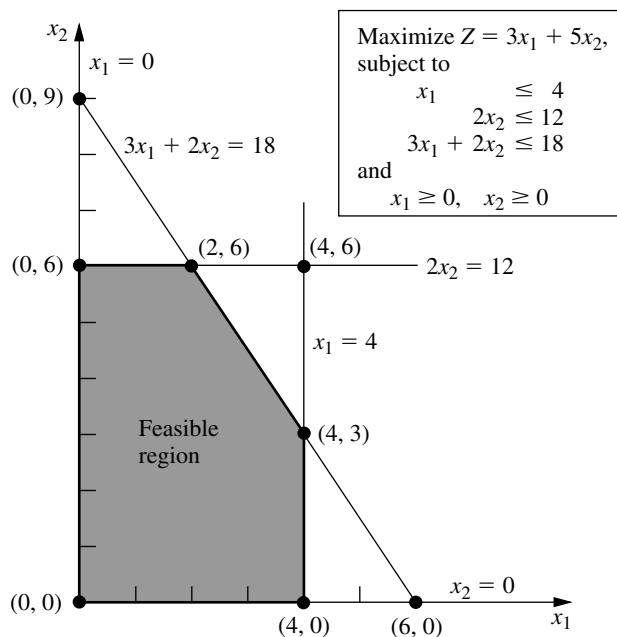
FIGURE 4.1
Constraint boundaries and corner-point solutions for the Wyndor Glass Co. problem.

of intersection are the **corner-point solutions** of the problem. The five that lie on the corners of the *feasible region*—(0, 0), (0, 6), (2, 6), (4, 3), and (4, 0)—are the *corner-point feasible solutions* (**CPF solutions**). [The other three—(0, 9), (4, 6), and (6, 0)—are called *corner-point infeasible solutions.*]

In this example, each corner-point solution lies at the intersection of *two* constraint boundaries. (For a linear programming problem with n decision variables, each of its corner-point solutions lies at the intersection of n constraint boundaries.[1]) Certain pairs of the CPF solutions in Fig. 4.1 share a constraint boundary, and other pairs do not. It will be important to distinguish between these cases by using the following general definitions.

> For any linear programming problem with n decision variables, two CPF solutions are **adjacent** to each other if they share $n - 1$ constraint boundaries. The two adjacent CPF solutions are connected by a line segment that lies on these same shared constraint boundaries. Such a line segment is referred to as an **edge** of the feasible region.

Since $n = 2$ in the example, two of its CPF solutions are adjacent if they share *one* constraint boundary; for example, (0, 0) and (0, 6) are adjacent because they share the $x_1 = 0$ constraint boundary. The feasible region in Fig. 4.1 has five edges, consisting of the five line segments forming the boundary of this region. Note that two edges emanate from each CPF solution. Thus, each CPF solution has two adjacent CPF solutions (each lying at the other end of one of the two edges), as enumerated in Table 4.1. (In each row

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[1]Although a corner-point solution is defined in terms of n constraint boundaries whose intersection gives this solution, it also is possible that one or more *additional* constraint boundaries pass through this same point.

TABLE 4.1 Adjacent CPF solutions for each CPF
solution of the Wyndor Glass Co. problem

CPF Solution	Its Adjacent CPF Solutions
(0, 0)	(0, 6) and (4, 0)
(0, 6)	(2, 6) and (0, 0)
(2, 6)	(4, 3) and (0, 6)
(4, 3)	(4, 0) and (2, 6)
(4, 0)	(0, 0) and (4, 3)

of this table, the CPF solution in the first column is adjacent to each of the two CPF solutions in the second column, but the two CPF solutions in the second column are *not* adjacent to each other.)

One reason for our interest in adjacent CPF solutions is the following general property about such solutions, which provides a very useful way of checking whether a CPF solution is an optimal solution.

> **Optimality test:** Consider any linear programming problem that possesses at least one optimal solution. If a CPF solution has no *adjacent* CPF solutions that are *better* (as measured by Z), then it *must* be an *optimal* solution.

Thus, for the example, (2, 6) must be optimal simply because its $Z = 36$ is larger than $Z = 30$ for (0, 6) and $Z = 27$ for (4, 3). (We will delve further into why this property holds in Sec. 5.1.) This optimality test is the one used by the simplex method for determining when an optimal solution has been reached.

Now we are ready to apply the simplex method to the example.

Solving the Example

Here is an outline of what the simplex method does (from a geometric viewpoint) to solve the Wyndor Glass Co. problem. At each step, first the conclusion is stated and then the reason is given in parentheses. (Refer to Fig. 4.1 for a visualization.)

Initialization: Choose (0, 0) as the *initial* CPF solution to examine. (This is a convenient choice because no calculations are required to identify this CPF solution.)

Optimality Test: Conclude that (0, 0) is *not* an optimal solution. (Adjacent CPF solutions are better.)

Iteration 1: Move to a better *adjacent* CPF solution, (0, 6), by performing the following three steps.

1. Considering the two edges of the feasible region that emanate from (0, 0), choose to move along the edge that leads up the x_2 axis. (With an objective function of $Z = 3x_1 + 5x_2$, moving up the x_2 axis increases Z at a faster rate than moving along the x_1 axis.)
2. Stop at the first new constraint boundary: $2x_2 = 12$. [Moving farther in the direction selected in step 1 leaves the feasible region; e.g., moving to the second new constraint boundary hit when moving in that direction gives (0, 9), which is a corner-point *infeasible* solution.]

3. Solve for the intersection of the new set of constraint boundaries: (0, 6). (The equations for these constraint boundaries, $x_1 = 0$ and $2x_2 = 12$, immediately yield this solution.)

   Optimality Test: Conclude that (0, 6) is *not* an optimal solution. (An adjacent CPF solution is better.)

   Iteration 2: Move to a better adjacent CPF solution, (2, 6), by performing the following three steps.

1. Considering the two edges of the feasible region that emanate from (0, 6), choose to move along the edge that leads to the right. (Moving along this edge increases Z, whereas backtracking to move back down the x_2 axis decreases Z.)
2. Stop at the first new constraint boundary encountered when moving in that direction: $3x_1 + 2x_2 = 12$. (Moving farther in the direction selected in step 1 leaves the feasible region.)
3. Solve for the intersection of the new set of constraint boundaries: (2, 6). (The equations for these constraint boundaries, $3x_1 + 2x_2 = 18$ and $2x_2 = 12$, immediately yield this solution.)

   Optimality Test: Conclude that (2, 6) *is* an optimal solution, so stop. (None of the adjacent CPF solutions are better.)

This sequence of CPF solutions examined is shown in Fig. 4.2, where each circled number identifies which iteration obtained that solution.

   Now let us look at the six key solution concepts of the simplex method that provide the rationale behind the above steps. (Keep in mind that these concepts also apply for solving problems with more than two decision variables where a graph like Fig. 4.2 is not available to help quickly find an optimal solution.)

The Key Solution Concepts

The first solution concept is based directly on the relationship between optimal solutions and CPF solutions given at the end of Sec. 3.2.

FIGURE 4.2
This graph shows the sequence of CPF solutions (⓪, ①, ②) examined by the simplex method for the Wyndor Glass Co. problem. The optimal solution (2, 6) is found after just three solutions are examined.
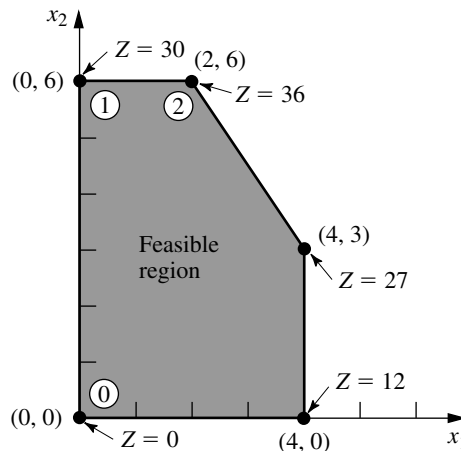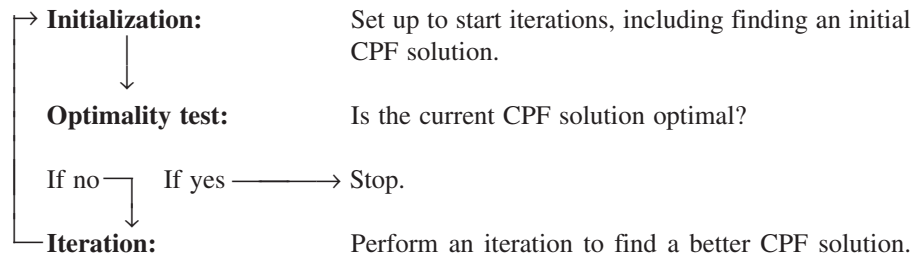
> **Solution concept 1:** The simplex method focuses solely on CPF solutions. For any problem with at least one optimal solution, finding one requires only finding a best CPF solution.[1]

Since the number of feasible solutions generally is infinite, reducing the number of solutions that need to be examined to a small finite number (just three in Fig. 4.2) is a tremendous simplification.

The next solution concept defines the flow of the simplex method.

> **Solution concept 2:** The simplex method is an *iterative algorithm* (a systematic solution procedure that keeps repeating a fixed series of steps, called an *iteration,* until a desired result has been obtained) with the following structure.

Initialization: Set up to start iterations, including finding an initial CPF solution.

Optimality test: Is the current CPF solution optimal?

If no ⌐   If yes ⟶ Stop.

Iteration: Perform an iteration to find a better CPF solution.

When the example was solved, note how this flow diagram was followed through two iterations until an optimal solution was found.

We next focus on how to get started.

> **Solution concept 3:** Whenever possible, the initialization of the simplex method chooses the *origin* (all decision variables equal to zero) to be the initial CPF solution. When there are too many decision variables to find an initial CPF solution graphically, this choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

Choosing the origin commonly is possible when all the decision variables have nonnegativity constraints, because the intersection of these constraint boundaries yields the origin as a corner-point solution. This solution then is a CPF solution *unless* it is *infeasible* because it violates one or more of the functional constraints. If it is infeasible, special procedures described in Sec. 4.6 are needed to find the initial CPF solution.

The next solution concept concerns the choice of a better CPF solution at each iteration.

> **Solution concept 4:** Given a CPF solution, it is much quicker computationally to gather information about its *adjacent* CPF solutions than about other CPF solutions. Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it *always* chooses a CPF solution that is *adjacent* to the current one. No other CPF solutions are considered. Consequently, the entire path followed to eventually reach an optimal solution is along the *edges* of the feasible region.

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[1]The only restriction is that the problem must possess CPF solutions. This is ensured if the feasible region is bounded.

The next focus is on which adjacent CPF solution to choose at each iteration.

Solution concept 5: After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution. Each of these edges leads to an *adjacent* CPF solution at the other end, but the simplex method does not even take the time to solve for the adjacent CPF solution. Instead, it simply identifies the *rate of improvement in Z* that would be obtained by moving along the edge. Among the edges with a *positive* rate of improvement in Z, it then chooses to move along the one with the *largest* rate of improvement in Z. The iteration is completed by first solving for the adjacent CPF solution at the other end of this one edge and then relabeling this adjacent CPF solution as the *current* CPF solution for the optimality test and (if needed) the next iteration.

At the first iteration of the example, moving from (0, 0) along the edge on the x_1 axis would give a rate of improvement in Z of 3 (Z increases by 3 per unit increase in x_1), whereas moving along the edge on the x_2 axis would give a rate of improvement in Z of 5 (Z increases by 5 per unit increase in x_2), so the decision is made to move along the latter edge. At the second iteration, the only edge emanating from (0, 6) that would yield a *positive* rate of improvement in Z is the edge leading to (2, 6), so the decision is made to move next along this edge.

The final solution concept clarifies how the optimality test is performed efficiently.

Solution concept 6: Solution concept 5 describes how the simplex method examines each of the edges of the feasible region that emanate from the current CPF solution. This examination of an edge leads to quickly identifying the rate of improvement in Z that would be obtained by moving along the edge toward the adjacent CPF solution at the other end. A *positive* rate of improvement in Z implies that the adjacent CPF solution is *better* than the current CPF solution, whereas a *negative* rate of improvement in Z implies that the adjacent CPF solution is *worse*. Therefore, the optimality test consists simply of checking whether *any* of the edges give a *positive* rate of improvement in Z. If *none* do, then the current CPF solution is optimal.

In the example, moving along *either* edge from (2, 6) decreases Z. Since we want to maximize Z, this fact immediately gives the conclusion that (2, 6) is optimal.

4.2   SETTING UP THE SIMPLEX METHOD

The preceding section stressed the geometric concepts that underlie the simplex method. However, this algorithm normally is run on a computer, which can follow only algebraic instructions. Therefore, it is necessary to translate the conceptually geometric procedure just described into a usable algebraic procedure. In this section, we introduce the *algebraic language* of the simplex method and relate it to the concepts of the preceding section.

The algebraic procedure is based on solving systems of equations. Therefore, the first step in setting up the simplex method is to convert the functional *inequality constraints* to equivalent *equality constraints*. (The nonnegativity constraints are left as inequalities because they are treated separately.) This conversion is accomplished by introducing **slack**

variables. To illustrate, consider the first functional constraint in the Wyndor Glass Co. example of Sec. 3.1

$$x_1 \leq 4.$$

The slack variable for this constraint is defined to be

$$x_3 = 4 - x_1,$$

which is the amount of slack in the left-hand side of the inequality. Thus,

$$x_1 + x_3 = 4.$$

Given this equation, $x_1 \leq 4$ if and only if $4 - x_1 = x_3 \geq 0$. Therefore, the original constraint $x_1 \leq 4$ is entirely *equivalent* to the pair of constraints

$$x_1 + x_3 = 4 \qquad \text{and} \qquad x_3 \geq 0.$$

Upon the introduction of slack variables for the other functional constraints, the original linear programming model for the example (shown below on the left) can now be replaced by the equivalent model (called the *augmented form* of the model) shown below on the right:

Original Form of the Model	*Augmented Form of the Model*[1]
Maximize $\quad Z = 3x_1 + 5x_2,$	Maximize $\quad Z = 3x_1 + 5x_2,$
subject to	subject to
$\qquad x_1 \qquad \leq \quad 4$	(1) $\quad x_1 \qquad + x_3 \qquad\qquad = \quad 4$
$\qquad\qquad 2x_2 \leq 12$	(2) $\qquad\qquad 2x_2 \qquad + x_4 \quad = \quad 12$
$\qquad 3x_1 + 2x_2 \leq 18$	(3) $\quad 3x_1 + 2x_2 \qquad\qquad + x_5 = \quad 18$
and	and
$\qquad x_1 \geq 0, \qquad x_2 \geq 0.$	$\qquad x_j \geq 0, \qquad \text{for } j = 1, 2, 3, 4, 5.$

Although both forms of the model represent exactly the same problem, the new form is much more convenient for algebraic manipulation and for identification of CPF solutions. We call this the **augmented form** of the problem because the original form has been *augmented* by some supplementary variables needed to apply the simplex method.

If a slack variable equals 0 in the current solution, then this solution lies on the constraint boundary for the corresponding functional constraint. A value greater than 0 means that the solution lies on the *feasible* side of this constraint boundary, whereas a value less than 0 means that the solution lies on the *infeasible* side of this constraint boundary. A demonstration of these properties is provided by the demonstration example in your OR Tutor entitled *Interpretation of the Slack Variables*.

The terminology used in the preceding section (corner-point solutions, etc.) applies to the original form of the problem. We now introduce the corresponding terminology for the augmented form.

> An **augmented solution** is a solution for the original variables (the *decision variables*) that has been augmented by the corresponding values of the *slack variables*.

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[1]The slack variables are not shown in the objective function because the coefficients there are 0.

For example, augmenting the solution (3, 2) in the example yields the augmented solution (3, 2, 1, 8, 5) because the corresponding values of the slack variables are $x_3 = 1$, $x_4 = 8$, and $x_5 = 5$.

A **basic solution** is an *augmented* corner-point solution.

To illustrate, consider the corner-point infeasible solution (4, 6) in Fig. 4.1. Augmenting it with the resulting values of the slack variables $x_3 = 0$, $x_4 = 0$, and $x_5 = -6$ yields the corresponding basic solution (4, 6, 0, 0, −6).

The fact that corner-point solutions (and so basic solutions) can be either feasible or infeasible implies the following definition:

A **basic feasible (BF) solution** is an *augmented* CPF solution.

Thus, the CPF solution (0, 6) in the example is equivalent to the BF solution (0, 6, 4, 0, 6) for the problem in augmented form.

The only difference between basic solutions and corner-point solutions (or between BF solutions and CPF solutions) is whether the values of the slack variables are included. For any basic solution, the corresponding corner-point solution is obtained simply by deleting the slack variables. Therefore, the geometric and algebraic relationships between these two solutions are very close, as described in Sec. 5.1.

Because the terms *basic solution* and *basic feasible solution* are very important parts of the standard vocabulary of linear programming, we now need to clarify their algebraic properties. For the augmented form of the example, notice that the system of functional constraints has 5 variables and 3 equations, so

Number of variables − number of equations = 5 − 3 = 2.

This fact gives us 2 *degrees of freedom* in solving the system, since any two variables can be chosen to be set equal to any arbitrary value in order to solve the three equations in terms of the remaining three variables.[1] The simplex method uses zero for this arbitrary value. Thus, two of the variables (called the *nonbasic variables*) are set equal to zero, and then the simultaneous solution of the three equations for the other three variables (called the *basic variables*) is a *basic solution*. These properties are described in the following general definitions.

A **basic solution** has the following properties:

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The *number of basic variables* equals the number of functional constraints (now equations). Therefore, the *number of nonbasic variables* equals the total number of variables *minus* the number of functional constraints.
3. The **nonbasic variables** are set equal to zero.
4. The values of the **basic variables** are obtained as the simultaneous solution of the system of equations (functional constraints in augmented form). (The set of basic variables is often referred to as **the basis.**)
5. If the basic variables satisfy the *nonnegativity constraints,* the basic solution is a **BF solution.**

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[1]This method of determining the number of degrees of freedom for a system of equations is valid as long as the system does not include any redundant equations. This condition always holds for the system of equations formed from the functional constraints in the augmented form of a linear programming model.

To illustrate these definitions, consider again the BF solution (0, 6, 4, 0, 6). This solution was obtained before by augmenting the CPF solution (0, 6). However, another way to obtain this same solution is to choose x_1 and x_4 to be the two nonbasic variables, and so the two variables are set equal to zero. The three equations then yield, respectively, $x_3 = 4$, $x_2 = 6$, and $x_5 = 6$ as the solution for the three basic variables, as shown below (with the basic variables in bold type):

$$x_1 = 0 \text{ and } x_4 = 0 \text{ so}$$

$$
\begin{array}{lllll}
(1) & x_1 & + \boldsymbol{x_3} & = 4 & \boldsymbol{x_3} = 4 \\
(2) & 2\boldsymbol{x_2} & + x_4 & = 12 & \boldsymbol{x_2} = 6 \\
(3) & 3x_1 + 2\boldsymbol{x_2} & + \boldsymbol{x_5} = 18 & & \boldsymbol{x_5} = 6
\end{array}
$$

Because all three of these basic variables are nonnegative, this *basic solution* (0, 6, 4, 0, 6) is indeed a *BF solution*.

Just as certain pairs of CPF solutions are *adjacent*, the corresponding pairs of BF solutions also are said to be adjacent. Here is an easy way to tell when two BF solutions are adjacent.

> Two BF solutions are **adjacent** if *all but one* of their *nonbasic variables* are the same. This implies that *all but one* of their *basic variables* also are the same, although perhaps with different numerical values.

Consequently, moving from the current BF solution to an adjacent one involves switching one variable from nonbasic to basic and vice versa for one other variable (and then adjusting the values of the basic variables to continue satisfying the system of equations).

To illustrate *adjacent BF solutions,* consider one pair of adjacent CPF solutions in Fig. 4.1: (0, 0) and (0, 6). Their augmented solutions, (0, 0, 4, 12, 18) and (0, 6, 4, 0, 6), automatically are adjacent BF solutions. However, you do not need to look at Fig. 4.1 to draw this conclusion. Another signpost is that their nonbasic variables, (x_1, x_2) and (x_1, x_4), are the same with just the one exception—x_2 has been replaced by x_4. Consequently, moving from (0, 0, 4, 12, 18) to (0, 6, 4, 0, 6) involves switching x_2 from nonbasic to basic and vice versa for x_4.

When we deal with the problem in augmented form, it is convenient to consider and manipulate the objective function equation at the same time as the new constraint equations. Therefore, before we start the simplex method, the problem needs to be rewritten once again in an equivalent way:

Maximize Z,

subject to

$$
\begin{array}{llll}
(0) & Z - 3x_1 - 5x_2 & = 0 \\
(1) & x_1 & + x_3 & = 4 \\
(2) & 2x_2 & + x_4 & = 12 \\
(3) & 3x_1 + 2x_2 & + x_5 = 18
\end{array}
$$

and

$$x_j \geq 0, \quad \text{for } j = 1, 2, \ldots, 5.$$

It is just as if Eq. (0) actually were one of the original constraints; but because it already is in equality form, no slack variable is needed. While adding one more equation, we also have added one more unknown (Z) to the system of equations. Therefore, when using Eqs. (1) to (3) to obtain a basic solution as described above, we use Eq. (0) to solve for Z at the same time.

Somewhat fortuitously, the model for the Wyndor Glass Co. problem fits *our standard form,* and all its functional constraints have nonnegative right-hand sides b_i. If this had not been the case, then additional adjustments would have been needed at this point before the simplex method was applied. These details are deferred to Sec. 4.6, and we now focus on the simplex method itself.

4.3 THE ALGEBRA OF THE SIMPLEX METHOD

We continue to use the prototype example of Sec. 3.1, as rewritten at the end of Sec. 4.2, for illustrative purposes. To start connecting the geometric and algebraic concepts of the simplex method, we begin by outlining side by side in Table 4.2 how the simplex method solves this example from both a geometric and an algebraic viewpoint. The geometric viewpoint (first presented in Sec. 4.1) is based on the *original form* of the model (no slack variables), so again refer to Fig. 4.1 for a visualization when you examine the second column of the table. Refer to the *augmented form* of the model presented at the end of Sec. 4.2 when you examine the third column of the table.

We now fill in the details for each step of the third column of Table 4.2.

Initialization

The choice of x_1 and x_2 to be the *nonbasic* variables (the variables set equal to zero) for the initial BF solution is based on solution concept 3 in Sec. 4.1. This choice eliminates the work required to solve for the *basic variables* (x_3, x_4, x_5) from the following system of equations (where the basic variables are shown in bold type):

$$x_1 = 0 \text{ and } x_2 = 0 \text{ so}$$

$$
\begin{array}{llll}
(1) & x_1 & + \boldsymbol{x_3} & = 4 & \boldsymbol{x_3} = 4 \\
(2) & 2x_2 & + \boldsymbol{x_4} & = 12 & \boldsymbol{x_4} = 12 \\
(3) & 3x_1 + 2x_2 & + \boldsymbol{x_5} = 18 & \boldsymbol{x_5} = 18
\end{array}
$$

Thus, the **initial BF solution** is (0, 0, 4, 12, 18).

Notice that this solution can be read immediately because each equation has just one basic variable, which has a coefficient of 1, and this basic variable does not appear in any other equation. You will soon see that when the set of basic variables changes, the simplex method uses an algebraic procedure (Gaussian elimination) to convert the equations to this same convenient form for reading every subsequent BF solution as well. This form is called **proper form from Gaussian elimination.**

Optimality Test

The objective function is

$$Z = 3x_1 + 5x_2,$$

TABLE 4.2 Geometric and algebraic interpretations of how the simplex method solves the Wyndor Glass Co. problem

Method Sequence	Geometric Interpretation	Algebraic Interpretation
Initialization	Choose (0, 0) to be the initial CPF solution.	Choose x_1 and x_2 to be the nonbasic variables (= 0) for the initial BF solution: (0, 0, 4, 12, 18).
Optimality test	Not optimal, because moving along either edge from (0, 0) increases Z.	Not optimal, because increasing either nonbasic variable (x_1 or x_2) increases Z.
Iteration 1		
Step 1	Move up the edge lying on the x_2 axis.	Increase x_2 while adjusting other variable values to satisfy the system of equations.
Step 2	Stop when the first new constraint boundary ($2x_2 = 12$) is reached.	Stop when the first basic variable (x_3, x_4, or x_5) drops to zero (x_4).
Step 3	Find the intersection of the new pair of constraint boundaries: (0, 6) is the new CPF solution.	With x_2 now a basic variable and x_4 now a nonbasic variable, solve the system of equations: (0, 6, 4, 0, 6) is the new BF solution.
Optimality test	Not optimal, because moving along the edge from (0, 6) to the right increases Z.	Not optimal, because increasing one nonbasic variable (x_1) increases Z.
Iteration 2		
Step 1	Move along this edge to the right.	Increase x_1 while adjusting other variable values to satisfy the system of equations.
Step 2	Stop when the first new constraint boundary ($3x_1 + 2x_2 = 18$) is reached.	Stop when the first basic variable (x_2, x_3, or x_5) drops to zero (x_5).
Step 3	Find the intersection of the new pair of constraint boundaries: (2, 6) is the new CPF solution.	With x_1 now a basic variable and x_5 now a nonbasic variable, solve the system of equations: (2, 6, 2, 0, 0) is the new BF solution.
Optimality test	(2, 6) is optimal, because moving along either edge from (2, 6) decreases Z.	(2, 6, 2, 0, 0) is optimal, because increasing either nonbasic variable (x_4 or x_5) decreases Z.

so $Z = 0$ for the initial BF solution. Because none of the basic variables (x_3, x_4, x_5) have a *nonzero* coefficient in this objective function, the coefficient of each nonbasic variable (x_1, x_2) gives the rate of improvement in Z if that variable were to be increased from zero (while the values of the basic variables are adjusted to continue satisfying the system of equations).[1] These rates of improvement (3 and 5) are *positive*. Therefore, based on solution concept 6 in Sec. 4.1, we conclude that (0, 0, 4, 12, 18) is not optimal.

For each BF solution examined after subsequent iterations, at least one basic variable has a nonzero coefficient in the objective function. Therefore, the optimality test then will use the new Eq. (0) to rewrite the objective function in terms of just the nonbasic variables, as you will see later.

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[1]Note that this interpretation of the coefficients of the x_j variables is based on these variables being on the right-hand side, $Z = 3x_1 + 5x_2$. When these variables are brought to the left-hand side for Eq. (0), $Z - 3x_1 - 5x_2 = 0$, the nonzero coefficients change their signs.

Determining the Direction of Movement (Step 1 of an Iteration)

Increasing one nonbasic variable from zero (while adjusting the values of the basic variables to continue satisfying the system of equations) corresponds to moving along one edge emanating from the current CPF solution. Based on solution concepts 4 and 5 in Sec. 4.1, the choice of which nonbasic variable to increase is made as follows:

$$Z = 3x_1 + 5x_2$$

Increase x_1?   Rate of improvement in $Z = 3$.
Increase x_2?   Rate of improvement in $Z = 5$.
$5 > 3$, so choose x_2 to increase.

As indicated next, we call x_2 the *entering basic variable* for iteration 1.

> At any iteration of the simplex method, the purpose of step 1 is to choose one *nonbasic variable* to increase from zero (while the values of the basic variables are adjusted to continue satisfying the system of equations). Increasing this nonbasic variable from zero will convert it to a *basic variable* for the next BF solution. Therefore, this variable is called the **entering basic variable** for the current iteration (because it is entering the basis).

Determining Where to Stop (Step 2 of an Iteration)

Step 2 addresses the question of how far to increase the entering basic variable x_2 before stopping. Increasing x_2 increases Z, so we want to go as far as possible without leaving the feasible region. The requirement to satisfy the functional constraints in augmented form (shown below) means that increasing x_2 (while keeping the nonbasic variable $x_1 = 0$) changes the values of some of the basic variables as shown on the right.

			$x_1 =$	$0,$	so
(1)	$x_1 \quad + x_3$	$= 4$	$x_3 =$	4	
(2)	$2x_2 \quad + x_4$	$= 12$	$x_4 =$	$12 - 2x_2$	
(3)	$3x_1 + 2x_2 \quad + x_5$	$= 18$	$x_5 =$	$18 - 2x_2.$	

The other requirement for feasibility is that all the variables be *nonnegative*. The nonbasic variables (including the entering basic variable) are nonnegative, but we need to check how far x_2 can be increased without violating the nonnegativity constraints for the basic variables.

$x_3 = 4 \geq 0 \qquad \Rightarrow$ no upper bound on x_2.

$x_4 = 12 - 2x_2 \geq 0 \Rightarrow x_2 \leq \dfrac{12}{2} = 6 \quad \leftarrow$ minimum.

$x_5 = 18 - 2x_2 \geq 0 \Rightarrow x_2 \leq \dfrac{18}{2} = 9.$

Thus, x_2 can be increased just to 6, at which point x_4 has dropped to 0. Increasing x_2 beyond 6 would cause x_4 to become negative, which would violate feasibility.

These calculations are referred to as the **minimum ratio test.** The objective of this test is to determine which basic variable drops to zero first as the entering basic variable is increased. We can immediately rule out the basic variable in any equation where the coefficient of the entering basic variable is zero or negative, since such a basic variable would not decrease as the entering basic variable is increased. [This is what happened

with x_3 in Eq. (1) of the example.] However, for each equation where the coefficient of the entering basic variable is *strictly positive* (> 0), this test calculates the *ratio* of the right-hand side to the coefficient of the entering basic variable. The basic variable in the equation with the *minimum ratio* is the one that drops to zero first as the entering basic variable is increased.

> At any iteration of the simplex method, step 2 uses the *minimum ratio test* to determine which basic variable drops to zero first as the entering basic variable is increased. Decreasing this basic variable to zero will convert it to a *nonbasic variable* for the next BF solution. Therefore, this variable is called the **leaving basic variable** for the current iteration (because it is leaving the basis).

Thus, x_4 is the leaving basic variable for iteration 1 of the example.

Solving for the New BF Solution (Step 3 of an Iteration)

Increasing $x_2 = 0$ to $x_2 = 6$ moves us from the *initial* BF solution on the left to the *new* BF solution on the right.

	Initial BF solution	New BF solution
Nonbasic variables:	$x_1 = 0, \quad x_2 = 0$	$x_1 = 0, \quad x_4 = 0$
Basic variables:	$x_3 = 4, \quad x_4 = 12, \quad x_5 = 18$	$x_3 = ?, \quad x_2 = 6, \quad x_5 = ?$

The purpose of step 3 is to convert the system of equations to a more convenient form (proper form from Gaussian elimination) for conducting the optimality test and (if needed) the next iteration with this new BF solution. In the process, this form also will identify the values of x_3 and x_5 for the new solution.

Here again is the complete original system of equations, where the *new* basic variables are shown in bold type (with Z playing the role of the basic variable in the objective function equation):

$$
\begin{aligned}
(0) \quad & \mathbf{Z} - 3x_1 - 5\mathbf{x_2} && && = 0 \\
(1) \quad & \quad\quad x_1 && + \mathbf{x_3} && = 4 \\
(2) \quad & \quad\quad\quad\quad 2\mathbf{x_2} && + x_4 && = 12 \\
(3) \quad & \quad\quad 3x_1 + 2\mathbf{x_2} && + x_5 && = 18.
\end{aligned}
$$

Thus, x_2 has replaced x_4 as the basic variable in Eq. (2). To solve this system of equations for Z, x_2, x_3, and x_5, we need to perform some **elementary algebraic operations** to reproduce the current pattern of coefficients of x_4 (0, 0, 1, 0) as the new coefficients of x_2. We can use either of two types of elementary algebraic operations:

1. Multiply (or divide) an equation by a nonzero constant.
2. Add (or subtract) a multiple of one equation to (or from) another equation.

To prepare for performing these operations, note that the coefficients of x_2 in the above system of equations are $-5, 0, 2,$ and 3, respectively, whereas we want these coefficients to become 0, 0, 1, and 0, respectively. To turn the coefficient of 2 in Eq. (2) into 1, we use the first type of elementary algebraic operation by dividing Eq. (2) by 2 to obtain

$$(2) \quad \mathbf{x_2} + \frac{1}{2}x_4 = 6.$$

To turn the coefficients of -5 and 3 into zeros, we need to use the second type of elementary algebraic operation. In particular, we add 5 times this new Eq. (2) to Eq. (0), and subtract 2 times this new Eq. (2) from Eq. (3). The resulting complete new system of equations is

$$(0) \quad \mathbf{Z} - 3x_1 \qquad\qquad + \frac{5}{2}x_4 \qquad = 30$$

$$(1) \qquad\quad x_1 \;\; + \mathbf{x}_3 \qquad\qquad\quad = \;\; 4$$

$$(2) \qquad\qquad\quad \mathbf{x}_2 \;\; + \frac{1}{2}x_4 \qquad = \;\; 6$$

$$(3) \qquad\quad 3x_1 \qquad\quad - \;\; x_4 + \mathbf{x}_5 = \;\; 6.$$

Since $x_1 = 0$ and $x_4 = 0$, the equations in this form immediately yield the new BF solution, $(x_1, x_2, x_3, x_4, x_5) = (0, 6, 4, 0, 6)$, which yields $Z = 30$.

This procedure for obtaining the simultaneous solution of a system of linear equations is called the *Gauss-Jordan method of elimination,* or **Gaussian elimination** for short.[1] The key concept for this method is the use of elementary algebraic operations to reduce the original system of equations to proper form from Gaussian elimination, where each basic variable has been eliminated from all but one equation (*its* equation) and has a coefficient of $+1$ in that equation.

Optimality Test for the New BF Solution

The current Eq. (0) gives the value of the objective function in terms of just the current nonbasic variables

$$Z = 30 + 3x_1 - \frac{5}{2}x_4.$$

Increasing either of these nonbasic variables from zero (while adjusting the values of the basic variables to continue satisfying the system of equations) would result in moving toward one of the two *adjacent* BF solutions. Because x_1 has a *positive* coefficient, increasing x_1 would lead to an adjacent BF solution that is better than the current BF solution, so the current solution is not optimal.

Iteration 2 and the Resulting Optimal Solution

Since $Z = 30 + 3x_1 - \frac{5}{2}x_4$, Z can be increased by increasing x_1, but not x_4. Therefore, step 1 chooses x_1 to be the entering basic variable.

For step 2, the current system of equations yields the following conclusions about how far x_1 can be increased (with $x_4 = 0$):

$$x_3 = 4 - x_1 \geq 0 \;\; \Rightarrow x_1 \leq \frac{4}{1} = 4.$$

$$x_2 = 6 \geq 0 \qquad\quad \Rightarrow \text{no upper bound on } x_1.$$

$$x_5 = 6 - 3x_1 \geq 0 \Rightarrow x_1 \leq \frac{6}{3} = 2 \quad \leftarrow \text{minimum.}$$

Therefore, the minimum ratio test indicates that x_5 is the leaving basic variable.

---

[1] Actually, there are some technical differences between the Gauss-Jordan method of elimination and Gaussian elimination, but we shall not make this distinction.

For step 3, with x_1 replacing x_5 as a basic variable, we perform elementary algebraic operations on the current system of equations to reproduce the current pattern of coefficients of x_5 (0, 0, 0, 1) as the new coefficients of x_1. This yields the following new system of equations:

$$(0) \quad Z \qquad\qquad + \frac{3}{2}x_4 + \quad x_5 = 36$$

$$(1) \qquad\qquad x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5 = \quad 2$$

$$(2) \qquad x_2 \quad + \frac{1}{2}x_4 \qquad\quad = \quad 6$$

$$(3) \quad x_1 \qquad - \frac{1}{3}x_4 + \frac{1}{3}x_5 = \quad 2.$$

Therefore, the next BF solution is $(x_1, x_2, x_3, x_4, x_5) = (2, 6, 2, 0, 0)$, yielding $Z = 36$. To apply the *optimality test* to this new BF solution, we use the current Eq. (0) to express Z in terms of just the current nonbasic variables,

$$Z = 36 - \frac{3}{2}x_4 - x_5.$$

Increasing either x_4 or x_5 would *decrease Z,* so neither adjacent BF solution is as good as the current one. Therefore, based on solution concept 6 in Sec. 4.1, the current BF solution must be optimal.

In terms of the original form of the problem (no slack variables), the optimal solution is $x_1 = 2$, $x_2 = 6$, which yields $Z = 3x_1 + 5x_2 = 36$.

To see another example of applying the simplex method, we recommend that you now view the demonstration entitled *Simplex Method—Algebraic Form* in your OR Tutor. This vivid demonstration simultaneously displays both the algebra and the geometry of the simplex method as it dynamically evolves step by step. Like the many other demonstration examples accompanying other sections of the book (including the next section), this computer demonstration highlights concepts that are difficult to convey on the printed page.

To further help you learn the simplex method efficiently, your OR Courseware includes a procedure entitled *Solve Interactively by the Simplex Method.* This routine performs nearly all the calculations while you make the decisions step by step, thereby enabling you to focus on concepts rather than get bogged down in a lot of number crunching. Therefore, you probably will want to use this routine for your homework on this section. The software will help you get started by letting you know whenever you make a mistake on the first iteration of a problem.

The next section includes a summary of the simplex method for a more convenient tabular form.

4.4 THE SIMPLEX METHOD IN TABULAR FORM

The algebraic form of the simplex method presented in Sec. 4.3 may be the best one for learning the underlying logic of the algorithm. However, it is not the most convenient form for performing the required calculations. When you need to solve a problem by hand (or

interactively with your OR Courseware), we recommend the *tabular form* described in this section.[1]

The tabular form of the simplex method records only the essential information, namely, (1) the coefficients of the variables, (2) the constants on the right-hand sides of the equations, and (3) the basic variable appearing in each equation. This saves writing the symbols for the variables in each of the equations, but what is even more important is the fact that it permits highlighting the numbers involved in arithmetic calculations and recording the computations compactly.

Table 4.3 compares the initial system of equations for the Wyndor Glass Co. problem in algebraic form (on the left) and in tabular form (on the right), where the table on the right is called a *simplex tableau.* The basic variable for each equation is shown in bold type on the left and in the first column of the simplex tableau on the right. [Although only the x_j variables are basic or nonbasic, Z plays the role of the basic variable for Eq. (0).] All variables *not* listed in this *basic variable* column (x_1, x_2) automatically are *nonbasic variables.* After we set $x_1 = 0$, $x_2 = 0$, the *right side* column gives the resulting solution for the basic variables, so that the initial BF solution is (x_1, x_2, x_3, x_4, x_5) = (0, 0, 4, 12, 18) which yields $Z = 0$.

> The *tabular form* of the simplex method uses a **simplex tableau** to compactly display the system of equations yielding the current BF solution. For this solution, each variable in the leftmost column equals the corresponding number in the rightmost column (and variables not listed equal zero). When the optimality test or an iteration is performed, the only relevant numbers are those to the right of the Z column. The term **row** refers to just a row of numbers to the right of the Z column (including the *right side* number), where row *i* corresponds to Eq. (*i*).

We summarize the tabular form of the simplex method below and, at the same time, briefly describe its application to the Wyndor Glass Co. problem. Keep in mind that the logic is identical to that for the algebraic form presented in the preceding section. Only the form for displaying both the current system of equations and the subsequent iteration has changed (plus we shall no longer bother to bring variables to the right-hand side of an equation before drawing our conclusions in the optimality test or in steps 1 and 2 of an iteration).

---

[1]A form more convenient for automatic execution on a computer is presented in Sec. 5.2.

TABLE 4.3 Initial system of equations for the Wyndor Glass Co. problem

(a) Algebraic Form	(b) Tabular Form								
	Basic Variable	**Eq.**	**Z**	**Coefficient of:**					**Right Side**
				x_1	x_2	x_3	x_4	x_5	
(0) **Z** $- 3x_1 - 5x_2$ $= 0$	Z	(0)	1	-3	-5	0	0	0	0
(1) $x_1 + \mathbf{x}_3 = 4$	x_3	(1)	0	1	0	1	0	0	4
(2) $2x_2 + \mathbf{x}_4 = 12$	x_4	(2)	0	0	2	0	1	0	12
(3) $3x_1 + 2x_2 + \mathbf{x}_5 = 18$	x_5	(3)	0	3	2	0	0	1	18

Summary of the Simplex Method (and Iteration 1 for the Example)

Initialization.   Introduce slack variables. Select the *decision variables* to be the *initial nonbasic variables* (set equal to zero) and the *slack variables* to be the *initial basic variables.* (See Sec. 4.6 for the necessary adjustments if the model is not in our standard form—maximization, only $\leq$ functional constraints, and all nonnegativity constraints— or if any b_i values are negative.)

For the Example: This selection yields the initial simplex tableau shown in Table 4.3b, so the initial BF solution is (0, 0, 4, 12, 18).

Optimality Test.   The current BF solution is optimal if and only if *every* coefficient in row 0 is nonnegative (≥ 0). If it is, stop; otherwise, go to an iteration to obtain the next BF solution, which involves changing one nonbasic variable to a basic variable (step 1) and vice versa (step 2) and then solving for the new solution (step 3).

For the Example: Just as $Z = 3x_1 + 5x_2$ indicates that increasing either x_1 or x_2 will increase Z, so the current BF solution is not optimal, the same conclusion is drawn from the equation $Z - 3x_1 - 5x_2 = 0$. These coefficients of -3 and -5 are shown in row 0 of Table 4.3b.

Iteration.   *Step 1:* Determine the *entering basic variable* by selecting the variable (automatically a nonbasic variable) with the *negative coefficient* having the largest absolute value (i.e., the "most negative" coefficient) in Eq. (0). Put a box around the column below this coefficient, and call this the **pivot column.**

For the Example: The most negative coefficient is -5 for x_2 (5 > 3), so x_2 is to be changed to a basic variable. (This change is indicated in Table 4.4 by the box around the x_2 column below -5.)

Step 2: Determine the *leaving basic variable* by applying the minimum ratio test.
Minimum Ratio Test

1. Pick out each coefficient in the pivot column that is strictly positive (> 0).
2. Divide each of these coefficients into the *right side* entry for the same row.
3. Identify the row that has the *smallest* of these ratios.
4. The basic variable for that row is the leaving basic variable, so replace that variable by the entering basic variable in the basic variable column of the next simplex tableau.

TABLE 4.4 Applying the minimum ratio test to determine the first leaving basic variable for the Wyndor Glass Co. problem

Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	0	0	
x_3	(1)	0	1	0	1	0	0	4	
x_4	(2)	0	0	2	0	1	0	$12 \rightarrow \dfrac{12}{2} = 6 \leftarrow$ minimum
x_5	(3)	0	3	2	0	0	1	$18 \rightarrow \dfrac{18}{2} = 9$

Put a box around this row and call it the **pivot row.** Also call the number that is in *both* boxes the **pivot number.**

For the Example: The calculations for the minimum ratio test are shown to the right of Table 4.4. Thus, row 2 is the pivot row (see the box around this row in the first simplex tableau of Table 4.5), and x_4 is the leaving basic variable. In the next simplex tableau (see the bottom of Table 4.5), x_2 replaces x_4 as the basic variable for row 2.

Step 3: Solve for the *new BF solution* by using **elementary row operations** (multiply or divide a row by a nonzero constant; add or subtract a multiple of one row to another row) to construct a new simplex tableau in proper form from Gaussian elimination below the current one, and then return to the optimality test. The specific elementary row operations that need to be performed are listed below.

1. Divide the pivot row by the pivot number. Use this *new* pivot row in steps 2 and 3.
2. For each other row (including row 0) that has a *negative* coefficient in the pivot column, *add* to this row the *product* of the absolute value of this coefficient and the new pivot row.
3. For each other row that has a *positive* coefficient in the pivot column, *subtract* from this row the *product* of this coefficient and the new pivot row.

For the Example: Since x_2 is replacing x_4 as a basic variable, we need to reproduce the first tableau's pattern of coefficients in the column of x_4 (0, 0, 1, 0) in the second tableau's column of x_2. To start, divide the pivot row (row 2) by the pivot number (2), which gives the new row 2 shown in Table 4.5. Next, we add to row 0 the product, 5 times the new row 2. Then we subtract from row 3 the product, 2 times the new row 2 (or equivalently, subtract from row 3 the *old* row 2). These calculations yield the new tableau shown in Table 4.6 for iteration 1. Thus, the new BF solution is (0, 6, 4, 0, 6), with $Z = 30$. We next return to the optimality test to check if the new BF solution is optimal. Since the new row 0 still has a negative coefficient (-3 for x_1), the solution is not optimal, and so at least one more iteration is needed.

TABLE 4.5 Simplex tableaux for the Wyndor Glass Co. problem after the first pivot row is divided by the first pivot number

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1						
	x_3	(1)	0						
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0						

TABLE 4.6 First two simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
					Coefficient of:				
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6

Iteration 2 for the Example and the Resulting Optimal Solution

The second iteration starts anew from the second tableau of Table 4.6 to find the next BF solution. Following the instructions for steps 1 and 2, we find x_1 as the entering basic variable and x_5 as the leaving basic variable, as shown in Table 4.7.

For step 3, we start by dividing the pivot row (row 3) in Table 4.7 by the pivot number (3). Next, we add to row 0 the product, 3 times the new row 3. Then we subtract the new row 3 from row 1.

We now have the set of tableaux shown in Table 4.8. Therefore, the new BF solution is (2, 6, 2, 0, 0), with $Z = 36$. Going to the optimality test, we find that this solution is *optimal* because none of the coefficients in row 0 is negative, so the algorithm is finished. Consequently, the optimal solution for the Wyndor Glass Co. problem (before slack variables are introduced) is $x_1 = 2$, $x_2 = 6$.

Now compare Table 4.8 with the work done in Sec. 4.3 to verify that these two forms of the simplex method really are *equivalent*. Then note how the algebraic form is superior for learning the logic behind the simplex method, but the tabular form organizes the

TABLE 4.7 Steps 1 and 2 of iteration 2 for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
					Coefficient of:					
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30	
	x_3	(1)	0	1	0	1	0	0	4	$\frac{4}{1} = 4$
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	
	x_5	(3)	0	3	0	0	−1	1	6	$\frac{6}{3} = 2 \leftarrow$ minimum

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of:						Right Side
			Z	x_1	x_2	x_3	x_4	x_5	
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	-1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

work being done in a considerably more convenient and compact form. We generally use the tabular form from now on.

An additional example of applying the simplex method in tabular form is available to you in the OR Tutor. See the demonstration entitled *Simplex Method—Tabular Form.*

4.5   TIE BREAKING IN THE SIMPLEX METHOD

You may have noticed in the preceding two sections that we never said what to do if the various choice rules of the simplex method do not lead to a clear-cut decision, because of either ties or other similar ambiguities. We discuss these details now.

Tie for the Entering Basic Variable

Step 1 of each iteration chooses the nonbasic variable having the *negative* coefficient with the *largest absolute value* in the current Eq. (0) as the entering basic variable. Now suppose that two or more nonbasic variables are tied for having the largest negative coefficient (in absolute terms). For example, this would occur in the first iteration for the Wyndor Glass Co. problem if its objective function were changed to $Z = 3x_1 + 3x_2$, so that the initial Eq. (0) became $Z - 3x_1 - 3x_2 = 0$. How should this tie be broken?

The answer is that the selection between these contenders may be made *arbitrarily.* The optimal solution will be reached eventually, regardless of the tied variable chosen, and there is no convenient method for predicting in advance which choice will lead there

sooner. In this example, the simplex method happens to reach the optimal solution $(2, 6)$ in three iterations with x_1 as the initial entering basic variable, versus two iterations if x_2 is chosen.

Tie for the Leaving Basic Variable—Degeneracy

Now suppose that two or more basic variables tie for being the leaving basic variable in step 2 of an iteration. Does it matter which one is chosen? Theoretically it does, and in a very critical way, because of the following sequence of events that could occur. First, all the tied basic variables reach zero simultaneously as the entering basic variable is increased. Therefore, the one or ones *not* chosen to be the leaving basic variable also will have a value of zero in the new BF solution. (Note that basic variables with a value of *zero* are called **degenerate,** and the same term is applied to the corresponding BF solution.) Second, if one of these degenerate basic variables retains its value of zero until it is chosen at a subsequent iteration to be a leaving basic variable, the corresponding entering basic variable also must remain zero (since it cannot be increased without making the leaving basic variable negative), so the value of Z must remain unchanged. Third, if Z may remain the same rather than increase at each iteration, the simplex method may then go around in a loop, repeating the same sequence of solutions periodically rather than eventually increasing Z toward an optimal solution. In fact, examples have been artificially constructed so that they do become entrapped in just such a perpetual loop.

Fortunately, although a perpetual loop is theoretically possible, it has rarely been known to occur in practical problems. If a loop were to occur, one could always get out of it by changing the choice of the leaving basic variable. Furthermore, special rules[1] have been constructed for breaking ties so that such loops are always avoided. However, these rules frequently are ignored in actual application, and they will not be repeated here. For your purposes, just break this kind of tie arbitrarily and proceed without worrying about the degenerate basic variables that result.

No Leaving Basic Variable—Unbounded Z

In step 2 of an iteration, there is one other possible outcome that we have not yet discussed, namely, that *no* variable qualifies to be the leaving basic variable.[2] This outcome would occur if the entering basic variable could be increased *indefinitely* without giving negative values to *any* of the current basic variables. In tabular form, this means that *every* coefficient in the pivot column (excluding row 0) is either negative or zero.

As illustrated in Table 4.9, this situation arises in the example displayed in Fig. 3.6 on p. 36. In this example, the last two functional constraints of the Wyndor Glass Co. problem have been overlooked and so are not included in the model. Note in Fig. 3.6 how x_2 can be increased indefinitely (thereby increasing Z indefinitely) without ever leaving the feasible region. Then note in Table 4.9 that x_2 is the entering basic variable but the

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[1]See R. Bland, "New Finite Pivoting Rules for the Simplex Method," *Mathematics of Operations Research,* **2:** 103–107, 1977.

[2]Note that the analogous case (no *entering* basic variable) cannot occur in step 1 of an iteration, because the optimality test would stop the algorithm first by indicating that an optimal solution had been reached.

TABLE 4.9 Initial simplex tableau for the Wyndor Glass Co. problem without the last two functional constraints

Basic Variable	Eq.	Z	Coefficient of: x_1	x_2	x_3	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	
x_3	(1)	0	1	0	1	4	None

With $x_1 = 0$ and x_2 increasing, $x_3 = 4 - 1x_1 - 0x_2 = 4 > 0$.

only coefficient in the pivot column is zero. Because the minimum ratio test uses only coefficients that are greater than zero, there is no ratio to provide a leaving basic variable.

The interpretation of a tableau like the one shown in Table 4.9 is that the constraints do not prevent the value of the objective function Z increasing indefinitely, so the simplex method would stop with the message that Z is *unbounded*. Because even linear programming has not discovered a way of making infinite profits, the real message for practical problems is that a mistake has been made! The model probably has been misformulated, either by omitting relevant constraints or by stating them incorrectly. Alternatively, a computational mistake may have occurred.

Multiple Optimal Solutions

We mentioned in Sec. 3.2 (under the definition of **optimal solution**) that a problem can have more than one optimal solution. This fact was illustrated in Fig. 3.5 by changing the objective function in the Wyndor Glass Co. problem to $Z = 3x_1 + 2x_2$, so that every point on the line segment between (2, 6) and (4, 3) is optimal. Thus, all optimal solutions are a *weighted average* of these two optimal CPF solutions

$$(x_1, x_2) = w_1(2, 6) + w_2(4, 3),$$

where the weights w_1 and w_2 are numbers that satisfy the relationships

$$w_1 + w_2 = 1 \quad \text{and} \quad w_1 \geq 0, \quad w_2 \geq 0.$$

For example, $w_1 = \frac{1}{3}$ and $w_2 = \frac{2}{3}$ give

$$(x_1, x_2) = \frac{1}{3}(2, 6) + \frac{2}{3}(4, 3) = \left(\frac{2}{3} + \frac{8}{3}, \quad \frac{6}{3} + \frac{6}{3}\right) = \left(\frac{10}{3}, \quad 4\right)$$

as one optimal solution.

In general, any weighted average of two or more solutions (vectors) where the weights are nonnegative and sum to 1 is called a **convex combination** of these solutions. Thus, every optimal solution in the example is a convex combination of (2, 6) and (4, 3).

This example is typical of problems with multiple optimal solutions.

As indicated at the end of Sec. 3.2, *any* linear programming problem with multiple optimal solutions (and a bounded feasible region) has at least two CPF solutions that are optimal. *Every* optimal solution is a convex combination of these optimal CPF solutions. Consequently, in augmented form, every optimal solution is a convex combination of the optimal BF solutions.

(Problems 4.5-5 and 4.5-6 guide you through the reasoning behind this conclusion.)

The simplex method automatically stops after *one* optimal BF solution is found. However, for many applications of linear programming, there are intangible factors not incorporated into the model that can be used to make meaningful choices between alternative optimal solutions. In such cases, these other optimal solutions should be identified as well. As indicated above, this requires finding all the other optimal BF solutions, and then every optimal solution is a convex combination of the optimal BF solutions.

After the simplex method finds one optimal BF solution, you can detect if there are any others and, if so, find them as follows:

> Whenever a problem has more than one optimal BF solution, at least one of the nonbasic variables has a coefficient of zero in the final row 0, so increasing any such variable will not change the value of Z. Therefore, these other optimal BF solutions can be identified (if desired) by performing additional iterations of the simplex method, each time choosing a nonbasic variable with a zero coefficient as the entering basic variable.[1]

To illustrate, consider again the case just mentioned, where the objective function in the Wyndor Glass Co. problem is changed to $Z = 3x_1 + 2x_2$. The simplex method obtains the first three tableaux shown in Table 4.10 and stops with an optimal BF solution. How-

---

[1]If such an iteration has no *leaving* basic variable, this indicates that the feasible region is unbounded and the entering basic variable can be increased indefinitely without changing the value of Z.

TABLE 4.10 Complete set of simplex tableaux to obtain all optimal BF solutions for the Wyndor Glass Co. problem with $c_2 = 2$

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side	Solution Optimal?
	Z	(0)	1	−3	−2	0	0	0	0	No
0	x_3	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	3	2	0	0	1	18	
	Z	(0)	1	0	−2	3	0	0	12	No
1	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	0	2	−3	0	1	6	
	Z	(0)	1	0	0	**0**	0	1	18	Yes
2	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	0	3	1	−1	6	
	x_2	(3)	0	0	1	$-\dfrac{3}{2}$	0	$\dfrac{1}{2}$	3	
	Z	(0)	1	0	0	0	**0**	1	18	Yes
Extra	x_1	(1)	0	1	0	0	$-\dfrac{1}{3}$	$\dfrac{1}{3}$	2	
	x_3	(2)	0	0	0	1	$\dfrac{1}{3}$	$-\dfrac{1}{3}$	2	
	x_2	(3)	0	0	1	0	$\dfrac{1}{2}$	0	6	

ever, because a nonbasic variable (x_3) then has a zero coefficient in row 0, we perform one more iteration in Table 4.10 to identify the other optimal BF solution. Thus, the two optimal BF solutions are (4, 3, 0, 6, 0) and (2, 6, 2, 0, 0), each yielding $Z = 18$. Notice that the last tableau also has a *nonbasic* variable (x_4) with a zero coefficient in row 0. This situation is inevitable because the extra iteration does not change row 0, so this leaving basic variable necessarily retains its zero coefficient. Making x_4 an entering basic variable now would only lead back to the third tableau. (Check this.) Therefore, these two are the only BF solutions that are optimal, and all *other* optimal solutions are a convex combination of these two.

$$(x_1, x_2, x_3, x_4, x_5) = w_1(2, 6, 2, 0, 0) + w_2(4, 3, 0, 6, 0),$$
$$w_1 + w_2 = 1, \qquad w_1 \geq 0, \qquad w_2 \geq 0.$$

4.6   ADAPTING TO OTHER MODEL FORMS

Thus far we have presented the details of the simplex method under the assumptions that the problem is in our standard form (maximize Z subject to functional constraints in $\leq$ form and nonnegativity constraints on all variables) and that $b_i \geq 0$ for all $i = 1, 2, \ldots, m$. In this section we point out how to make the adjustments required for other legitimate forms of the linear programming model. You will see that all these adjustments can be made during the initialization, so the rest of the simplex method can then be applied just as you have learned it already.

The only serious problem introduced by the other forms for functional constraints (the $=$ or $\geq$ forms, or having a negative right-hand side) lies in identifying an *initial BF solution*. Before, this initial solution was found very conveniently by letting the slack variables be the initial basic variables, so that each one just equals the *nonnegative* right-hand side of its equation. Now, something else must be done. The standard approach that is used for all these cases is the **artificial-variable technique.** This technique constructs a more convenient *artificial problem* by introducing a dummy variable (called an *artificial variable*) into each constraint that needs one. This new variable is introduced just for the purpose of being the initial basic variable for that equation. The usual nonnegativity constraints are placed on these variables, and the objective function also is modified to impose an exorbitant penalty on their having values larger than zero. The iterations of the simplex method then automatically force the artificial variables to disappear (become zero), one at a time, until they are all gone, after which the *real* problem is solved.

To illustrate the artificial-variable technique, first we consider the case where the only nonstandard form in the problem is the presence of one or more equality constraints.

Equality Constraints

Any equality constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

actually is equivalent to a pair of inequality constraints:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i.$$

However, rather than making this substitution and thereby increasing the number of constraints, it is more convenient to use the artificial-variable technique. We shall illustrate this technique with the following example.

Example.   Suppose that the Wyndor Glass Co. problem in Sec. 3.1 is modified to *require* that Plant 3 be used at full capacity. The only resulting change in the linear programming model is that the third constraint, $3x_1 + 2x_2 \leq 18$, instead becomes an equality constraint

$$3x_1 + 2x_2 = 18,$$

so that the complete model becomes the one shown in the upper right-hand corner of Fig. 4.3. This figure also shows in darker ink the feasible region which now consists of *just* the line segment connecting (2, 6) and (4, 3).

After the slack variables still needed for the inequality constraints are introduced, the system of equations for the augmented form of the problem becomes
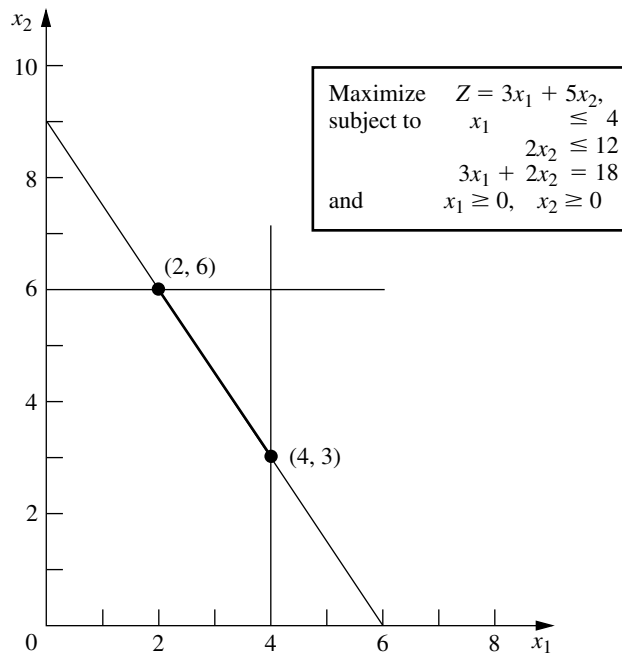
$$
\begin{align}
(0) \quad & Z - 3x_1 - 5x_2 && = 0 \\
(1) \quad & x_1 + x_3 && = 4 \\
(2) \quad & 2x_2 \phantom{{}+ x_3} + x_4 && = 12 \\
(3) \quad & 3x_1 + 2x_2 && = 18.
\end{align}
$$

Unfortunately, these equations do not have an obvious initial BF solution because there is no longer a slack variable to use as the initial basic variable for Eq. (3). It is necessary to find an initial BF solution to start the simplex method.

This difficulty can be circumvented in the following way.

FIGURE 4.3
When the third functional constraint becomes an equality constraint, the feasible region for the Wyndor Glass Co. problem becomes the line segment between (2, 6) and (4, 3).

Maximize $Z = 3x_1 + 5x_2,$
subject to $x_1 \qquad\quad \leq\ 4$
$ 2x_2 \leq 12$
$ 3x_1 + 2x_2 = 18$
and $x_1 \geq 0, \quad x_2 \geq 0$

Obtaining an Initial BF Solution. The procedure is to construct an **artificial problem** that has the same optimal solution as the real problem by making two modifications of the real problem.

1. Apply the **artificial-variable technique** by introducing a *nonnegative* **artificial variable** (call it $\bar{x}_5$)[1] into Eq. (3), just as if it were a slack variable

   (3) $3x_1 + 2x_2 + \bar{x}_5 = 18.$

2. Assign an *overwhelming penalty* to having $\bar{x}_5 > 0$ by changing the objective function $Z = 3x_1 + 5x_2$ to

   $Z = 3x_1 + 5x_2 - M\bar{x}_5,$

   where M symbolically represents a *huge* positive number. (This method of forcing $\bar{x}_5$ to be $\bar{x}_5 = 0$ in the optimal solution is called the **Big M method.**)

Now find the optimal solution for the real problem by applying the simplex method to the artificial problem, starting with the following initial BF solution:

   Initial BF Solution
   Nonbasic variables: $x_1 = 0,$ $x_2 = 0$
   Basic variables: $x_3 = 4,$ $x_4 = 12,$ $\bar{x}_5 = 18.$

   Because $\bar{x}_5$ plays the role of the slack variable for the third constraint in the artificial problem, this constraint is equivalent to $3x_1 + 2x_2 \leq 18$ (just as for the original Wyndor Glass Co. problem in Sec. 3.1). We show below the resulting artificial problem (before augmenting) next to the real problem.

The Real Problem	*The Artificial Problem*
	Define $\bar{x}_5 = 18 - 3x_1 - 2x_2.$
Maximize $Z = 3x_1 + 5x_2,$	Maximize $Z = 3x_1 + 5x_2 - M\bar{x}_5,$
subject to	subject to
$x_1 \qquad \leq \; 4$	$x_1 \qquad\qquad \leq \; 4$
$2x_2 \leq 12$	$2x_2 \qquad \leq \; 12$
$3x_1 + 2x_2 = 18$	$3x_1 + 2x_2 \qquad \leq \; 18$
and	(so $3x_1 + 2x_2 + \bar{x}_5 = 18$)
$x_1 \geq 0, \qquad x_2 \geq 0.$	and
	$x_1 \geq 0, \qquad x_2 \geq 0, \qquad \bar{x}_5 \geq 0.$

Therefore, just as in Sec. 3.1, the feasible region for (x_1, x_2) for the artificial problem is the one shown in Fig. 4.4. The only portion of this feasible region that coincides with the feasible region for the real problem is where $\bar{x}_5 = 0$ (so $3x_1 + 2x_2 = 18$).

Figure 4.4 also shows the order in which the simplex method examines the CPF solutions (or BF solutions after augmenting), where each circled number identifies which iteration obtained that solution. Note that the simplex method moves counterclockwise here

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[1]We shall always label the artificial variables by putting a bar over them.
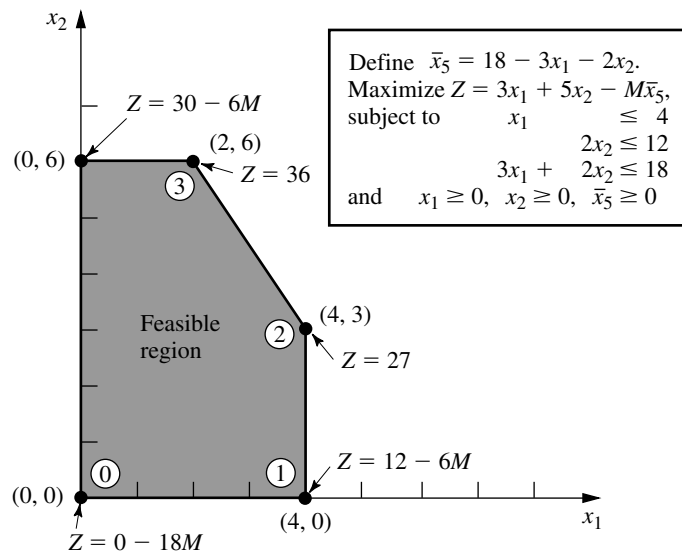
FIGURE 4.4
This graph shows the feasible region and the sequence of CPF solutions (⓪, ①, ②, ③) examined by the simplex method for the artificial problem that corresponds to the real problem of Fig. 4.3.

whereas it moved clockwise for the original Wyndor Glass Co. problem (see Fig. 4.2). The reason for this difference is the extra term $-M\bar{x}_5$ in the objective function for the artificial problem.

Before applying the simplex method and demonstrating that it follows the path shown in Fig. 4.4, the following preparatory step is needed.

Converting Equation (0) to Proper Form. The system of equations after the artificial problem is augmented is

$$
\begin{array}{llllll}
(0) & Z - 3x_1 - 5x_2 & & & + M\bar{x}_5 = & 0 \\
(1) & x_1 & + \mathbf{x}_3 & & = & 4 \\
(2) & 2x_2 & + \mathbf{x}_4 & & = & 12 \\
(3) & 3x_1 + 2x_2 & & + \bar{\mathbf{x}}_5 = & 18
\end{array}
$$

where the initial basic variables $(x_3, x_4, \bar{x}_5)$ are shown in bold type. However, this system is not yet in proper form from Gaussian elimination because a basic variable $\bar{x}_5$ has a nonzero coefficient in Eq. (0). Recall that all basic variables must be algebraically eliminated from Eq. (0) before the simplex method can either apply the optimality test or find the entering basic variable. This elimination is necessary so that the negative of the coefficient of each nonbasic variable will give the rate at which Z would increase if that nonbasic variable were to be increased from 0 while adjusting the values of the basic variables accordingly.

To algebraically eliminate $\bar{x}_5$ from Eq. (0), we need to subtract from Eq. (0) the product, M times Eq. (3).

$$
\begin{array}{ll}
& Z - 3x_1 - 5x_2 + M\bar{x}_5 = \quad 0 \\
& \underline{-M(3x_1 + 2x_2 + \quad \bar{x}_5 = 18)} \\
\text{New (0)} \quad & Z - (3M + 3)x_1 - (2M + 5)x_2 = -18M.
\end{array}
$$

Application of the Simplex Method.   This new Eq. (0) gives Z in terms of *just* the nonbasic variables (x_1, x_2),

$$Z = -18M + (3M + 3)x_1 + (2M + 5)x_2.$$

Since $3M + 3 > 2M + 5$ (remember that M represents a huge number), increasing x_1 increases Z at a faster rate than increasing x_2 does, so x_1 is chosen as the entering basic variable. This leads to the move from $(0, 0)$ to $(4, 0)$ at iteration 1, shown in Fig. 4.4, thereby increasing Z by $4(3M + 3)$.

The quantities involving M never appear in the system of equations except for Eq. (0), so they need to be taken into account only in the optimality test and when an entering basic variable is determined. One way of dealing with these quantities is to assign some particular (huge) numerical value to M and use the resulting coefficients in Eq. (0) in the usual way. However, this approach may result in significant rounding errors that invalidate the optimality test. Therefore, it is better to do what we have just shown, namely, to express each coefficient in Eq. (0) as a linear function $aM + b$ of the *symbolic* quantity M by separately recording and updating the current numerical value of (1) the *multiplicative* factor a and (2) the *additive* term b. Because M is assumed to be so large that b always is negligible compared with M when $a \neq 0$, the decisions in the optimality test and the choice of the entering basic variable are made by using just the *multiplicative* factors in the usual way, except for breaking ties with the *additive* factors.

Using this approach on the example yields the simplex tableaux shown in Table 4.11. Note that the artificial variable $\bar{x}_5$ is a *basic variable* $(\bar{x}_5 > 0)$ in the first two tableaux and a *nonbasic variable* $(\bar{x}_5 = 0)$ in the last two. Therefore, the first two BF solutions for this artificial problem are *infeasible* for the real problem whereas the last two also are BF solutions for the real problem.

This example involved only one equality constraint. If a linear programming model has more than one, each is handled in just the same way. (If the right-hand side is negative, multiply through both sides by -1 first.)

Negative Right-Hand Sides

The technique mentioned in the preceding sentence for dealing with an equality constraint with a negative right-hand side (namely, multiply through both sides by -1) also works for any inequality constraint with a negative right-hand side. Multiplying through both sides of an inequality by -1 also reverses the direction of the inequality; i.e., $\leq$ changes to $\geq$ or vice versa. For example, doing this to the constraint

$$x_1 - x_2 \leq -1 \qquad \text{(that is, } x_1 \leq x_2 - 1\text{)}$$

gives the equivalent constraint

$$-x_1 + x_2 \geq 1 \qquad \text{(that is, } x_2 - 1 \geq x_1\text{)}$$

but now the right-hand side is positive. Having nonnegative right-hand sides for all the functional constraints enables the simplex method to begin, because (after augmenting) these right-hand sides become the respective values of the *initial basic variables,* which must satisfy nonnegativity constraints.

TABLE 4.11 Complete set of simplex tableaux for the problem shown in Fig. 4.4

| Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	Right Side
0	Z	(0)	1	$-3M-3$	$-2M-5$	0	0	0	$-18M$
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	$\bar{x}_5$	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	0	$-2M-5$	$3M+3$	0	0	$-6M+12$
	x_1	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	$\bar{x}_5$	(3)	0	0	2	-3	0	1	6
2	Z	(0)	1	0	0	$-\dfrac{9}{2}$	0	$M+\dfrac{5}{2}$	27
	x_1	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	0	3	1	-1	6
	x_2	(3)	0	0	1	$-\dfrac{3}{2}$	0	$\dfrac{1}{2}$	3
Extra	Z	(0)	1	0	0	0	$\dfrac{3}{2}$	$M+1$	36
	x_1	(1)	0	1	0	0	$-\dfrac{1}{3}$	$\dfrac{1}{3}$	2
	x_3	(2)	0	0	0	1	$\dfrac{1}{3}$	$-\dfrac{1}{3}$	2
	x_2	(3)	0	0	1	0	$\dfrac{1}{2}$	0	6

We next focus on how to augment $\geq$ constraints, such as $-x_1 + x_2 \geq 1$, with the help of the artificial-variable technique.

Functional Constraints in $\geq$ Form

To illustrate how the artificial-variable technique deals with functional constraints in $\geq$ form, we will use the model for designing Mary's radiation therapy, as presented in Sec. 3.4. For your convenience, this model is repeated below, where we have placed a box around the constraint of special interest here.

Radiation Therapy Example

Minimize $\quad Z = 0.4x_1 + 0.5x_2,$

subject to

$$0.3x_1 + 0.1x_2 \leq 2.7$$

$$0.5x_1 + 0.5x_2 = 6$$

$$\boxed{0.6x_1 + 0.4x_2 \geq 6}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$
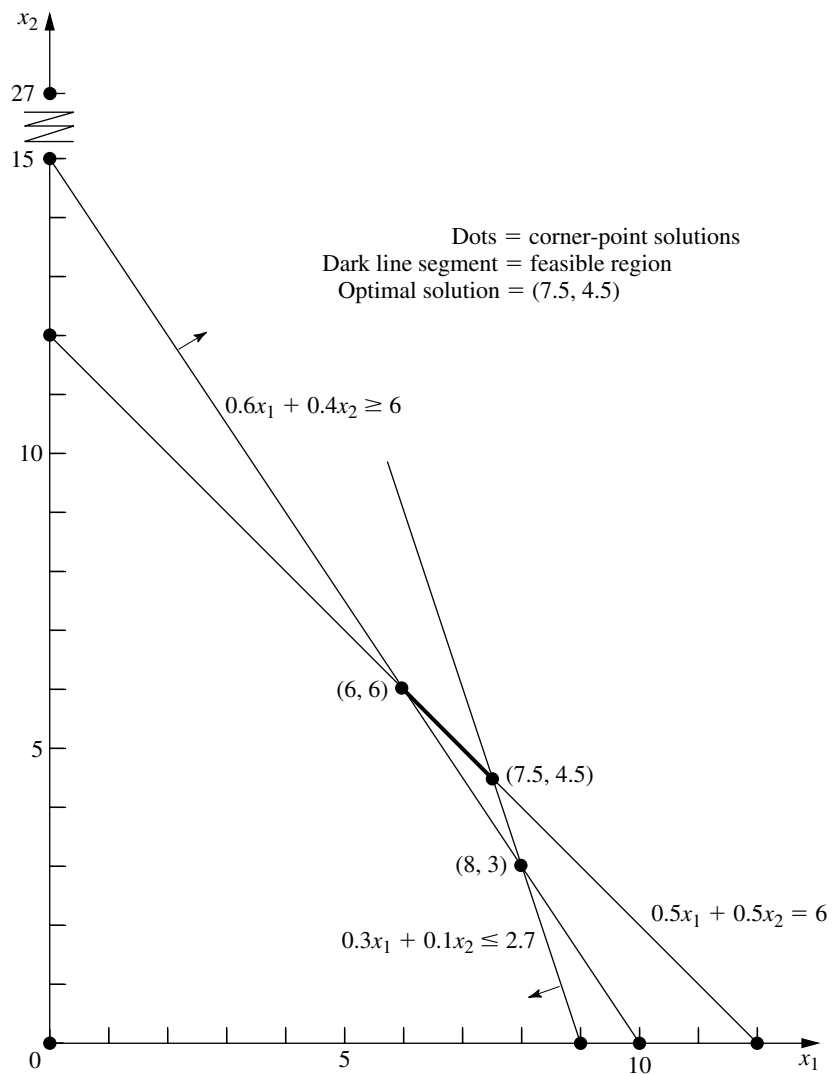
FIGURE 4.5
Graphical display of the
radiation therapy example
and its corner-point
solutions.

The graphical solution for this example (originally presented in Fig. 3.12) is repeated here in a slightly different form in Fig. 4.5. The three lines in the figure, along with the two axes, constitute the five constraint boundaries of the problem. The dots lying at the intersection of a pair of constraint boundaries are the *corner-point solutions*. The only two corner-point *feasible* solutions are (6, 6) and (7.5, 4.5), and the feasible region is the line segment connecting these two points. The optimal solution is $(x_1, x_2) = (7.5, 4.5)$, with $Z = 5.25$.

We soon will show how the simplex method solves this problem by directly solving the corresponding artificial problem. However, first we must describe how to deal with the third constraint.

Our approach involves introducing *both* a surplus variable x_5 (defined as $x_5 = 0.6x_1 + 0.4x_2 - 6$) and an artificial variable $\bar{x}_6$, as shown next.

$$
\begin{aligned}
&0.6x_1 + 0.4x_2 && \geq 6 \\
\rightarrow \quad &0.6x_1 + 0.4x_2 - x_5 && = 6 && (x_5 \geq 0) \\
\rightarrow \quad &0.6x_1 + 0.4x_2 - x_5 + \bar{x}_6 = 6 && && (x_5 \geq 0, \bar{x}_6 \geq 0).
\end{aligned}
$$

Here x_5 is called a **surplus variable** because it subtracts the surplus of the left-hand side over the right-hand side to convert the inequality constraint to an equivalent equality constraint. Once this conversion is accomplished, the artificial variable is introduced just as for any equality constraint.

After a slack variable x_3 is introduced into the first constraint, an artificial variable $\bar{x}_4$ is introduced into the second constraint, and the Big M method is applied, so the complete artificial problem (in augmented form) is

Minimize $\quad Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6,$

subject to
$$
\begin{aligned}
0.3x_1 + 0.1x_2 + x_3 && &= 2.7 \\
0.5x_1 + 0.5x_2 && + \bar{x}_4 &= 6 \\
0.6x_1 + 0.4x_2 && - x_5 + \bar{x}_6 &= 6
\end{aligned}
$$

and $\quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad \bar{x}_4 \geq 0, \quad x_5 \geq 0, \quad \bar{x}_6 \geq 0.$

Note that the coefficients of the artificial variables in the objective function are $+M$, instead of $-M$, because we now are minimizing Z. Thus, even though $\bar{x}_4 > 0$ and/or $\bar{x}_6 > 0$ is possible for a feasible solution for the artificial problem, the huge unit penalty of $+M$ prevents this from occurring in an optimal solution.

As usual, introducing artificial variables enlarges the feasible region. Compare below the original constraints for the real problem with the corresponding constraints on (x_1, x_2) for the artificial problem.

Constraints on (x_1, x_2) *for the Real Problem*	*Constraints on (x_1, x_2)* *for the Artificial Problem*
$0.3x_1 + 0.1x_2 \leq 2.7$	$0.3x_1 + 0.1x_2 \leq 2.7$
$0.5x_1 + 0.5x_2 = 6$	$0.5x_1 + 0.5x_2 \leq 6$ (= holds when $\bar{x}_4 = 0$)
$0.6x_1 + 0.4x_2 \geq 6$	No such constraint (except when $\bar{x}_6 = 0$)
$x_1 \geq 0, \quad x_2 \geq 0$	$x_1 \geq 0, \quad x_2 \geq 0$

Introducing the artificial variable $\bar{x}_4$ to play the role of a slack variable in the second constraint allows values of (x_1, x_2) *below* the $0.5x_1 + 0.5x_2 = 6$ line in Fig. 4.5. Introducing x_5 and $\bar{x}_6$ into the third constraint of the real problem (and moving these variables to the right-hand side) yields the equation

$$0.6x_1 + 0.4x_2 = 6 + x_5 - \bar{x}_6.$$

Because both x_5 and $\bar{x}_6$ are constrained only to be nonnegative, their difference $x_5 - \bar{x}_6$ can be any positive or negative number. Therefore, $0.6x_1 + 0.4x_2$ can have any value, which has the effect of eliminating the third constraint from the artificial problem and allowing points on either side of the $0.6x_1 + 0.4x_2 = 6$ line in Fig. 4.5. (We keep the third constraint in the system of equations only because it will become relevant again later, after the Big M method forces $\bar{x}_6$ to be zero.) Consequently, the feasible region for the ar-

tificial problem is the entire polyhedron in Fig. 4.5 whose vertices are (0, 0), (9, 0), (7.5, 4.5), and (0, 12).

Since the origin now is feasible for the artificial problem, the simplex method starts with (0, 0) as the initial CPF solution, i.e., with $(x_1, x_2, x_3, \bar{x}_4, x_5, \bar{x}_6) = (0, 0, 2.7, 6, 0, 6)$ as the initial BF solution. (Making the origin feasible as a convenient starting point for the simplex method is the whole point of creating the artificial problem.) We soon will trace the entire path followed by the simplex method from the origin to the optimal solution for both the artificial and real problems. But, first, how does the simplex method handle *minimization?*

Minimization

One straightforward way of minimizing Z with the simplex method is to exchange the roles of the positive and negative coefficients in row 0 for both the optimality test and step 1 of an iteration. However, rather than changing our instructions for the simplex method for this case, we present the following simple way of converting any minimization problem to an equivalent maximization problem:

$$\text{Minimizing} \qquad Z = \sum_{j=1}^{n} c_j x_j$$

is equivalent to

$$\text{maximizing} \qquad -Z = \sum_{j=1}^{n} (-c_j) x_j;$$

i.e., the two formulations yield the same optimal solution(s).

The two formulations are equivalent because the smaller Z is, the larger $-Z$ is, so the solution that gives the *smallest* value of Z in the entire feasible region must also give the *largest* value of $-Z$ in this region.

Therefore, in the radiation therapy example, we make the following change in the formulation:

$$\begin{aligned}
&\text{Minimize} &&Z = 0.4x_1 + 0.5x_2 \\
\rightarrow \quad &\text{Maximize} &&-Z = -0.4x_1 - 0.5x_2.
\end{aligned}$$

After artificial variables $\bar{x}_4$ and $\bar{x}_6$ are introduced and then the Big M method is applied, the corresponding conversion is

$$\begin{aligned}
&\text{Minimize} &&Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6 \\
\rightarrow \quad &\text{Maximize} &&-Z = -0.4x_1 - 0.5x_2 - M\bar{x}_4 - M\bar{x}_6.
\end{aligned}$$

Solving the Radiation Therapy Example

We now are nearly ready to apply the simplex method to the radiation therapy example. By using the maximization form just obtained, the entire system of equations is now

$$\begin{aligned}
(0) \qquad &-Z + 0.4x_1 + 0.5x_2 &&+ M\bar{x}_4 &&+ M\bar{x}_6 = 0 \\
(1) \qquad & 0.3x_1 + 0.1x_2 + x_3 &&&& = 2.7
\end{aligned}$$

$$\text{(2)} \qquad 0.5x_1 + 0.5x_2 \qquad + \quad \bar{x}_4 \qquad\qquad = 6$$
$$\text{(3)} \qquad 0.6x_1 + 0.4x_2 \qquad\qquad - x_5 + \quad \bar{x}_6 = 6.$$

The basic variables $(x_3, \bar{x}_4, \bar{x}_6)$ for the initial BF solution (for this artificial problem) are shown in bold type.

Note that this system of equations is not yet in proper form from Gaussian elimination, as required by the simplex method, since the basic variables $\bar{x}_4$ and $\bar{x}_6$ still need to be algebraically eliminated from Eq. (0). Because $\bar{x}_4$ and $\bar{x}_6$ both have a coefficient of M, Eq. (0) needs to have subtracted from it *both* M times Eq. (2) *and* M times Eq. (3). The calculations for all the coefficients (and the right-hand sides) are summarized below, where the vectors are the relevant rows of the simplex tableau corresponding to the above system of equations.

Row 0:

[0.4,		0.5,	0,	M,	0,	M,	0]
$-M[0.5,$		0.5,	0,	1,	0,	0,	6]
$-M[0.6,$		0.4,	0,	0,	-1,	1,	6]
New row 0 = $[-1.1M + 0.4,$	$-0.9M + 0.5,$		0,	0,	M,	0,	$-12M]$

The resulting initial simplex tableau, ready to begin the simplex method, is shown at the top of Table 4.12. Applying the simplex method in just the usual way then yields the sequence of simplex tableaux shown in the rest of Table 4.12. For the optimality test and the selection of the entering basic variable at each iteration, the quantities involving M are treated just as discussed in connection with Table 4.11. Specifically, whenever M is present, only its multiplicative factor is used, unless there is a tie, in which case the tie is broken by using the corresponding additive terms. Just such a tie occurs in the last selection of an entering basic variable (see the next-to-last tableau), where the coefficients of x_3 and x_5 in row 0 both have the same multiplicative factor of $-\frac{5}{3}$. Comparing the additive terms, $\frac{11}{6} < \frac{7}{3}$ leads to choosing x_5 as the entering basic variable.

Note in Table 4.12 the progression of values of the artificial variables $\bar{x}_4$ and $\bar{x}_6$ and of Z. We start with large values, $\bar{x}_4 = 6$ and $\bar{x}_6 = 6$, with $Z = 12M$ $(-Z = -12M)$. The first iteration greatly reduces these values. The Big M method succeeds in driving $\bar{x}_6$ to zero (as a new nonbasic variable) at the second iteration and then in doing the same to $\bar{x}_4$ at the next iteration. With both $\bar{x}_4 = 0$ and $\bar{x}_6 = 0$, the basic solution given in the last tableau is guaranteed to be feasible for the real problem. Since it passes the optimality test, it also is optimal.

Now see what the Big M method has done graphically in Fig. 4.6. The feasible region for the artificial problem initially has four CPF solutions—(0, 0), (9, 0), (0, 12), and (7.5, 4.5)—and then replaces the first three with two new CPF solutions—(8, 3), (6, 6)—after $\bar{x}_6$ decreases to $\bar{x}_6 = 0$ so that $0.6x_1 + 0.4x_2 \geq 6$ becomes an additional constraint. (Note that the three replaced CPF solutions—(0, 0), (9, 0), and (0, 12)—actually were corner-point *infeasible* solutions for the real problem shown in Fig. 4.5.) Starting with the origin as the convenient initial CPF solution for the artificial problem, we move around the boundary to three other CPF solutions—(9, 0), (8, 3), and (7.5, 4.5). The last of these is the first one that also is feasible for the real problem. Fortuitously, this first feasible solution also is optimal, so no additional iterations are needed.

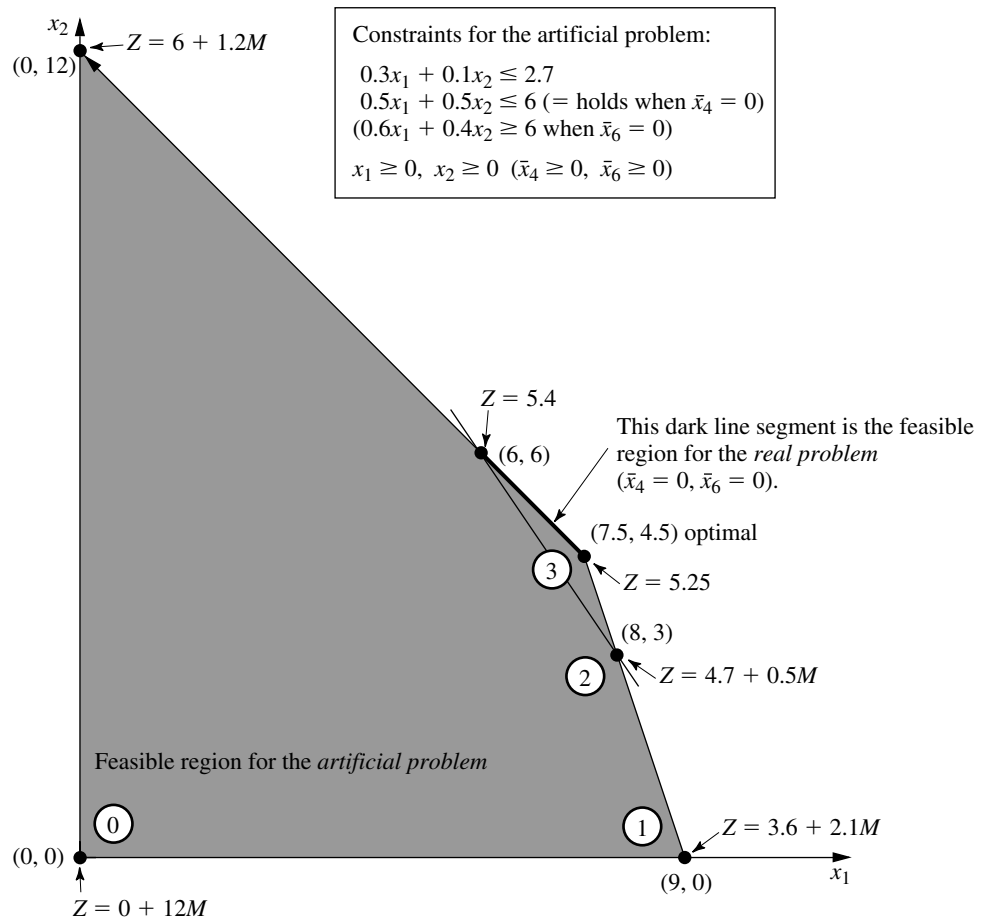**TABLE 4.12** The Big M method for the radiation therapy example

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	−1	$-1.1M + 0.4$	$-0.9M + 0.5$	0	0	M	0	$-12M$
	x_3	(1)	0	0.3	0.1	1	0	0	0	2.7
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	−1	1	6
1	Z	(0)	−1	0	$-\dfrac{16}{30}M + \dfrac{11}{30}$	$\dfrac{11}{3}M - \dfrac{4}{3}$	0	M	0	$-2.1M - 3.6$
	x_1	(1)	0	1	$\dfrac{1}{3}$	$\dfrac{10}{3}$	0	0	0	9
	$\bar{x}_4$	(2)	0	0	$\dfrac{1}{3}$	$-\dfrac{5}{3}$	1	0	0	1.5
	$\bar{x}_6$	(3)	0	0	0.2	−2	0	−1	1	0.6
2	Z	(0)	−1	0	0	$-\dfrac{5}{3}M + \dfrac{7}{3}$	0	$-\dfrac{5}{3}M + \dfrac{11}{6}$	$\dfrac{8}{3}M - \dfrac{11}{6}$	$-0.5M - 4.7$
	x_1	(1)	0	1	0	$\dfrac{20}{3}$	0	$\dfrac{5}{3}$	$-\dfrac{5}{3}$	8
	$\bar{x}_4$	(2)	0	0	0	$\dfrac{5}{3}$	1	$\dfrac{5}{3}$	$-\dfrac{5}{3}$	0.5
	x_2	(3)	0	0	1	−10	0	−5	5	3
3	Z	(0)	−1	0	0	0.5	$M - 1.1$	0	M	−5.25
	x_1	(1)	0	1	0	5	−1	0	0	7.5
	x_5	(2)	0	0	0	1	0.6	1	−1	0.3
	x_2	(3)	0	0	1	−5	3	0	0	4.5

For other problems with artificial variables, it may be necessary to perform additional iterations to reach an optimal solution after the first feasible solution is obtained for the real problem. (This was the case for the example solved in Table 4.11.) Thus, the Big M method can be thought of as having two phases. In the *first phase,* all the artificial variables are driven to zero (because of the penalty of M per unit for being greater than zero) in order to reach an initial BF solution for the *real* problem. In the *second phase,* all the artificial variables are kept at zero (because of this same penalty) while the simplex method generates a sequence of BF solutions for the real problem that leads to an optimal solution. The *two-phase method* described next is a streamlined procedure for performing these two phases directly, without even introducing M explicitly.

The Two-Phase Method

For the radiation therapy example just solved in Table 4.12, recall its real objective function

Real problem:　　Minimize　　$Z = 0.4x_1 + 0.5x_2.$

Constraints for the artificial problem:

$0.3x_1 + 0.1x_2 \leq 2.7$
$0.5x_1 + 0.5x_2 \leq 6$ (= holds when $\bar{x}_4 = 0$)
$(0.6x_1 + 0.4x_2 \geq 6$ when $\bar{x}_6 = 0)$

$x_1 \geq 0, \; x_2 \geq 0 \; (\bar{x}_4 \geq 0, \; \bar{x}_6 \geq 0)$

$Z = 6 + 1.2M$ (0, 12)

$Z = 5.4$

This dark line segment is the feasible region for the *real problem* ($\bar{x}_4 = 0$, $\bar{x}_6 = 0$).

(6, 6)

(7.5, 4.5) optimal

③ $Z = 5.25$

(8, 3)

② $Z = 4.7 + 0.5M$

Feasible region for the *artificial problem*

⓪

① $Z = 3.6 + 2.1M$

(0, 0)

(9, 0) x_1

$Z = 0 + 12M$

FIGURE 4.6
This graph shows the feasible region and the sequence of CPF solutions (⓪, ①, ②, ③) examined by the simplex method (with the Big *M* method) for the artificial problem that corresponds to the real problem of Fig. 4.5.

However, the Big *M* method uses the following objective function (or its equivalent in maximization form) throughout the entire procedure:

Big M method: Minimize $Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6$.

Since the first two coefficients are negligible compared to *M*, the two-phase method is able to drop *M* by using the following two objective functions with completely different definitions of *Z* in turn.

Two-phase method:

Phase 1:   Minimize $Z = \bar{x}_4 + \bar{x}_6$ (until $\bar{x}_4 = 0$, $\bar{x}_6 = 0$).
Phase 2:   Minimize $Z = 0.4x_1 + 0.5x_2$ (with $\bar{x}_4 = 0$, $\bar{x}_6 = 0$).

The phase 1 objective function is obtained by dividing the Big *M* method objective function by *M* and then dropping the negligible terms. Since phase 1 concludes by obtaining

a BF solution for the real problem (one where $\bar{x}_4 = 0$ and $\bar{x}_6 = 0$), this solution is then used as the *initial* BF solution for applying the simplex method to the real problem (with its real objective function) in phase 2.

Before solving the example in this way, we summarize the general method.

Summary of the Two-Phase Method.   *Initialization:* Revise the constraints of the original problem by introducing artificial variables as needed to obtain an obvious initial BF solution for the *artificial problem.*

Phase 1: The objective for this phase is to find a BF solution for the *real problem.* To do this,

Minimize $Z = \Sigma$ artificial variables, subject to revised constraints.

The optimal solution obtained for this problem (with $Z = 0$) will be a BF solution for the real problem.

Phase 2: The objective for this phase is to find an *optimal solution* for the real problem. Since the artificial variables are not part of the real problem, these variables can now be dropped (they are all zero now anyway).[1] Starting from the BF solution obtained at the end of phase 1, use the simplex method to solve the real problem.

For the example, the problems to be solved by the simplex method in the respective phases are summarized below.

Phase 1 Problem (Radiation Therapy Example):

Minimize $Z = \bar{x}_4 + \bar{x}_6,$

subject to

$$
\begin{aligned}
0.3x_1 + 0.1x_2 + x_3 \qquad\qquad &= 2.7 \\
0.5x_1 + 0.5x_2 \qquad + \bar{x}_4 \qquad &= 6 \\
0.6x_1 + 0.4x_2 \qquad\quad - x_5 + \bar{x}_6 &= 6
\end{aligned}
$$

and

$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0, \qquad \bar{x}_4 \geq 0, \qquad x_5 \geq 0, \qquad \bar{x}_6 \geq 0.$

Phase 2 Problem (Radiation Therapy Example):

Minimize $Z = 0.4x_1 + 0.5x_2,$

subject to

$$
\begin{aligned}
0.3x_1 + 0.1x_2 + x_3 \quad &= 2.7 \\
0.5x_1 + 0.5x_2 \qquad &= 6 \\
0.6x_1 + 0.4x_2 \quad - x_5 &= 6
\end{aligned}
$$

and

$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0, \qquad x_5 \geq 0.$

---

[1]We are skipping over three other possibilities here: (1) artificial variables > 0 (discussed in the next subsection), (2) artificial variables that are degenerate basic variables, and (3) retaining the artificial variables as nonbasic variables in phase 2 (and not allowing them to become basic) as an aid to subsequent postoptimality analysis. Your OR Courseware allows you to explore these possibilities.

The only differences between these two problems are in the objective function and in the inclusion (phase 1) or exclusion (phase 2) of the artificial variables $\bar{x}_4$ and $\bar{x}_6$. Without the artificial variables, the phase 2 problem does not have an obvious *initial BF solution.* The sole purpose of solving the phase 1 problem is to obtain a BF solution with $\bar{x}_4 = 0$ and $\bar{x}_6 = 0$ so that this solution (without the artificial variables) can be used as the initial BF solution for phase 2.

Table 4.13 shows the result of applying the simplex method to this phase 1 problem. [Row 0 in the initial tableau is obtained by converting Minimize $Z = \bar{x}_4 + \bar{x}_6$ to Maximize $(-Z) = -\bar{x}_4 - \bar{x}_6$ and then using *elementary row operations* to eliminate the basic variables $\bar{x}_4$ and $\bar{x}_6$ from $-Z + \bar{x}_4 + \bar{x}_6 = 0$.] In the next-to-last tableau, there is a tie for the *entering basic variable* between x_3 and x_5, which is broken arbitrarily in favor of x_3. The solution obtained at the end of phase 1, then, is $(x_1, x_2, x_3, \bar{x}_4, x_5, \bar{x}_6) = (6, 6, 0.3, 0, 0, 0)$ or, after $\bar{x}_4$ and $\bar{x}_6$ are dropped, $(x_1, x_2, x_3, x_5) = (6, 6, 0.3, 0)$.

As claimed in the summary, this solution from phase 1 is indeed a BF solution for the *real* problem (the phase 2 problem) because it is the solution (after you set $x_5 = 0$) to the system of equations consisting of the three functional constraints for the phase 2 problem. In fact, after deleting the $\bar{x}_4$ and $\bar{x}_6$ columns as well as row 0 for each iteration, Table

TABLE 4.13 Phase 1 of the two-phase method for the radiation therapy example

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	−1	−1.1	−0.9	0	0	1	0	−12
	x_3	(1)	0	0.3	0.1	1	0	0	0	2.7
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	−1	1	6
1	Z	(0)	−1	0	$-\dfrac{16}{30}$	$\dfrac{11}{3}$	0	1	0	−2.1
	x_1	(1)	0	1	$\dfrac{1}{3}$	$\dfrac{10}{3}$	0	0	0	9
	$\bar{x}_4$	(2)	0	0	$\dfrac{1}{3}$	$-\dfrac{5}{3}$	1	0	0	1.5
	$\bar{x}_6$	(3)	0	0	0.2	−2	0	−1	1	0.6
2	Z	(0)	−1	0	0	$-\dfrac{5}{3}$	0	$-\dfrac{5}{3}$	$\dfrac{8}{3}$	−0.5
	x_1	(1)	0	1	0	$\dfrac{20}{3}$	0	$\dfrac{5}{3}$	$-\dfrac{5}{3}$	8
	$\bar{x}_4$	(2)	0	0	0	$\dfrac{5}{3}$	1	$\dfrac{5}{3}$	$-\dfrac{5}{3}$	0.5
	x_2	(3)	0	0	1	−10	0	−5	5	3
3	Z	(0)	−1	0	0	0	1	0	1	0
	x_1	(1)	0	1	0	0	−4	−5	5	6
	x_3	(2)	0	0	0	1	$\dfrac{3}{5}$	1	−1	0.3
	x_2	(3)	0	0	1	0	6	5	−5	6

4.13 shows one way of using Gaussian elimination to solve this system of equations by reducing the system to the form displayed in the final tableau.

Table 4.14 shows the preparations for beginning phase 2 after phase 1 is completed. Starting from the final tableau in Table 4.13, we drop the artificial variables ($\bar{x}_4$ and $\bar{x}_6$), substitute the phase 2 objective function ($-Z = -0.4x_1 - 0.5x_2$ in maximization form) into row 0, and then restore the proper form from Gaussian elimination (by algebraically eliminating the basic variables x_1 and x_2 from row 0). Thus, row 0 in the last tableau is obtained by performing the following *elementary row operations* in the next-to-last tableau: from row 0 subtract both the product, 0.4 times row 1, and the product, 0.5 times row 3. Except for the deletion of the two columns, note that rows 1 to 3 never change. The only adjustments occur in row 0 in order to replace the phase 1 objective function by the phase 2 objective function.

The last tableau in Table 4.14 is the initial tableau for applying the simplex method to the phase 2 problem, as shown at the top of Table 4.15. Just one iteration then leads to the optimal solution shown in the second tableau: $(x_1, x_2, x_3, x_5) = (7.5, 4.5, 0, 0.3)$. This solution is the desired optimal solution for the real problem of interest rather than the artificial problem constructed for phase 1.

Now we see what the two-phase method has done graphically in Fig. 4.7. Starting at the origin, phase 1 examines a total of four CPF solutions for the artificial problem. The first three actually were corner-point infeasible solutions for the real problem shown in Fig. 4.5. The fourth CPF solution, at (6, 6), is the first one that also is feasible for the real

**TABLE 4.14** Preparing to begin phase 2 for the radiation therapy example

	Basic Variable	Eq.	Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
						Coefficient of:				
Final Phase 1 tableau	Z	(0)	−1	0	0	0	1	0	1	0
	x_1	(1)	0	1	0	0	−4	−5	5	6
	x_3	(2)	0	0	0	1	$\frac{3}{5}$	1	−1	0.3
	x_2	(3)	0	0	1	0	6	5	−5	6
Drop $\bar{x}_4$ and $\bar{x}_6$	Z	(0)	−1	0	0	0		0		0
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6
Substitute phase 2 objective function	Z	(0)	−1	0.4	0.5	0		0		0
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6
Restore proper form from Gaussian elimination	Z	(0)	−1	0	0	0		−0.5		−5.4
	x_1	(1)	0	1	0	0		−5		6
	x_3	(2)	0	0	0	1		1		0.3
	x_2	(3)	0	0	1	0		5		6

TABLE 4.15 Phase 2 of the two-phase method for the radiation therapy example

	Basic			Coefficient of:				Right
Iteration	Variable	Eq.	Z	x_1	x_2	x_3	x_5	Side
	Z	(0)	-1	0	0	0	-0.5	-5.4
0	x_1	(1)	0	1	0	0	-5	6
	x_3	(2)	0	0	0	1	1	0.3
	x_2	(3)	0	0	1	0	5	6
	Z	(0)	-1	0	0	0.5	0	-5.25
1	x_1	(1)	0	1	0	5	0	7.5
	x_5	(2)	0	0	0	1	1	0.3
	x_2	(3)	0	0	1	-5	0	4.5

FIGURE 4.7
This graph shows the sequence of CPF solutions for phase 1 (⓪, ①, ②, ③) and then for phase 2 (⓪, ①) when the two-phase method is applied to the radiation therapy example.
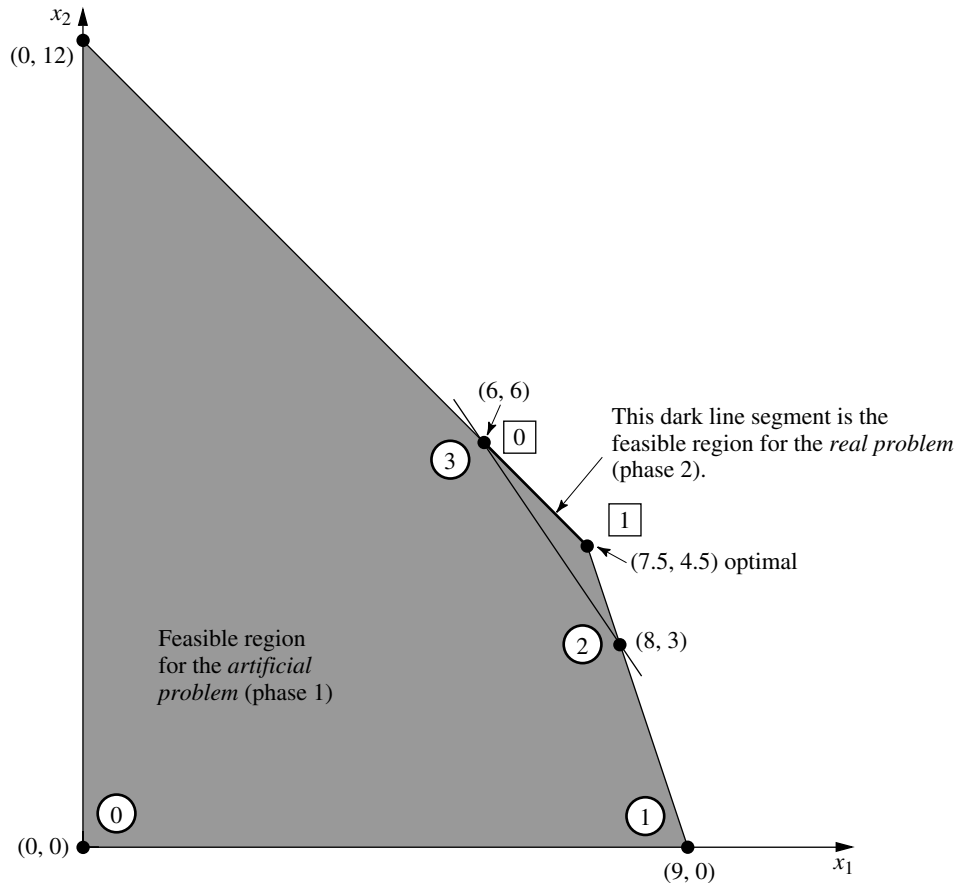
problem, so it becomes the initial CPF solution for phase 2. One iteration in phase 2 leads to the optimal CPF solution at (7.5, 4.5).

If the tie for the entering basic variable in the next-to-last tableau of Table 4.13 had been broken in the other way, then phase 1 would have gone directly from (8, 3) to (7.5, 4.5). After (7.5, 4.5) was used to set up the initial simplex tableau for phase 2, the *optimality test* would have revealed that this solution was optimal, so no iterations would be done.

It is interesting to compare the Big M and two-phase methods. Begin with their objective functions.

Big M Method:

Minimize $Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6.$

Two-Phase Method:

Phase 1:   Minimize $Z = \bar{x}_4 + \bar{x}_6.$
Phase 2:   Minimize $Z = 0.4x_1 + 0.5x_2.$

Because the $M\bar{x}_4$ and $M\bar{x}_6$ terms dominate the $0.4x_1$ and $0.5x_2$ terms in the objective function for the Big M method, this objective function is essentially equivalent to the phase 1 objective function as long as $\bar{x}_4$ and/or $\bar{x}_6$ is greater than zero. Then, when both $\bar{x}_4 = 0$ and $\bar{x}_6 = 0$, the objective function for the Big M method becomes completely equivalent to the phase 2 objective function.

Because of these virtual equivalencies in objective functions, the Big M and two-phase methods generally have the same sequence of BF solutions. The one possible exception occurs when there is a tie for the entering basic variable in phase 1 of the two-phase method, as happened in the third tableau of Table 4.13. Notice that the first three tableaux of Tables 4.12 and 4.13 are almost identical, with the only difference being that the multiplicative factors of M in Table 4.12 become the sole quantities in the corresponding spots in Table 4.13. Consequently, the additive terms that broke the tie for the entering basic variable in the third tableau of Table 4.12 were not present to break this same tie in Table 4.13. The result for this example was an extra iteration for the two-phase method. Generally, however, the advantage of having the additive factors is minimal.

The two-phase method streamlines the Big M method by using only the multiplicative factors in phase 1 and by dropping the artificial variables in phase 2. (The Big M method could combine the multiplicative and additive factors by assigning an actual huge number to M, but this might create numerical instability problems.) For these reasons, the two-phase method is commonly used in computer codes.

No Feasible Solutions

So far in this section we have been concerned primarily with the fundamental problem of identifying an initial BF solution when an obvious one is not available. You have seen how the artificial-variable technique can be used to construct an artificial problem and obtain an initial BF solution for this artificial problem instead. Use of either the Big M method or the two-phase method then enables the simplex method to begin its pilgrim-

age toward the BF solutions, and ultimately toward the optimal solution, for the *real* problem.

However, you should be wary of a certain pitfall with this approach. There may be no obvious choice for the initial BF solution for the very good reason that there are no feasible solutions at all! Nevertheless, by constructing an artificial feasible solution, there is nothing to prevent the simplex method from proceeding as usual and ultimately reporting a supposedly optimal solution.

Fortunately, the artificial-variable technique provides the following signpost to indicate when this has happened:

> If the original problem has *no feasible solutions,* then either the Big M method or phase 1 of the two-phase method yields a final solution that has at least one artificial variable *greater* than zero. Otherwise, they *all* equal zero.

To illustrate, let us change the first constraint in the radiation therapy example (see Fig. 4.5) as follows:

$$0.3x_1 + 0.1x_2 \leq 2.7 \quad \rightarrow \quad 0.3x_1 + 0.1x_2 \leq 1.8,$$

so that the problem no longer has any feasible solutions. Applying the Big M method just as before (see Table 4.12) yields the tableaux shown in Table 4.16. (Phase 1 of the two-phase method yields the same tableaux except that each expression involving M is replaced by just the multiplicative factor.) Hence, the Big M method normally would be indicating that the optimal solution is $(3, 9, 0, 0, 0, 0.6)$. However, since an artificial variable $\bar{x}_6 = 0.6 > 0$, the real message here is that the problem has no feasible solutions.

TABLE 4.16 The Big M method for the revision of the radiation therapy example that has no feasible solutions

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	$\bar{x}_4$	x_5	$\bar{x}_6$	Right Side
0	Z	(0)	-1	$-1.1M + 0.4$	$-0.9M + 0.5$	0	0	M	0	$-12M$
	x_3	(1)	0	0.3	0.1	1	0	0	0	1.8
	$\bar{x}_4$	(2)	0	0.5	0.5	0	1	0	0	6
	$\bar{x}_6$	(3)	0	0.6	0.4	0	0	-1	1	6
1	Z	(0)	-1	0	$-\frac{16}{30}M + \frac{11}{30}$	$\frac{11}{3}M - \frac{4}{3}$	0	M	0	$-5.4M - 2.4$
	x_1	(1)	0	1	$\frac{1}{3}$	$\frac{10}{3}$	0	0	0	6
	$\bar{x}_4$	(2)	0	0	$\frac{1}{3}$	$-\frac{5}{3}$	1	0	0	3
	$\bar{x}_6$	(3)	0	0	0.2	-2	0	-1	1	2.4
2	Z	(0)	-1	0	0	$M + 0.5$	$1.6M - 1.1$	M	0	$-0.6M - 5.7$
	x_1	(1)	0	1	0	5	-1	0	0	3
	x_2	(2)	0	0	1	-5	3	0	0	9
	$\bar{x}_6$	(3)	0	0	0	-1	-0.6	-1	1	0.6

Variables Allowed to Be Negative

In most practical problems, negative values for the decision variables would have no physical meaning, so it is necessary to include nonnegativity constraints in the formulations of their linear programming models. However, this is not always the case. To illustrate, suppose that the Wyndor Glass Co. problem is changed so that product 1 already is in production, and the first decision variable x_1 represents the *increase* in its production rate. Therefore, a negative value of x_1 would indicate that product 1 is to be cut back by that amount. Such reductions might be desirable to allow a larger production rate for the new, more profitable product 2, so negative values should be allowed for x_1 in the model.

Since the procedure for determining the *leaving basic variable* requires that all the variables have nonnegativity constraints, any problem containing variables allowed to be negative must be converted to an *equivalent* problem involving only nonnegative variables before the simplex method is applied. Fortunately, this conversion can be done. The modification required for each variable depends upon whether it has a (negative) lower bound on the values allowed. Each of these two cases is now discussed.

Variables with a Bound on the Negative Values Allowed. Consider any decision variable x_j that is allowed to have negative values which satisfy a constraint of the form

$$x_j \geq L_j,$$

where L_j is some negative constant. This constraint can be converted to a nonnegativity constraint by making the change of variables

$$x_j' = x_j - L_j, \qquad \text{so} \qquad x_j' \geq 0.$$

Thus, $x_j' + L_j$ would be substituted for x_j throughout the model, so that the redefined decision variable x_j' cannot be negative. (This same technique can be used when L_j is *positive* to convert a functional constraint $x_j \geq L_j$ to a nonnegativity constraint $x_j' \geq 0$.)

To illustrate, suppose that the current production rate for product 1 in the Wyndor Glass Co. problem is 10. With the definition of x_1 just given, the complete model at this point is the same as that given in Sec. 3.1 except that the nonnegativity constraint $x_1 \geq 0$ is replaced by

$$x_1 \geq -10.$$

To obtain the equivalent model needed for the simplex method, this decision variable would be redefined as the *total* production rate of product 1

$$x_j' = x_1 + 10,$$

which yields the changes in the objective function and constraints as shown:

$Z = 3x_1 + 5x_2$	$Z = 3(x_1' - 10) + 5x_2$	$Z = -30 + 3x_1' + 5x_2$
$x_1 \qquad \leq 4$	$x_1' - 10 \qquad \leq 4$	$x_1' \qquad \leq 14$
$2x_2 \leq 12$	$2x_2 \leq 12$	$2x_2 \leq 12$
$3x_1 + 2x_2 \leq 18$	$3(x_1' - 10) + 2x_2 \leq 18$	$3x_1' + 2x_2 \leq 48$
$x_1 \geq -10, \qquad x_2 \geq 0$	$x_1' - 10 \geq -10, \qquad x_2 \geq 0$	$x_1' \geq 0, \qquad x_2 \geq 0$

$\rightarrow$ between first and second; $\rightarrow$ between second and third.

Variables with No Bound on the Negative Values Allowed. In the case where x_j does *not* have a lower-bound constraint in the model formulated, another approach is required: x_j is replaced throughout the model by the *difference* of two new *nonnegative* variables

$$x_j = x_j^+ - x_j^-, \qquad \text{where } x_j^+ \geq 0, \, x_j^- \geq 0.$$

Since x_j^+ and x_j^- can have any nonnegative values, this difference $x_j^+ - x_j^-$ can have *any* value (positive or negative), so it is a legitimate substitute for x_j in the model. But after such substitutions, the simplex method can proceed with just nonnegative variables.

The new variables x_j^+ and x_j^- have a simple interpretation. As explained in the next paragraph, each BF solution for the new form of the model necessarily has the property that *either* $x_j^+ = 0$ or $x_j^- = 0$ (or both). Therefore, at the optimal solution obtained by the simplex method (a BF solution),

$$x_j^+ = \begin{cases} x_j & \text{if } x_j \geq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$x_j^- = \begin{cases} |x_j| & \text{if } x_j \leq 0, \\ 0 & \text{otherwise;} \end{cases}$$

so that x_j^+ represents the positive part of the decision variable x_j and x_j^- its negative part (as suggested by the superscripts).

For example, if $x_j = 10$, the above expressions give $x_j^+ = 10$ and $x_j^- = 0$. This same value of $x_j = x_j^+ - x_j^- = 10$ also would occur with larger values of x_j^+ and x_j^- such that $x_j^+ = x_j^- + 10$. Plotting these values of x_j^+ and x_j^- on a two-dimensional graph gives a line with an endpoint at $x_j^+ = 10$, $x_j^- = 0$ to avoid violating the nonnegativity constraints. This endpoint is the only corner-point solution on the line. Therefore, only this endpoint can be part of an overall CPF solution or BF solution involving all the variables of the model. This illustrates why each BF solution necessarily has either $x_j^+ = 0$ or $x_j^- = 0$ (or both).

To illustrate the use of the x_j^+ and x_j^-, let us return to the example on the preceding page where x_1 is redefined as the increase over the current production rate of 10 for product 1 in the Wyndor Glass Co. problem.

However, now suppose that the $x_1 \geq -10$ constraint was not included in the original model because it clearly would not change the optimal solution. (In some problems, certain variables do not need explicit lower-bound constraints because the functional constraints already prevent lower values.) Therefore, before the simplex method is applied, x_1 would be replaced by the difference

$$x_1 = x_1^+ - x_1^-, \qquad \text{where } x_1^+ \geq 0, \, x_1^- \geq 0,$$

as shown:

Maximize	$Z = 3x_1 + 5x_2,$					Maximize	$Z = 3x_1^+ - 3x_1^- + 5x_2,$	
subject to	x_1	≤ 4				subject to	$x_1^+ - x_1^- \leq 4$	
		$2x_2 \leq 12$			$\rightarrow$		$2x_2 \leq 12$	
	$3x_1 + 2x_2 \leq 18$						$3x_1^+ - 3x_1^- + 2x_2 \leq 18$	
	$x_2 \geq 0$ (only)						$x_1^+ \geq 0, \quad x_1^- \geq 0, \quad x_2 \geq 0$	

From a computational viewpoint, this approach has the disadvantage that the new equivalent model to be used has more variables than the original model. In fact, if *all* the original variables lack lower-bound constraints, the new model will have *twice* as many variables. Fortunately, the approach can be modified slightly so that the number of variables is increased by only one, regardless of how many original variables need to be replaced. This modification is done by replacing each such variable x_j by

$$x_j = x_j' - x'', \qquad \text{where } x_j' \geq 0, \ x'' \geq 0,$$

instead, where x'' is the *same* variable for all relevant j. The interpretation of x'' in this case is that $-x''$ is the current value of the *largest* (in absolute terms) negative original variable, so that x_j' is the amount by which x_j exceeds this value. Thus, the simplex method now can make some of the x_j' variables larger than zero even when $x'' > 0$.

4.7   POSTOPTIMALITY ANALYSIS

We stressed in Secs. 2.3, 2.4, and 2.5 that *postoptimality analysis*—the analysis done *after* an optimal solution is obtained for the initial version of the model—constitutes a very major and very important part of most operations research studies. The fact that postoptimality analysis is very important is particularly true for typical linear programming applications. In this section, we focus on the role of the simplex method in performing this analysis.

Table 4.17 summarizes the typical steps in postoptimality analysis for linear programming studies. The rightmost column identifies some algorithmic techniques that involve the simplex method. These techniques are introduced briefly here with the technical details deferred to later chapters.

Reoptimization

As discussed in Sec. 3.7, linear programming models that arise in practice commonly are very large, with hundreds or thousands of functional constraints and decision variables. In such cases, many variations of the basic model may be of interest for considering different scenarios. Therefore, after having found an optimal solution for one version of a linear programming model, we frequently must solve again (often many times) for the so-

TABLE 4.17 Postoptimality analysis for linear programming

Task	Purpose	Technique
Model debugging	Find errors and weaknesses in model	Reoptimization
Model validation	Demonstrate validity of final model	See Sec. 2.4
Final managerial decisions on resource allocations (the b_i values)	Make appropriate division of organizational resources between activities under study and other important activities	Shadow prices
Evaluate estimates of model parameters	Determine crucial estimates that may affect optimal solution for further study	Sensitivity analysis
Evaluate trade-offs between model parameters	Determine best trade-off	Parametric linear programming

lution of a slightly different version of the model. We nearly always have to solve again several times during the model debugging stage (described in Secs. 2.3 and 2.4), and we usually have to do so a large number of times during the later stages of postoptimality analysis as well.

One approach is simply to reapply the simplex method from scratch for each new version of the model, even though each run may require hundreds or even thousands of iterations for large problems. However, a *much more efficient* approach is to *reoptimize.* Reoptimization involves deducing how changes in the model get carried along to the *final* simplex tableau (as described in Secs. 5.3 and 6.6). This revised tableau and the optimal solution for the prior model are then used as the *initial tableau* and the *initial basic solution* for solving the new model. If this solution is feasible for the new model, then the simplex method is applied in the usual way, starting from this initial BF solution. If the solution is not feasible, a related algorithm called the *dual simplex method* (described in Sec. 7.1) probably can be applied to find the new optimal solution,[1] starting from this initial basic solution.

The big advantage of this **reoptimization technique** over re-solving from scratch is that an optimal solution for the revised model probably is going to be *much* closer to the prior optimal solution than to an initial BF solution constructed in the usual way for the simplex method. Therefore, assuming that the model revisions were modest, only a few iterations should be required to reoptimize instead of the hundreds or thousands that may be required when you start from scratch. In fact, the optimal solutions for the prior and revised models are frequently the same, in which case the reoptimization technique requires only one application of the optimality test and *no* iterations.

Shadow Prices

Recall that linear programming problems often can be interpreted as allocating resources to activities. In particular, when the functional constraints are in $\leq$ form, we interpreted the b_i (the right-hand sides) as the amounts of the respective resources being made available for the activities under consideration. In many cases, there may be some latitude in the amounts that will be made available. If so, the b_i values used in the initial (validated) model actually may represent management's *tentative initial decision* on how much of the organization's resources will be provided to the activities considered in the model instead of to other important activities under the purview of management. From this broader perspective, some of the b_i values can be increased in a revised model, but only if a sufficiently strong case can be made to management that this revision would be beneficial.

Consequently, information on the economic contribution of the resources to the measure of performance (Z) for the current study often would be extremely useful. The simplex method provides this information in the form of *shadow prices* for the respective resources.

> The **shadow price** for resource i (denoted by y_i^*) measures the *marginal value* of this resource, i.e., the rate at which Z could be increased by (slightly) increasing the amount of

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[1]The one requirement for using the dual simplex method here is that the *optimality test* is still passed when applied to row 0 of the *revised* final tableau. If not, then still another algorithm called the *primal-dual method* can be used instead.

this resource (b_i) being made available.[1,2] The simplex method identifies this shadow price by $y_i^* =$ coefficient of the ith slack variable in row 0 of the final simplex tableau.

To illustrate, for the Wyndor Glass Co. problem,

Resource $i =$ production capacity of Plant i ($i = 1, 2, 3$) being made available to the two new products under consideration,

$b_i =$ hours of production time per week being made available in Plant i for these new products.

Providing a substantial amount of production time for the new products would require adjusting production times for the current products, so choosing the b_i value is a difficult managerial decision. The tentative initial decision has been

$$b_1 = 4, \qquad b_2 = 12, \qquad b_3 = 18,$$

as reflected in the basic model considered in Sec. 3.1 and in this chapter. However, management now wishes to evaluate the effect of changing any of the b_i values.

The shadow prices for these three resources provide just the information that management needs. The final tableau in Table 4.8 (see p. 128) yields

$$y_1^* = 0 = \text{shadow price for resource 1,}$$

$$y_2^* = \frac{3}{2} = \text{shadow price for resource 2,}$$

$$y_3^* = 1 = \text{shadow price for resource 3.}$$

With just two decision variables, these numbers can be verified by checking graphically that individually increasing any b_i by 1 indeed would increase the optimal value of Z by y_i^*. For example, Fig. 4.8 demonstrates this increase for resource 2 by reapplying the graphical method presented in Sec. 3.1. The optimal solution, (2, 6) with $Z = 36$, changes to $(\frac{5}{3}, \frac{13}{2})$ with $Z = 37\frac{1}{2}$ when b_2 is increased by 1 (from 12 to 13), so that

$$y_2^* = \Delta Z = 37\frac{1}{2} - 36 = \frac{3}{2}.$$

Since Z is expressed in thousands of dollars of profit per week, $y_2^* = \frac{3}{2}$ indicates that adding 1 more hour of production time per week in Plant 2 for these two new products would increase their total profit by $1,500 per week. Should this actually be done? It depends on the marginal profitability of other products currently using this production time. If there is a current product that contributes less than $1,500 of weekly profit per hour of weekly production time in Plant 2, then some shift of production time to the new products would be worthwhile.

We shall continue this story in Sec. 6.7, where the Wyndor OR team uses shadow prices as part of its *sensitivity analysis* of the model.

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[1]The increase in b_i must be sufficiently small that the current set of basic variables remains optimal since this rate (marginal value) changes if the set of basic variables changes.

[2]In the case of a functional constraint in $\geq$ or $=$ form, its shadow price is again defined as the rate at which Z could be increased by (slightly) increasing the value of b_i, although the interpretation of b_i now would normally be something other than the amount of a resource being made available.
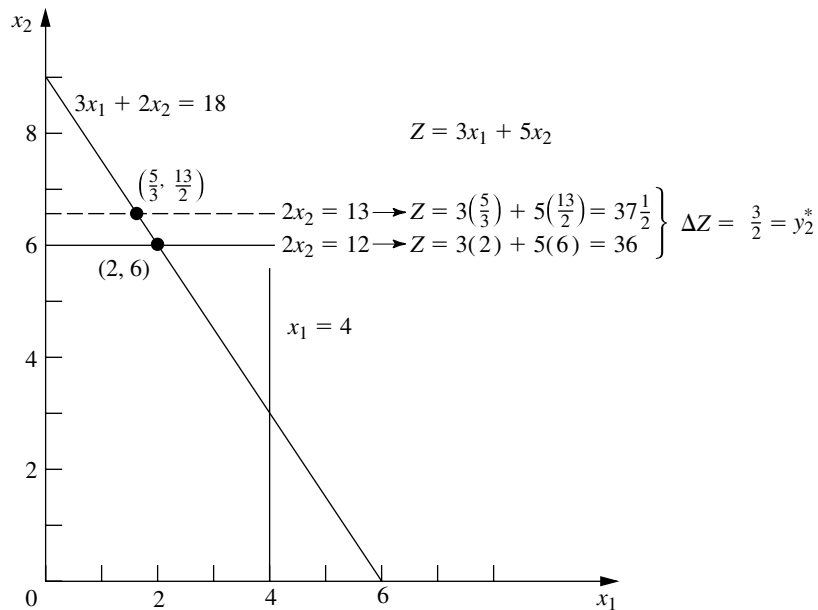
FIGURE 4.8
This graph shows that the shadow price is $y_2^* = \frac{3}{2}$ for resource 2 for the Wyndor Glass Co. problem. The two dots are the optimal solutions for $b_2 = 12$ or $b_2 = 13$, and plugging these solutions into the objective function reveals that increasing b_2 by 1 increases Z by $y_2^* = \frac{3}{2}$.

Figure 4.8 demonstrates that $y_2^* = \frac{3}{2}$ is the rate at which Z could be increased by increasing b_2 slightly. However, it also demonstrates the common phenomenon that this interpretation holds only for a small increase in b_2. Once b_2 is increased beyond 18, the optimal solution stays at $(0, 9)$ with no further increase in Z. (At that point, the set of basic variables in the optimal solution has changed, so a new final simplex tableau will be obtained with new shadow prices, including $y_2^* = 0$.)

Now note in Fig. 4.8 why $y_1^* = 0$. Because the constraint on resource 1, $x_1 \leq 4$, is *not binding* on the optimal solution $(2, 6)$, there is a *surplus* of this resource. Therefore, increasing b_1 beyond 4 cannot yield a new optimal solution with a larger value of Z.

By contrast, the constraints on resources 2 and 3, $2x_2 \leq 12$ and $3x_1 + 2x_2 \leq 18$, are **binding constraints** (constraints that hold with equality at the optimal solution). Because the limited supply of these resources ($b_2 = 12$, $b_3 = 18$) *binds Z* from being increased further, they have *positive* shadow prices. Economists refer to such resources as *scarce goods,* whereas resources available in surplus (such as resource 1) are *free goods* (resources with a zero shadow price).

The kind of information provided by shadow prices clearly is valuable to management when it considers reallocations of resources within the organization. It also is very helpful when an increase in b_i can be achieved only by going outside the organization to purchase more of the resource in the marketplace. For example, suppose that Z represents *profit* and that the unit profits of the activities (the c_j values) include the costs (at regular prices) of all the resources consumed. Then a *positive* shadow price of y_i^* for resource i means that the total profit Z can be increased by y_i^* by purchasing 1 more unit of this resource at its regular price. Alternatively, if a *premium* price must be paid for the resource

in the marketplace, then y_i^* represents the *maximum* premium (excess over the regular price) that would be worth paying.[1]

The theoretical foundation for shadow prices is provided by the duality theory described in Chap. 6.

Sensitivity Analysis

When discussing the *certainty assumption* for linear programming at the end of Sec. 3.3, we pointed out that the values used for the model parameters (the a_{ij}, b_i, and c_j identified in Table 3.3) generally are just *estimates* of quantities whose true values will not become known until the linear programming study is implemented at some time in the future. A main purpose of sensitivity analysis is to identify the **sensitive parameters** (i.e., those that cannot be changed without changing the optimal solution). The sensitive parameters are the parameters that need to be estimated with special care to minimize the risk of obtaining an erroneous optimal solution. They also will need to be monitored particularly closely as the study is implemented. If it is discovered that the true value of a sensitive parameter differs from its estimated value in the model, this immediately signals a need to change the solution.

How are the sensitive parameters identified? In the case of the b_i, you have just seen that this information is given by the shadow prices provided by the simplex method. In particular, if $y_i^* > 0$, then the optimal solution changes if b_i is changed, so b_i is a sensitive parameter. However, $y_i^* = 0$ implies that the optimal solution is not sensitive to at least small changes in b_i. Consequently, if the value used for b_i is an estimate of the amount of the resource that will be available (rather than a managerial decision), then the b_i values that need to be monitored more closely are those with *positive* shadow prices—especially those with *large* shadow prices.

When there are just two variables, the sensitivity of the various parameters can be analyzed graphically. For example, in Fig. 4.9, $c_1 = 3$ can be changed to any other value from 0 to 7.5 without the optimal solution changing from (2, 6). (The reason is that any value of c_1 within this range keeps the slope of $Z = c_1 x_1 + 5x_2$ between the slopes of the lines $2x_2 = 12$ and $3x_1 + 2x_2 = 18$.) Similarly, if $c_2 = 5$ is the only parameter changed, it can have any value greater than 2 without affecting the optimal solution. Hence, neither c_1 nor c_2 is a sensitive parameter.

The easiest way to analyze the sensitivity of each of the a_{ij} parameters graphically is to check whether the corresponding constraint is *binding* at the optimal solution. Because $x_1 \leq 4$ is *not* a binding constraint, any sufficiently small change in its coefficients ($a_{11} = 1$, $a_{12} = 0$) is not going to change the optimal solution, so these are *not* sensitive parameters. On the other hand, both $2x_2 \leq 12$ and $3x_1 + 2x_2 \leq 18$ are *binding constraints,* so changing *any* one of their coefficients ($a_{21} = 0$, $a_{22} = 2$, $a_{31} = 3$, $a_{32} = 2$) is going to change the optimal solution, and therefore these are sensitive parameters.

Typically, greater attention is given to performing sensitivity analysis on the b_i and c_j parameters than on the a_{ij} parameters. On real problems with hundreds or thousands of constraints and variables, the effect of changing one a_{ij} value is usually negligible, but

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[1]If the unit profits do *not* include the costs of the resources consumed, then y_i^* represents the maximum *total* unit price that would be worth paying to increase b_i.

FIGURE 4.9
This graph demonstrates the sensitivity analysis of c_1 and c_2 for the Wyndor Glass Co. problem. Starting with the original objective function line [where $c_1 = 3$, $c_2 = 5$, and the optimal solution is (2, 6)], the other two lines show the extremes of how much the slope of the objective function line can change and still retain (2, 6) as an optimal solution. Thus, with $c_2 = 5$, the allowable range for c_1 is $0 \leq c_1 \leq 7.5$. With $c_1 = 3$, the allowable range for c_2 is $c_2 \geq 2$.
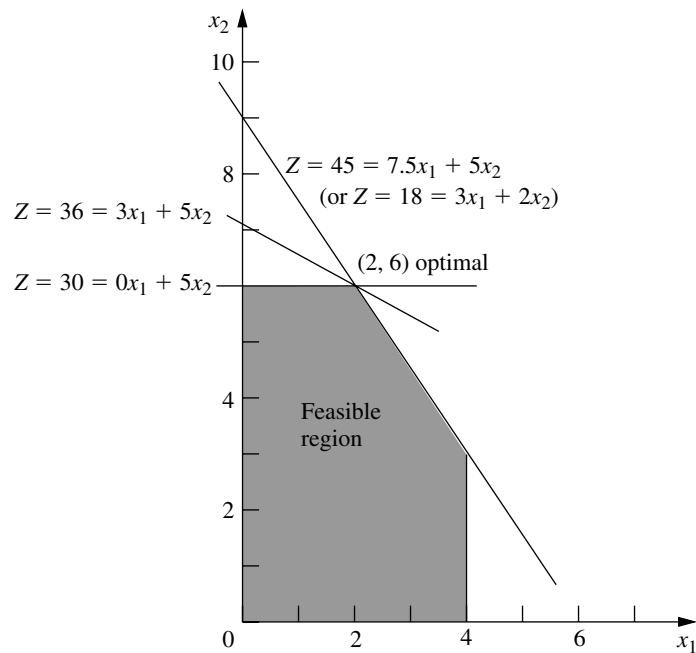
changing one b_i or c_j value can have real impact. Furthermore, in many cases, the a_{ij} values are determined by the technology being used (the a_{ij} values are sometimes called *technological coefficients*), so there may be relatively little (or no) uncertainty about their final values. This is fortunate, because there are far more a_{ij} parameters than b_i and c_j parameters for large problems.

For problems with more than two (or possibly three) decision variables, you cannot analyze the sensitivity of the parameters graphically as was just done for the Wyndor Glass Co. problem. However, you can extract the same kind of information from the simplex method. Getting this information requires using the *fundamental insight* described in Sec. 5.3 to deduce the changes that get carried along to the final simplex tableau as a result of changing the value of a parameter in the original model. The rest of the procedure is described and illustrated in Secs. 6.6 and 6.7.

Using Excel to Generate Sensitivity Analysis Information

Sensitivity analysis normally is incorporated into software packages based on the simplex method. For example, the Excel Solver will generate sensitivity analysis information upon request. As was shown in Fig. 3.19 (see page 72), when the Solver gives the message that it has found a solution, it also gives on the right a list of three reports that can be provided. By selecting the second one (labeled "Sensitivity") after solving the Wyndor Glass Co. problem, you will obtain the *sensitivity report* shown in Fig. 4.10. The upper table in this report provides sensitivity analysis information about the decision variables and their coefficients in the objective function. The lower table does the same for the functional constraints and their right-hand sides.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C9	Solution Doors	2	0	3	4.5	3
D9	Solution Windows	6	0	5	1E+30	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E5	Plant 1 Totals	2	0	4	1E+30	2
E6	Plant 2 Totals	12	1.5	12	6	6
E7	Plant 3 Totals	18	1	18	6	6

FIGURE 4.10
The sensitivity report provided by the Excel Solver for the Wyndor Glass Co. problem.

Look first at the upper table in this figure. The "Final Value" column indicates the optimal solution. The next column gives the *reduced costs*. (We will not discuss these reduced costs now because the information they provide can also be gleaned from the rest of the upper table.) The next three columns provide the information needed to identify the *allowable range to stay optimal* for each coefficient c_j in the objective function.

For any c_j, its **allowable range to stay optimal** is the range of values for this coefficient over which the current optimal solution remains optimal, assuming no change in the other coefficients.

The "Objective Coefficient" column gives the current value of each coefficient, and then the next two columns give the *allowable increase* and the *allowable decrease* from this value to remain within the allowable range. Therefore,

$$3 - 3 \le c_1 \le 3 + 4.5, \quad \text{so} \quad 0 \le c_1 \le 7.5$$

is the allowable range for c_1 over which the current optimal solution will stay optimal (assuming $c_2 = 5$), just as was found graphically in Fig. 4.9. Similarly, since Excel uses $1E + 30$ (10^{30}) to represent infinity,

$$5 - 3 \le c_2 \le 5 + \infty, \quad \text{so} \quad 2 \le c_2$$

is the allowable range to stay optimal for c_2.

The fact that both the allowable increase and the allowable decrease are greater than zero for the coefficient of both decision variables provides another useful piece of information, as described below.

When the upper table in the sensitivity report generated by the Excel Solver indicates that both the allowable increase and the allowable decrease are greater than zero for every objective coefficient, this is a signpost that the optimal solution in the "Final Value" column is the only optimal solution. Conversely, having any allowable increase or allowable decrease equal to zero is a signpost that there are multiple optimal solutions. Changing the corresponding coefficient a tiny amount beyond the zero allowed and re-solving provides another optimal CPF solution for the original model.

Now consider the lower table in Fig. 4.10 that focuses on sensitivity analysis for the three functional constraints. The "Final Value" column gives the value of each constraint's left-hand side for the optimal solution. The next two columns give the shadow price and the current value of the right-hand side (b_i) for each constraint. When just one b_i value is then changed, the last two columns give the *allowable increase* or *allowable decrease* in order to remain within its *allowable range to stay feasible.*

> For any b_i, its **allowable range to stay feasible** is the range of values for this right-hand side over which the current optimal BF solution (with adjusted values[1] for the basic variables) remains feasible, assuming no change in the other right-hand sides.

Thus, using the lower table in Fig. 4.10, combining the last two columns with the current values of the right-hand sides gives the following allowable ranges to stay feasible:

$$2 \leq b_1$$
$$6 \leq b_2 \leq 18$$
$$12 \leq b_3 \leq 24.$$

This sensitivity report generated by the Excel Solver is typical of the sensitivity analysis information provided by linear programming software packages. You will see in Appendix 4.1 that LINDO provides essentially the same report. MPL/CPLEX does also when it is requested with the Solution File dialogue box. Once again, this information obtained algebraically also can be derived from graphical analysis for this two-variable problem. (See Prob. 4.7-1.) For example, when b_2 is increased from 12 in Fig. 4.8, the originally optimal CPF solution at the intersection of two constraint boundaries $2x_2 = b_2$ and $3x_1 + 2x_2 = 18$ will remain feasible (including $x_1 \geq 0$) only for $b_2 \leq 18$.

The latter part of Chap. 6 will delve into this type of analysis more deeply.

Parametric Linear Programming

Sensitivity analysis involves changing one parameter at a time in the original model to check its effect on the optimal solution. By contrast, **parametric linear programming** (or **parametric programming** for short) involves the systematic study of how the optimal solution changes as *many* of the parameters change *simultaneously* over some range. This study can provide a very useful extension of sensitivity analysis, e.g., to check the effect of "correlated" parameters that change together due to exogenous factors such as the state of the economy. However, a more important application is the investigation of *trade-offs* in parameter values. For example, if the c_j values represent the unit profits of the respective activities, it may be possible to increase some of the c_j values at the expense of decreasing others by an appropriate shifting of personnel and equipment among activities. Similarly, if the b_i values represent the amounts of the respective resources being made available, it may be possible to increase some of the b_i values by agreeing to accept decreases in some of the others. The analysis of such possibilities is discussed and illustrated at the end of Sec. 6.7.

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[1]Since the values of the basic variables are obtained as the simultaneous solution of a system of equations (the functional constraints in augmented form), at least some of these values change if one of the right-hand sides changes. However, the adjusted values of the current set of basic variables still will satisfy the nonnegativity constraints, and so still will be feasible, as long as the new value of this right-hand side remains within its allowable range to stay feasible. If the adjusted basic solution is still feasible, it also will still be optimal. We shall elaborate further in Sec. 6.7.

In some applications, the main purpose of the study is to determine the most appropriate trade-off between two basic factors, such as *costs* and *benefits*. The usual approach is to express one of these factors in the objective function (e.g., minimize total cost) and incorporate the other into the constraints (e.g., benefits ≥ minimum acceptable level), as was done for the Nori & Leets Co. air pollution problem in Sec. 3.4. Parametric linear programming then enables systematic investigation of what happens when the initial tentative decision on the trade-off (e.g., the minimum acceptable level for the benefits) is changed by improving one factor at the expense of the other.

The algorithmic technique for parametric linear programming is a natural extension of that for sensitivity analysis, so it, too, is based on the simplex method. The procedure is described in Sec. 7.2.

4.8   COMPUTER IMPLEMENTATION

If the electronic computer had never been invented, undoubtedly you would have never heard of linear programming and the simplex method. Even though it is possible to apply the simplex method by hand to solve tiny linear programming problems, the calculations involved are just too tedious to do this on a routine basis. However, the simplex method is ideally suited for execution on a computer. It is the computer revolution that has made possible the widespread application of linear programming in recent decades.

Implementation of the Simplex Method

Computer codes for the simplex method now are widely available for essentially all modern computer systems. These codes commonly are part of a sophisticated software package for mathematical programming that includes many of the procedures described in subsequent chapters (including those used for postoptimality analysis).

These production computer codes do not closely follow either the algebraic form or the tabular form of the simplex method presented in Secs. 4.3 and 4.4. These forms can be streamlined considerably for computer implementation. Therefore, the codes use instead a *matrix form* (usually called the *revised simplex method*) that is especially well suited for the computer. This form accomplishes exactly the same things as the algebraic or tabular form, but it does this while computing and storing only the numbers that are actually needed for the current iteration; and then it carries along the essential data in a more compact form. The revised simplex method is described in Sec. 5.2.

The simplex method is used routinely to solve surprisingly large linear programming problems. For example, powerful desktop computers (especially workstations) commonly are used to solve problems with *many thousand* functional constraints and a larger number of decision variables. We now are beginning to hear reports of successfully solved problems ranging into the hundreds of thousands of functional constraints and millions of decision variables.[1] For certain *special types* of linear programming problems (such as the

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[1]Do not try this at home. Attacking such a massive problem requires an especially sophisticated linear programming system that uses the latest techniques for exploiting sparcity in the coefficient matrix as well as other special techniques (e.g., *crashing techniques* for quickly finding an advanced initial BF solution). When problems are re-solved periodically after minor updating of the data, much time often is saved by using (or modifying) the last optimal solution to provide the initial BF solution for the new run.

transportation, assignment, and minimum cost flow problems to be described later in the book), even larger problems now can be solved by *specialized* versions of the simplex method.

Several factors affect how long it will take to solve a linear programming problem by the general simplex method. The most important one is the *number of ordinary functional constraints*. In fact, computation time tends to be roughly proportional to the cube of this number, so that doubling this number may multiply the computation time by a factor of approximately 8. By contrast, the number of variables is a relatively minor factor.[1] Thus, doubling the number of variables probably will not even double the computation time. A third factor of some importance is the *density* of the table of constraint coefficients (i.e., the *proportion* of the coefficients that are *not* zero), because this affects the computation time *per iteration*. (For large problems encountered in practice, it is common for the density to be under 5 percent, or even under 1 percent, and this much "sparcity" tends to greatly accelerate the simplex method.) One common rule of thumb for the *number of iterations* is that it tends to be roughly twice the number of functional constraints.

With large linear programming problems, it is inevitable that some mistakes and faulty decisions will be made initially in formulating the model and inputting it into the computer. Therefore, as discussed in Sec. 2.4, a thorough process of testing and refining the model (*model validation*) is needed. The usual end product is not a single static model that is solved once by the simplex method. Instead, the OR team and management typically consider a long series of variations on a basic model (sometimes even thousands of variations) to examine different scenarios as part of postoptimality analysis. This entire process is greatly accelerated when it can be carried out *interactively* on a *desktop computer*. And, with the help of both mathematical programming modeling languages and improving computer technology, this now is becoming common practice.

Until the mid-1980s, linear programming problems were solved almost exclusively on *mainframe computers*. Since then, there has been an explosion in the capability of doing linear programming on desktop computers, including personal computers as well as workstations. Workstations, including some with parallel processing capabilities, now are commonly used instead of mainframe computers to solve massive linear programming models. The fastest personal computers are not lagging far behind, although solving huge models usually requires additional memory.

Linear Programming Software Featured in This Book

A considerable number of excellent software packages for linear programming and its extensions now are available to fill a variety of needs. One that is widely regarded to be a particularly powerful package for solving massive problems is **CPLEX,** a product of ILOG, Inc., located in Silicon Valley. For more than a decade, CPLEX has helped to lead the way in solving larger and larger linear programming problems. An extensive research and development effort has enabled a series of upgrades with dramatic increases in efficiency. CPLEX 6.5 released in March 1999 provided another order-of-magnitude improvement. This software package has successfully solved real linear programming problems arising in industry with as many as 2 million functional constraints and a comparable number of

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[1]This statement assumes that the revised simplex method described in Sec. 5.2 is being used.

decision variables! CPLEX 6.5 often uses the simplex method and its variants (such as the dual simplex method presented in Sec. 7.1) to solve these massive problems. In addition to the simplex method, CPLEX 6.5 also features some other powerful weapons for attacking linear programming problems. One is a lightning-fast algorithm that uses the *interior-point approach* introduced in the next section. This algorithm can solve some huge general linear programming problems that the simplex method cannot (and vice versa). Another feature is the *network simplex method* (described in Sec. 9.7) that can solve even larger special types of linear programming problems. CPLEX 6.5 also extends beyond linear programming by including state-of-the-art algorithms for *integer programming* (Chap. 12) and *quadratic programming* (Sec. 13.7).

Because it often is used to solve really large problems, CPLEX normally is used in conjunction with a mathematical programming *modeling language.* As described in Sec. 3.7, modeling languages are designed for efficiently formulating large linear programming models (and related models) in a compact way, after which a solver is called upon to solve the model. Several of the prominent modeling languages support CPLEX as a solver. ILOG also has recently introduced its own modeling language, called *OPL Studio,* that can be used with CPLEX. (A trial version of OPL Studio is available at ILOG's website, www.ilog.com.)

As we mentioned in Sec. 3.7, the student version of CPLEX is included in your OR Courseware as the solver for the MPL modeling language. This version features the simplex method for solving linear programming problems.

LINDO (short for *Linear, INteractive, and Discrete Optimizer*) is another prominent software package for linear programming and its extensions. A product of LINDO Systems based in Chicago, LINDO has an even longer history than CPLEX. Although not as powerful as CPLEX, the largest version of LINDO has solved problems with tens of thousands of functional constraints and hundreds of thousands of decision variables. Its long-time popularity is partially due to its ease of use. For relatively small (textbook-sized) problems, the model can be entered and solved in an intuitive straightforward manner, so LINDO provides a convenient tool for students to use. However, LINDO lacks some of the capabilities of modeling languages for dealing with large linear programming problems. For such problems, it may be more efficient to use the LINGO modeling language to formulate the model and then to call the solver it shares with LINDO to solve the model.

You can download the student version of LINDO from the website, www.lindo.com. Appendix 4.1 provides an introduction to how to use LINDO. The CD-ROM also includes a LINDO tutorial, as well as LINDO formulations for all the examples in this book to which it can be applied.

Spreadsheet-based solvers are becoming increasingly popular for linear programming and its extensions. Leading the way are the solvers produced by Frontline Systems for Microsoft Excel, Lotus 1-2-3, and Corel Quattro Pro. In addition to the basic solver shipped with these packages, two more powerful upgrades—*Premium Solver* and *Premium Solver Plus*—also are available. Because of the widespread use of spreadsheet packages such as Microsoft Excel today, these solvers are introducing large numbers of people to the potential of linear programming for the first time. For textbook-sized linear programming problems (and considerably larger problems as well), spreadsheets provide a convenient way to formulate and solve the model, as described in Sec. 3.6. The more powerful spreadsheet solvers can solve fairly large models with many thousand decision variables. How-

ever, when the spreadsheet grows to an unwieldy size, a good modeling language and its solver may provide a more efficient approach to formulating and solving the model.

Spreadsheets provide an excellent communication tool, especially when dealing with typical managers who are very comfortable with this format but not with the algebraic formulations of OR models. Therefore, optimization software packages and modeling languages now can commonly import and export data and results in a spreadsheet format. For example, the MPL modeling language now includes an enhancement (called the *OptiMax 2000 Component Library*) that enables the modeler to create the feel of a spreadsheet model for the user of the model while still using MPL to formulate the model very efficiently. (The student version of OptiMax 2000 is included in your OR Courseware.)

Premium Solver is one of the Excel add-ins included on the CD-ROM. You can install this add-in to obtain a much better performance than with the standard Excel Solver.

Consequently, all the software, tutorials, and examples packed on the CD-ROM are providing you with several attractive software options for linear programming.

Available Software Options for Linear Programming.

1. Demonstration examples (in OR Tutor) and interactive routines for efficiently learning the simplex method.
2. Excel and its Premium Solver for formulating and solving linear programming models in a spreadsheet format.
3. MPL/CPLEX for efficiently formulating and solving large linear programming models.
4. LINGO and its solver (shared with LINDO) for an alternative way of efficiently formulating and solving large linear programming models.
5. LINDO for formulating and solving linear programming models in a straightforward way.

Your instructor may specify which software to use. Whatever the choice, you will be gaining experience with the kind of state-of-the-art software that is used by OR professionals.

4.9 THE INTERIOR-POINT APPROACH TO SOLVING LINEAR PROGRAMMING PROBLEMS

The most dramatic new development in operations research during the 1980s was the discovery of the interior-point approach to solving linear programming problems. This discovery was made in 1984 by a young mathematician at AT&T Bell Laboratories, Narendra Karmarkar, when he successfully developed a new algorithm for linear programming with this kind of approach. Although this particular algorithm experienced only mixed success in competing with the simplex method, the key solution concept described below appeared to have great potential for solving *huge* linear programming problems beyond the reach of the simplex method. Many top researchers subsequently worked on modifying Karmarkar's algorithm to fully tap this potential. Much progress has been made (and continues to be made), and a number of powerful algorithms using the interior-point approach have been developed. Today, the more powerful software packages that are designed for solving really large linear programming problems (such as CPLEX) include at least one algorithm using the interior-point approach along with the simplex method. As research continues on these algorithms, their computer implementations continue to improve. This has spurred renewed research on the simplex method, and its computer implementations continue to improve as well (recall the dramatic advance by CPLEX 6.5

cited in the preceding section). The competition between the two approaches for supremacy in solving huge problems is continuing.

Now let us look at the key idea behind Karmarkar's algorithm and its subsequent variants that use the interior-point approach.

The Key Solution Concept

Although radically different from the simplex method, Karmarkar's algorithm does share a few of the same characteristics. It is an *iterative* algorithm. It gets started by identifying a feasible *trial solution*. At each iteration, it moves from the current trial solution to a better trial solution in the feasible region. It then continues this process until it reaches a trial solution that is (essentially) optimal.

The big difference lies in the nature of these trial solutions. For the simplex method, the trial solutions are *CPF solutions* (or BF solutions after augmenting), so all movement is along edges on the *boundary* of the feasible region. For Karmarkar's algorithm, the trial solutions are **interior points,** i.e., points *inside* the boundary of the feasible region. For this reason, Karmarkar's algorithm and its variants are referred to as **interior-point algorithms.**

To illustrate, Fig. 4.11 shows the path followed by the interior-point algorithm in your OR Courseware when it is applied to the Wyndor Glass Co. problem, starting from the

FIGURE 4.11
The curve from (1, 2) to (2, 6) shows a typical path followed by an interior-point algorithm, right through the *interior* of the feasible region for the Wyndor Glass Co. problem.
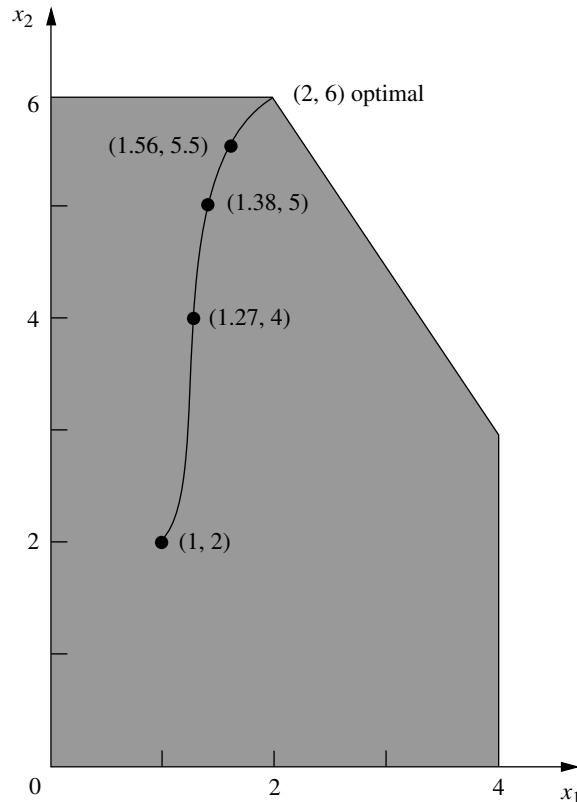
TABLE 4.18 Output of interior-point algorithm in OR Courseware for Wyndor Glass Co. problem

Iteration	x_1	x_2	Z
0	1	2	13
1	1.27298	4	23.8189
2	1.37744	5	29.1323
3	1.56291	5.5	32.1887
4	1.80268	5.71816	33.9989
5	1.92134	5.82908	34.9094
6	1.96639	5.90595	35.429
7	1.98385	5.95199	35.7115
8	1.99197	5.97594	35.8556
9	1.99599	5.98796	35.9278
10	1.99799	5.99398	35.9639
11	1.999	5.99699	35.9819
12	1.9995	5.9985	35.991
13	1.99975	5.99925	35.9955
14	1.99987	5.99962	35.9977
15	1.99994	5.99981	35.9989

initial trial solution (1, 2). Note how all the trial solutions (dots) shown on this path are inside the boundary of the feasible region as the path approaches the optimal solution (2, 6). (All the subsequent trial solutions not shown also are inside the boundary of the feasible region.) Contrast this path with the path followed by the simplex method around the boundary of the feasible region from (0, 0) to (0, 6) to (2, 6).

Table 4.18 shows the actual output from your OR Courseware for this problem.[1] (Try it yourself.) Note how the successive trial solutions keep getting closer and closer to the optimal solution, but never literally get there. However, the deviation becomes so infinitesimally small that the final trial solution can be taken to be the optimal solution for all practical purposes.

Section 7.4 presents the details of the specific interior-point algorithm that is implemented in your OR Courseware.

Comparison with the Simplex Method

One meaningful way of comparing interior-point algorithms with the simplex method is to examine their theoretical properties regarding computational complexity. Karmarkar has proved that the original version of his algorithm is a **polynomial time algorithm**; i.e., the time required to solve *any* linear programming problem can be bounded above by a polynomial function of the size of the problem. Pathological counterexamples have been constructed to demonstrate that the simplex method does not possess this property, so it is an **exponential time algorithm** (i.e., the required time can be bounded above only by an exponential function of the problem size). This difference in *worst-case performance*

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[1]The routine is called *Solve Automatically by the Interior-Point Algorithm*. The option menu provides two choices for a certain parameter of the algorithm α (defined in Sec. 7.4). The choice used here is the default value of $\alpha = 0.5$.

is noteworthy. However, it tells us nothing about their comparison in average performance on real problems, which is the more crucial issue.

The two basic factors that determine the performance of an algorithm on a real problem are the *average computer time per iteration* and the *number of iterations*. Our next comparisons concern these factors.

Interior-point algorithms are far more complicated than the simplex method. Considerably more extensive computations are required for each iteration to find the next trial solution. Therefore, the computer time per iteration for an interior-point algorithm is many times longer than that for the simplex method.

For fairly small problems, the numbers of iterations needed by an interior-point algorithm and by the simplex method tend to be somewhat comparable. For example, on a problem with 10 functional constraints, roughly 20 iterations would be typical for either kind of algorithm. Consequently, on problems of similar size, the total computer time for an interior-point algorithm will tend to be many times longer than that for the simplex method.

On the other hand, a key advantage of interior-point algorithms is that large problems do not require many more iterations than small problems. For example, a problem with 10,000 functional constraints probably will require well under 100 iterations. Even considering the very substantial computer time per iteration needed for a problem of this size, such a small number of iterations makes the problem quite tractable. By contrast, the simplex method might need 20,000 iterations and so might not finish within a reasonable amount of computer time. Therefore, interior-point algorithms often are faster than the simplex method for such huge problems.

The reason for this very large difference in the number of iterations on huge problems is the difference in the paths followed. At each iteration, the simplex method moves from the current CPF solution to an adjacent CPF solution along an edge on the boundary of the feasible region. Huge problems have an astronomical number of CPF solutions. The path from the initial CPF solution to an optimal solution may be a very circuitous one around the boundary, taking only a small step each time to the next adjacent CPF solution, so a huge number of steps may be required to reach an optimal solution. By contrast, an interior-point algorithm bypasses all this by shooting through the interior of the feasible region toward an optimal solution. Adding more functional constraints adds more constraint boundaries to the feasible region, but has little effect on the number of trial solutions needed on this path through the interior. This makes it possible for interior-point algorithms to solve problems with a huge number of functional constraints.

A final key comparison concerns the ability to perform the various kinds of postoptimality analysis described in Sec. 4.7. The simplex method and its extensions are very well suited to and are widely used for this kind of analysis. Unfortunately, the interior-point approach currently has limited capability in this area.[1] Given the great importance of postoptimality analysis, this is a crucial drawback of interior-point algorithms. However, we point out next how the simplex method can be combined with the interior-point approach to overcome this drawback.

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[1]However, research aimed at increasing this capability continues to make progress. For example, see H. J. Greenberg, "Matrix Sensitivity Analysis from an Interior Solution of a Linear Program," *INFORMS Journal on Computing,* **11:** 316–327, 1999, and its references.

The Complementary Roles of the Simplex Method and the Interior-Point Approach

Ongoing research is continuing to provide substantial improvements in computer implementations of both the simplex method (including its variants) and interior-point algorithms. Therefore, any predictions about their future roles are risky. However, we do summarize below our current assessment of their complementary roles.

The simplex method (and its variants) continues to be the standard algorithm for the routine use of linear programming. It continues to be the most efficient algorithm for problems with less than a few hundred functional constraints. It also is the most efficient for some (but not all) problems with up to several thousand functional constraints and nearly an unlimited number of decision variables, so most users are continuing to use the simplex method for such problems. However, as the number of functional constraints increases even further, it becomes increasingly likely that an interior-point approach will be the most efficient, so it often is now used instead. As the size grows into the tens of thousands of functional constraints, the interior-point approach may be the only one capable of solving the problem. However, this certainly is not always the case. As mentioned in the preceding section, the latest state-of-the-art software (CPLEX 6.5) is successfully using the simplex method and its variants to solve some truly massive problems with hundreds of thousands, or even millions of functional constraints and decision variables.

These generalizations about how the interior-point approach and the simplex method should compare for various problem sizes will not hold across the board. The specific software packages and computer equipment being used have a major impact. The comparison also is affected considerably by the *specific type* of linear programming problem being solved. As time goes on, we should learn much more about how to identify specific types which are better suited for one kind of algorithm.

One of the by-products of the emergence of the interior-point approach has been a major renewal of efforts to improve the efficiency of computer implementations of the simplex method. As we indicated, impressive progress has been made in recent years, and more lies ahead. At the same time, ongoing research and development of the interior-point approach will further increase its power, and perhaps at a faster rate than for the simplex method.

Improving computer technology, such as massive parallel processing (a huge number of computer units operating in parallel on different parts of the same problem), also will substantially increase the size of problem that either kind of algorithm can solve. However, it now appears that the interior-point approach has much greater potential to take advantage of parallel processing than the simplex method does.

As discussed earlier, a key disadvantage of the interior-point approach is its limited capability for performing postoptimality analysis. To overcome this drawback, researchers have been developing procedures for switching over to the simplex method after an interior-point algorithm has finished. Recall that the trial solutions obtained by an interior-point algorithm keep getting closer and closer to an optimal solution (the best CPF solution), but never quite get there. Therefore, a switching procedure requires identifying a CPF solution (or BF solution after augmenting) that is very close to the final trial solution.

For example, by looking at Fig. 4.11, it is easy to see that the final trial solution in Table 4.18 is very near the CPF solution (2, 6). Unfortunately, on problems with thou-

sands of decision variables (so no graph is available), identifying a nearby CPF (or BF) solution is a very challenging and time-consuming task. However, good progress has been made in developing procedures to do this.

Once this nearby BF solution has been found, the optimality test for the simplex method is applied to check whether this actually is the optimal BF solution. If it is not optimal, some iterations of the simplex method are conducted to move from this BF solution to an optimal solution. Generally, only a very few iterations (perhaps one) are needed because the interior-point algorithm has brought us so close to an optimal solution. Therefore, these iterations should be done quite quickly, even on problems that are too huge to be solved from scratch. After an optimal solution is actually reached, the simplex method and its variants are applied to help perform postoptimality analysis.

Because of the difficulties involved in applying a switching procedure (including the extra computer time), some practitioners prefer to just use the simplex method from the outset. This makes good sense when you only occasionally encounter problems that are large enough for an interior-point algorithm to be modestly faster (before switching) than the simplex method. This modest speed-up would not justify both the extra computer time for a switching procedure and the high cost of acquiring (and learning to use) a software package based on the interior-point approach. However, for organizations which frequently must deal with extremely large linear programming problems, acquiring a state-of-the-art software package of this kind (including a switching procedure) probably is worthwhile. For sufficiently huge problems, the only available way of solving them may be with such a package.

Applications of huge linear programming models sometimes lead to savings of millions of dollars. Just one such application can pay many times over for a state-of-the-art software package based on the interior-point approach plus switching over to the simplex method at the end.

4.10 CONCLUSIONS

The simplex method is an efficient and reliable algorithm for solving linear programming problems. It also provides the basis for performing the various parts of postoptimality analysis very efficiently.

Although it has a useful geometric interpretation, the simplex method is an algebraic procedure. At each iteration, it moves from the current BF solution to a better, adjacent BF solution by choosing both an entering basic variable and a leaving basic variable and then using Gaussian elimination to solve a system of linear equations. When the current solution has no adjacent BF solution that is better, the current solution is optimal and the algorithm stops.

We presented the full algebraic form of the simplex method to convey its logic, and then we streamlined the method to a more convenient tabular form. To set up for starting the simplex method, it is sometimes necessary to use artificial variables to obtain an initial BF solution for an artificial problem. If so, either the Big M method or the two-phase method is used to ensure that the simplex method obtains an optimal solution for the real problem.

Computer implementations of the simplex method and its variants have become so powerful that they now are frequently used to solve linear programming problems with

many thousand functional constraints and decision variables, and occasionally vastly larger problems. Interior-point algorithms also provide a powerful tool for solving very large problems.

APPENDIX 4.1   AN INTRODUCTION TO USING LINDO

The LINDO software is designed to be easy to learn and to use, especially for small problems of the size you will encounter in this book. In addition to linear programming, it also can be used to solve both integer programming problems (Chap. 12) and quadratic programming problems (Sec. 13.7). Our focus in this appendix is on its use for linear programming.

LINDO allows you to enter a model in a straightforward algebraic way. For example, here is a nice way of entering the LINDO model for the Wyndor Glass Co. example introduced in Sec. 3.1.

```
! Wyndor Glass Co. Problem. LINDO model
! X1 = batches of product 1 per week
! X2 = batches of product 2 per week

! Profit, in 1000 of dollars

MAX Profit) 3 X1 + 5 X2

Subject to

! Production time
Plant1) X1 <= 4
Plant2) 2 X2 <= 12
Plant3) 3 X1 + 2 X2 <= 18
END
```

In addition to the basic model, this formulation includes several clarifying comments, where each comment is indicated by starting with an exclamation point. Thus, the first three lines give the title and the definitions of the decision variables. The decision variables can be either lowercase or uppercase, but uppercase usually is used so the variables won't be dwarfed by the following "subscripts." Another option is to use a suggestive word (or abbreviation of a word), such as the name of the product being produced, to represent the decision variable throughout the model, provided the word does not exceed eight letters.

The fifth line of the LINDO formulation indicates that the objective of the model is to maximize the objective function, $3x_1 + 5x_2$. The word Profit followed by a parenthesis clarifies that this quantity being maximized is profit. The comment on the fourth line further clarifies that the objective function is expressed in units of thousands of dollars. The number 1000 in this comment does not have the usual comma in front of the last three digits because LINDO does not accept commas. (It also does not accept parentheses in algebraic expressions.)

The comment on the seventh line points out that the following constraints are on the production times being used. The next three lines start by giving a name (followed by a parenthesis) for each of the functional constraints. These constraints are written in the usual way except for the inequality signs. Because many keyboards do not include $\leq$ and $\geq$ signs, LINDO interprets either $<$ or $<=$ as $\leq$ and either $>$ or $>=$ as $\geq$. (On systems that include $\leq$ and $\geq$ signs, LINDO will not recognize them.)

The end of the constraints is signified by the word END. No nonnegativity constraints are stated because LINDO automatically assumes that all the variables have these constraints. If, say, x_1 had not had a nonnegativity constraint, this would have to be indicated by typing FREE X1 on the next line below END.

To solve this model in the Windows version of LINDO, either select the Solve command from the Solve menu or press the Solve button on the toolbar. On a platform other than Windows, simply type GO followed by a return at the colon prompt. Figure A4.1 shows the resulting solution report delivered by LINDO.

Both the top line and bottom line in this figure indicate that an optimal solution was found at iteration 2 of the simplex method. Next comes the value of the objective function for this solution. Below this, we have the values of x_1 and x_2 for the optimal solution.

The column to the right of these values gives the **reduced costs.** We have not discussed reduced costs in this chapter because the information they provide can also be gleaned from the *allowable range to stay optimal* for the coefficients in the objective function, and these allowable ranges also are readily available (as you will see in the next figure). When the variable is a *basic variable* in the optimal solution (as for both variables in the Wyndor problem), its reduced cost automatically is 0. When the variable is a *nonbasic variable,* its reduced cost provides some interesting information. This variable is 0 because its coefficient in the objective function is too small (when maximizing the objective function) or too large (when minimizing) to justify undertaking the activity represented by the variable. The reduced cost indicates how much this coefficient can be *increased* (when maximizing) or *decreased* (when minimizing) before the optimal solution would change and this variable would become a *basic variable.* However, recall that this same information already is available from the allowable range to stay optimal for the coefficient of this variable in the objective function. The reduced cost (for a nonbasic variable) is just the *allowable increase* (when maximizing) from the current value of this coefficient to remain within its allowable range to stay optimal or the *allowable decrease* (when minimizing).

Below the variable values and reduced costs in Fig. A4.1, we next have information about the three functional constraints. The *Slack or Surplus* column gives the difference between the two sides of each constraint. The *Dual Prices* column gives, by another name, the *shadow prices* discussed in Sec. 4.7 for these constraints.[1] (This alternate name comes from the fact found in Sec. 6.1 that these shadow prices are just the optimal values of the *dual* variables introduced in Chap. 6.)

When LINDO provides you with this solution report, it also asks you whether you want to do range (sensitivity) analysis. Answering yes (by pressing the Y key) provides you with the additional *range report* shown in Fig. A4.2. This report is identical to the last three columns of the

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[1]However, beware that LINDO uses a different sign convention from the common one adopted here (see the second footnote for the definition of shadow price in Sec. 4.7), so that for minimization problems, its shadow prices (dual prices) are the negative of ours.

FIGURE A4.1
The solution report provided by LINDO for the Wyndor Glass Co. problem.

```
LP OPTIMUM FOUND AT STEP 2

   OBJECTIVE FUNCTION VALUE

   Profit)  36.00000

VARIABLE       VALUE        REDUCED COST
    X1       2.000000         .000000
    X2       6.000000         .000000

   ROW      SLACK OR SURPLUS     DUAL PRICES
Plant1)         2.000000            .000000
Plant2)          .000000           1.500000
Plant3)          .000000           1.000000

NO. ITERATIONS= 2
```

RANGES IN WHICH THE BASIS IS UNCHANGED:

```
                OBJ COEFFICIENT RANGES
VARIABLE      CURRENT      ALLOWABLE     ALLOWABLE
                COEF        INCREASE      DECREASE
     X1      3.000000      4.500000      3.000000
     X2      5.000000      INFINITY      3.000000

                RIGHTHAND  SIDE RANGES
  ROW         CURRENT      ALLOWABLE     ALLOWABLE
                RHS         INCREASE      DECREASE
Plant1      4.000000      INFINITY      2.000000
Plant2     12.000000      6.000000      6.000000
Plant3     18.000000      6.000000      6.000000
```

FIGURE A4.2
The range report provided
by LINDO for the Wyndor
Glass Co. problem.

tables in the *sensitivity report* generated by the Excel Solver, as shown earlier in Fig. 4.10. Thus, as already discussed in Sec. 4.7, the first two rows of this range report indicate that the *allowable range to stay optimal* for each coefficient in the objective function (assuming no other change in the model) is

$$0 \leq c_1 \leq 7.5$$
$$2 \leq c_2$$

Similarly, the last three rows indicate that the *allowable range to stay feasible* for each right-hand side (assuming no other change in the model) is

$$2 \leq b_1$$
$$6 \leq b_2 \leq 18$$
$$12 \leq b_3 \leq 24$$

To print your results with the Windows version of LINDO, you simply need to use the Print command to send the contents of the active window to the printer. If you are running LINDO on a platform other than Windows, you can use the DIVERT command (followed by the file name) to send screen output to a file, which can then print from either the operating system or a word processing package.

These are the basics for getting started with LINDO. The LINDO tutorial on the CD-ROM also provides some additional details. The LINGO/LINDO files on the CD-ROM for various chapters show the LINDO formulations for numerous examples. In addition, LINDO includes a Help menu to provide guidance. These resources should enable you to apply LINDO to any linear programming problem you will encounter in this book. (We will discuss applications to other problem types in Chaps. 12 and 13.) For more advanced applications, the LINDO *User's Manual* (Selected Reference 4 for this chapter) might be needed.

SELECTED REFERENCES

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3. Dantzig, G.B., and M.N. Thapa: *Linear Programming 1: Introduction,* Springer, New York, 1997.

4. *LINDO User's Manual,* LINDO Systems, Inc., Chicago, IL, e-mail: info@lindo.com, 1999.

5. Vanderbei, R. J.: *Linear Programming: Foundations and Extensions,* Kluwer Academic Publishers, Boston, MA, 1996.

LEARNING AIDS FOR THIS CHAPTER IN YOUR OR COURSEWARE

Demonstration Examples in OR Tutor:

Interpretation of the Slack Variables
Simplex Method—Algebraic Form
Simplex Method—Tabular Form

Interactive Routines:

Enter or Revise a General Linear Programming Model
Set Up for the Simplex Method—Interactive Only
Solve Interactively by the Simplex Method

An Automatic Routine:

Solve Automatically by the Interior-Point Algorithm

An Excel Add-In:

Premium Solver

Files (Chapter 3) for Solving the Wyndor and Radiation Therapy Examples:

Excel File
LINGO/LINDO File
MPL/CPLEX File

See **Appendix 1** for documentation of the software.

PROBLEMS

The symbols to the left of some of the problems (or their parts) have the following meaning:

D: The corresponding demonstration example listed above may be helpful.

I: We suggest that you use the corresponding interactive routine listed above (the printout records your work).

C: Use the computer with any of the software options available to you (or as instructed by your instructor) to solve the problem automatically. (See Sec. 4.8 for a listing of the options featured in this book and on the CD-ROM.)

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

4.1-1. Consider the following problem.

$$\text{Maximize} \quad Z = x_1 + 2x_2,$$

subject to

$$x_1 \qquad \leq 2$$
$$x_2 \leq 2$$
$$x_1 + x_2 \leq 3$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

(a) Plot the feasible region and circle all the CPF solutions.

(b) For each CPF solution, identify the pair of constraint boundary equations that it satisfies.

(c) For each CPF solution, use this pair of constraint boundary equations to solve algebraically for the values of x_1 and x_2 at the corner point.

(d) For each CPF solution, identify its adjacent CPF solutions.

(e) For each pair of adjacent CPF solutions, identify the constraint boundary they share by giving its equation.

4.1-2. Consider the following problem.

$$\text{Maximize} \quad Z = 3x_1 + 2x_2,$$

subject to

$$2x_1 + x_2 \leq 6$$
$$x_1 + 2x_2 \leq 6$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

(a) Use the graphical method to solve this problem. Circle all the corner points on the graph.

(b) For each CPF solution, identify the pair of constraint boundary equations it satisfies.

(c) For each CPF solution, identify its adjacent CPF solutions.

(d) Calculate Z for each CPF solution. Use this information to identify an optimal solution.

(e) Describe graphically what the simplex method does step by step to solve the problem.

4.1-3. A certain linear programming model involving two activities has the feasible region shown below.
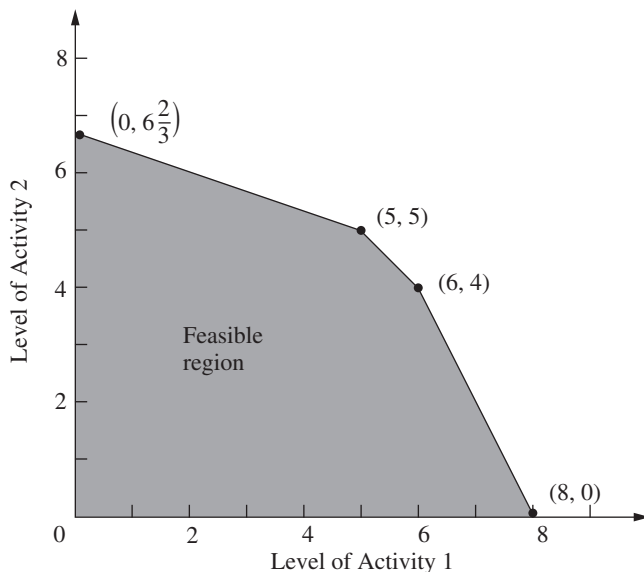
The objective is to maximize the total profit from the two activities. The unit profit for activity 1 is $1,000 and the unit profit for activity 2 is $2,000.

(a) Calculate the total profit for each CPF solution. Use this information to find an optimal solution.

(b) Use the solution concepts of the simplex method given in Sec. 4.1 to identify the sequence of CPF solutions that would be examined by the simplex method to reach an optimal solution.

4.1-4.* Consider the linear programming model (given in the back of the book) that was formulated for Prob. 3.2-3.

(a) Use graphical analysis to identify all the *corner-point solutions* for this model. Label each as either feasible or infeasible.

(b) Calculate the value of the objective function for each of the CPF solutions. Use this information to identify an optimal solution.

(c) Use the solution concepts of the simplex method given in Sec. 4.1 to identify which sequence of CPF solutions might be examined by the simplex method to reach an optimal solution. (*Hint*: There are *two* alternative sequences to be identified for this particular model.)

4.1-5. Repeat Prob. 4.1-4 for the following problem.

$$\text{Maximize} \quad Z = x_1 + 2x_2,$$

subject to

$$x_1 + 3x_2 \leq 8$$
$$x_1 + x_2 \leq 4$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

4.1-6. Repeat Prob. 4.1-4 for the following problem.

$$\text{Maximize} \quad Z = 3x_1 + 2x_2,$$

subject to

$$x_1 \quad\quad \leq 4$$
$$x_1 + 3x_2 \leq 15$$
$$2x_1 + x_2 \leq 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

4.1-7. Describe graphically what the simplex method does step by step to solve the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 3x_2,$$

subject to

$$-3x_1 + x_2 \leq 1$$
$$4x_1 + 2x_2 \leq 20$$

$$4x_1 - x_2 \leq 10$$
$$-x_1 + 2x_2 \leq 5$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

4.1-8. Describe graphically what the simplex method does step by step to solve the following problem.

Minimize $Z = 5x_1 + 7x_2,$

subject to

$$2x_1 + 3x_2 \geq 42$$
$$3x_1 + 4x_2 \geq 60$$
$$x_1 + x_2 \geq 18$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

4.1-9. Label each of the following statements about linear programming problems as true or false, and then justify your answer.
(a) For minimization problems, if the objective function evaluated at a CPF solution is no larger than its value at every adjacent CPF solution, then that solution is optimal.
(b) Only CPF solutions can be optimal, so the number of optimal solutions cannot exceed the number of CPF solutions.
(c) If multiple optimal solutions exist, then an optimal CPF solution may have an adjacent CPF solution that also is optimal (the same value of Z).

4.1-10. The following statements give inaccurate paraphrases of the six solution concepts presented in Sec. 4.1. In each case, explain what is wrong with the statement.
(a) The best CPF solution always is an optimal solution.
(b) An iteration of the simplex method checks whether the current CPF solution is optimal and, if not, moves to a new CPF solution.
(c) Although any CPF solution can be chosen to be the initial CPF solution, the simplex method always chooses the origin.
(d) When the simplex method is ready to choose a new CPF solution to move to from the current CPF solution, it only considers adjacent CPF solutions because one of them is likely to be an optimal solution.
(e) To choose the new CPF solution to move to from the current CPF solution, the simplex method identifies all the adjacent CPF solutions and determines which one gives the largest rate of improvement in the value of the objective function.

4.2-1. Reconsider the model in Prob. 4.1-4.
(a) Introduce slack variables in order to write the functional constraints in augmented form.

(b) For each CPF solution, identify the corresponding BF solution by calculating the values of the slack variables. For each BF solution, use the values of the variables to identify the nonbasic variables and the basic variables.
(c) For each BF solution, demonstrate (by plugging in the solution) that, after the nonbasic variables are set equal to zero, this BF solution also is the simultaneous solution of the system of equations obtained in part (a).

4.2-2. Reconsider the model in Prob. 4.1-5. Follow the instructions of Prob. 4.2-1 for parts (a), (b), and (c).
(d) Repeat part (b) for the corner-point infeasible solutions and the corresponding basic infeasible solutions.
(e) Repeat part (c) for the basic infeasible solutions.

4.2-3. Follow the instructions of Prob. 4.2-1 for the model in Prob. 4.1-6.

D,I **4.3-1.** Work through the simplex method (in algebraic form) step by step to solve the model in Prob. 4.1-4.

4.3-2. Reconsider the model in Prob. 4.1-5.
(a) Work through the simplex method (in algebraic form) *by hand* to solve this model.
D,I **(b)** Repeat part (a) with the corresponding interactive routine in your OR Tutor.
C **(c)** Verify the optimal solution you obtained by using a software package based on the simplex method.

4.3-3. Follow the instructions of Prob. 4.3-2 for the model in Prob. 4.1-6.

D,I **4.3-4.*** Work through the simplex method (in algebraic form) step by step to solve the following problem.

Maximize $Z = 4x_1 + 3x_2 + 6x_3,$

subject to

$$3x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 3x_3 \leq 40$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **4.3-5.** Work through the simplex method (in algebraic form) step by step to solve the following problem.

Maximize $Z = x_1 + 2x_2 + 4x_3,$

subject to

$$3x_1 + x_2 + 5x_3 \leq 10$$
$$x_1 + 4x_2 + x_3 \leq 8$$
$$2x_1 + 2x_3 \leq 7$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **4.3-6.** Work through the simplex method (in algebraic form) step by step to solve the following problem.

Maximize $\quad Z = x_1 + 2x_2 + 2x_3,$

subject to

$$5x_1 + 2x_2 + 3x_3 \leq 15$$
$$x_1 + 4x_2 + 2x_3 \leq 12$$
$$2x_1 \qquad + \ x_3 \leq \ 8$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

4.3-7. Consider the following problem.

Maximize $\quad Z = 5x_1 + 3x_2 + 4x_3,$

subject to

$$2x_1 + x_2 + \ x_3 \leq 20$$
$$3x_1 + x_2 + 2x_3 \leq 30$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

You are given the information that the *nonzero* variables in the optimal solution are x_2 and x_3.

(a) Describe how you can use this information to adapt the simplex method to solve this problem in the minimum possible number of iterations (when you start from the usual initial BF solution). Do *not* actually perform any iterations.

(b) Use the procedure developed in part (a) to solve this problem by hand. (Do *not* use your OR Courseware.)

4.3-8. Consider the following problem.

Maximize $\quad Z = 2x_1 + 4x_2 + 3x_3,$

subject to

$$x_1 + 3x_2 + 2x_3 \leq 30$$
$$x_1 + \ x_2 + \ x_3 \leq 24$$
$$3x_1 + 5x_2 + 3x_3 \leq 60$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

You are given the information that $x_1 > 0$, $x_2 = 0$, and $x_3 > 0$ in the optimal solution.

(a) Describe how you can use this information to adapt the simplex method to solve this problem in the minimum possible number of iterations (when you start from the usual initial BF solution). Do *not* actually perform any iterations.

(b) Use the procedure developed in part (a) to solve this problem by hand. (Do *not* use your OR Courseware.)

4.3-9. Label each of the following statements as true or false, and then justify your answer by referring to specific statements (with page citations) in the chapter.

(a) The simplex method's rule for choosing the entering basic variable is used because it always leads to the *best* adjacent BF solution (largest Z).

(b) The simplex method's minimum ratio rule for choosing the leaving basic variable is used because making another choice with a larger ratio would yield a basic solution that is not feasible.

(c) When the simplex method solves for the next BF solution, elementary algebraic operations are used to eliminate each nonbasic variable from all but one equation (*its* equation) and to give it a coefficient of $+1$ in that one equation.

D,I **4.4-1.** Repeat Prob. 4.3-1, using the tabular form of the simplex method.

D,I,C **4.4-2.** Repeat Prob. 4.3-2, using the tabular form of the simplex method.

D,I,C **4.4-3.** Repeat Prob. 4.3-3, using the tabular form of the simplex method.

4.4-4. Consider the following problem.

Maximize $\quad Z = 2x_1 + x_2,$

subject to

$$x_1 + x_2 \leq \ 40$$
$$4x_1 + x_2 \leq 100$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

(a) Solve this problem graphically in a freehand manner. Also identify all the CPF solutions.

(b) Now repeat part (a) when using a ruler to draw the graph carefully.

D (c) Use hand calculations to solve this problem by the simplex method in algebraic form.

D,I (d) Now use your OR Courseware to solve this problem interactively by the simplex method in algebraic form.

D (e) Use hand calculations to solve this problem by the simplex method in tabular form.

D,I (f) Now use your OR Courseware to solve this problem interactively by the simplex method in tabular form.

C (g) Use a software package based on the simplex method to solve the problem.

4.4-5. Repeat Prob. 4.4-4 for the following problem.

Maximize $Z = 2x_1 + 3x_2,$

subject to

$$x_1 + 2x_2 \leq 30$$
$$x_1 + \ x_2 \leq 20$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

4.4-6. Consider the following problem.

Maximize $Z = 2x_1 + 4x_2 + 3x_3,$

subject to

$$3x_1 + 4x_2 + 2x_3 \leq 60$$
$$2x_1 + \ x_2 + 2x_3 \leq 40$$
$$x_1 + 3x_2 + 2x_3 \leq 80$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **(a)** Work through the simplex method step by step in algebraic form.

D,I **(b)** Work through the simplex method step by step in tabular form.

C **(c)** Use a software package based on the simplex method to solve the problem.

4.4-7. Consider the following problem.

Maximize $Z = 3x_1 + 5x_2 + 6x_3,$

subject to

$$2x_1 + \ x_2 + \ x_3 \leq 4$$
$$x_1 + 2x_2 + \ x_3 \leq 4$$
$$x_1 + \ x_2 + 2x_3 \leq 4$$
$$x_1 + \ x_2 + \ x_3 \leq 3$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **(a)** Work through the simplex method step by step in algebraic form.

D,I **(b)** Work through the simplex method in tabular form.

C **(c)** Use a computer package based on the simplex method to solve the problem.

4.4-8. Consider the following problem.

Maximize $Z = 2x_1 - x_2 + x_3,$

subject to

$$x_1 - x_2 + 3x_3 \leq \ 4$$
$$2x_1 + x_2 \qquad \leq 10$$
$$x_1 - x_2 - \ x_3 \leq \ 7$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **(a)** Work through the simplex method step by step in algebraic form to solve this problem.

D,I **(b)** Work through the simplex method step by step in tabular form to solve the problem.

C **(c)** Use a computer package based on the simplex method to solve the problem.

D,I **4.4-9.** Work through the simplex method step by step (in tabular form) to solve the following problem.

Maximize $Z = 2x_1 - x_2 + x_3,$

subject to

$$3x_1 + x_2 + \ x_3 \leq 6$$
$$x_1 - x_2 + 2x_3 \leq 1$$
$$x_1 + x_2 - \ x_3 \leq 2$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **4.4-10.** Work through the simplex method step by step to solve the following problem.

Maximize $Z = -x_1 + x_2 + 2x_3,$

subject to

$$x_1 + 2x_2 - \ x_3 \leq 20$$
$$-2x_1 + 4x_2 + 2x_3 \leq 60$$
$$2x_1 + 3x_2 + \ x_3 \leq 50$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

4.5-1. Consider the following statements about linear programming and the simplex method. Label each statement as true or false, and then justify your answer.

(a) In a particular iteration of the simplex method, if there is a tie for which variable should be the leaving basic variable, then the next BF solution must have at least one basic variable equal to zero.

(b) If there is no leaving basic variable at some iteration, then the problem has no feasible solutions.

(c) If at least one of the basic variables has a coefficient of zero in row 0 of the final tableau, then the problem has multiple optimal solutions.

(d) If the problem has multiple optimal solutions, then the problem must have a bounded feasible region.

4.5-2. Suppose that the following constraints have been provided for a linear programming model with decision variables x_1 and x_2.

$$-x_1 + 3x_2 \leq 30$$
$$-3x_1 + x_2 \leq 30$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

(a) Demonstrate graphically that the feasible region is unbounded.
(b) If the objective is to maximize $Z = -x_1 + x_2$, does the model have an optimal solution? If so, find it. If not, explain why not.
(c) Repeat part (b) when the objective is to maximize $Z = x_1 - x_2$.
(d) For objective functions where this model has no optimal solution, does this mean that there are no good solutions according to the model? Explain. What probably went wrong when formulating the model?
D,I **(e)** Select an objective function for which this model has no optimal solution. Then work through the simplex method step by step to demonstrate that Z is unbounded.
C **(f)** For the objective function selected in part (e), use a software package based on the simplex method to determine that Z is unbounded.

4.5-3. Follow the instructions of Prob. 4.5-2 when the constraints are the following:

$$2x_1 - x_2 \leq 20$$
$$x_1 - 2x_2 \leq 20$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

D,I **4.5-4.** Consider the following problem.

Maximize $\qquad Z = 5x_1 + x_2 + 3x_3 + 4x_4,$

subject to

$$x_1 - 2x_2 + 4x_3 + 3x_4 \leq 20$$
$$-4x_1 + 6x_2 + 5x_3 - 4x_4 \leq 40$$
$$2x_1 - 3x_2 + 3x_3 + 8x_4 \leq 50$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0, \qquad x_4 \geq 0.$$

Work through the simplex method step by step to demonstrate that Z is unbounded.

4.5-5. A basic property of any linear programming problem with a bounded feasible region is that every feasible solution can be expressed as a convex combination of the CPF solutions (perhaps in more than one way). Similarly, for the augmented form of the problem, every feasible solution can be expressed as a convex combination of the BF solutions.

(a) Show that *any* convex combination of *any* set of feasible solutions must be a feasible solution (so that any convex combination of CPF solutions must be feasible).
(b) Use the result quoted in part (a) to show that any convex combination of BF solutions must be a feasible solution.

4.5-6. Using the facts given in Prob. 4.5-5, show that the following statements must be true for any linear programming problem that has a bounded feasible region and multiple optimal solutions:
(a) Every convex combination of the optimal BF solutions must be optimal.
(b) No other feasible solution can be optimal.

4.5-7. Consider a two-variable linear programming problem whose CPF solutions are (0, 0), (6, 0), (6, 3), (3, 3), and (0, 2). (See Prob. 3.2-2 for a graph of the feasible region.)
(a) Use the graph of the feasible region to identify all the constraints for the model.
(b) For each pair of adjacent CPF solutions, give an example of an objective function such that all the points on the line segment between these two corner points are multiple optimal solutions.
(c) Now suppose that the objective function is $Z = -x_1 + 2x_2$. Use the graphical method to find all the optimal solutions.
D,I **(d)** For the objective function in part (c), work through the simplex method step by step to find all the optimal BF solutions. Then write an algebraic expression that identifies all the optimal solutions.

D,I **4.5-8.** Consider the following problem.

Maximize $\qquad Z = x_1 + x_2 + x_3 + x_4,$

subject to

$$x_1 + x_2 \leq 3$$
$$x_3 + x_4 \leq 2$$

and

$$x_j \geq 0, \qquad \text{for } j = 1, 2, 3, 4.$$

Work through the simplex method step by step to find *all* the optimal BF solutions.

4.6-1.* Consider the following problem.

Maximize $\qquad Z = 2x_1 + 3x_2,$

subject to

$$x_1 + 2x_2 \leq 4$$
$$x_1 + x_2 = 3$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

(a) Solve this problem graphically.

(b) Using the Big M method, construct the complete first simplex tableau for the simplex method and identify the corresponding initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.

I **(c)** Continue from part (b) to work through the simplex method step by step to solve the problem.

4.6-2. Consider the following problem.

Maximize $Z = 4x_1 + 2x_2 + 3x_3 + 5x_4,$

subject to

$$2x_1 + 3x_2 + 4x_3 + 2x_4 = 300$$
$$8x_1 + x_2 + x_3 + 5x_4 = 300$$

and

$$x_j \geq 0, \quad \text{for } j = 1, 2, 3, 4.$$

(a) Using the Big M method, construct the complete first simplex tableau for the simplex method and identify the corresponding initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.

I **(b)** Work through the simplex method step by step to solve the problem.

(c) Using the two-phase method, construct the complete first simplex tableau for phase 1 and identify the corresponding initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.

I **(d)** Work through phase 1 step by step.

(e) Construct the complete first simplex tableau for phase 2.

I **(f)** Work through phase 2 step by step to solve the problem.

(g) Compare the sequence of BF solutions obtained in part (b) with that in parts (d) and (f). Which of these solutions are feasible only for the artificial problem obtained by introducing artificial variables and which are actually feasible for the real problem?

C **(h)** Use a software package based on the simplex method to solve the problem.

4.6-3. Consider the following problem.

Minimize $Z = 3x_1 + 2x_2,$

subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

(a) Solve this problem graphically.

(b) Using the Big M method, construct the complete first simplex tableau for the simplex method and identify the corresponding

initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.

I **(c)** Work through the simplex method step by step to solve the problem.

4.6-4.* Consider the following problem.

Minimize $Z = 2x_1 + 3x_2 + x_3,$

subject to

$$x_1 + 4x_2 + 2x_3 \geq 8$$
$$3x_1 + 2x_2 \geq 6$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

(a) Reformulate this problem to fit our standard form for a linear programming model presented in Sec. 3.2.

I **(b)** Using the Big M method, work through the simplex method step by step to solve the problem.

I **(c)** Using the two-phase method, work through the simplex method step by step to solve the problem.

(d) Compare the sequence of BF solutions obtained in parts (b) and (c). Which of these solutions are feasible only for the artificial problem obtained by introducing artificial variables and which are actually feasible for the real problem?

C **(e)** Use a software package based on the simplex method to solve the problem.

4.6-5. For the Big M method, explain why the simplex method never would choose an artificial variable to be an entering basic variable once all the artificial variables are nonbasic.

4.6-6. Consider the following problem.

Maximize $Z = 90x_1 + 70x_2,$

subject to

$$2x_1 + x_2 \leq 2$$
$$x_1 - x_2 \geq 2$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

(a) Demonstrate graphically that this problem has no feasible solutions.

C **(b)** Use a computer package based on the simplex method to determine that the problem has no feasible solutions.

I **(c)** Using the Big M method, work through the simplex method step by step to demonstrate that the problem has no feasible solutions.

I **(d)** Repeat part (c) when using phase 1 of the two-phase method.

4.6-7. Follow the instructions of Prob. 4.6-6 for the following problem.

Minimize $Z = 5{,}000x_1 + 7{,}000x_2$,

subject to

$$-2x_1 + x_2 \geq 1$$
$$x_1 - 2x_2 \geq 1$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

4.6-8. Consider the following problem.

Maximize $Z = 2x_1 + 5x_2 + 3x_3$,

subject to

$$x_1 - 2x_2 + x_3 \geq 20$$
$$2x_1 + 4x_2 + x_3 = 50$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

(a) Using the Big M method, construct the complete first simplex tableau for the simplex method and identify the corresponding initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.
I (b) Work through the simplex method step by step to solve the problem.
I (c) Using the two-phase method, construct the complete first simplex tableau for phase 1 and identify the corresponding initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.
I (d) Work through phase 1 step by step.
(e) Construct the complete first simplex tableau for phase 2.
I (f) Work through phase 2 step by step to solve the problem.
(g) Compare the sequence of BF solutions obtained in part (b) with that in parts (d) and (f). Which of these solutions are feasible only for the artificial problem obtained by introducing artificial variables and which are actually feasible for the real problem?
C (h) Use a software package based on the simplex method to solve the problem.

4.6-9. Consider the following problem.

Minimize $Z = 2x_1 + x_2 + 3x_3$,

subject to

$$5x_1 + 2x_2 + 7x_3 = 420$$
$$3x_1 + 2x_2 + 5x_3 \geq 280$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

I (a) Using the two-phase method, work through phase 1 step by step.
C (b) Use a software package based on the simplex method to formulate and solve the phase 1 problem.
I (c) Work through phase 2 step by step to solve the original problem.
C (d) Use a computer code based on the simplex method to solve the original problem.

4.6-10.* Consider the following problem.

Minimize $Z = 3x_1 + 2x_2 + 4x_3$,

subject to

$$2x_1 + x_2 + 3x_3 = 60$$
$$3x_1 + 3x_2 + 5x_3 \geq 120$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

I (a) Using the Big M method, work through the simplex method step by step to solve the problem.
I (b) Using the two-phase method, work through the simplex method step by step to solve the problem.
(c) Compare the sequence of BF solutions obtained in parts (a) and (b). Which of these solutions are feasible only for the artificial problem obtained by introducing artificial variables and which are actually feasible for the real problem?
C (d) Use a software package based on the simplex method to solve the problem.

4.6-11. Follow the instructions of Prob. 4.6-10 for the following problem.

Minimize $Z = 3x_1 + 2x_2 + 7x_3$,

subject to

$$-x_1 + x_2 \qquad = 10$$
$$2x_1 - x_2 + x_3 \geq 10$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

4.6-12. Follow the instructions of Prob. 4.6-10 for the following problem.

Minimize $Z = 3x_1 + 2x_2 + x_3$,

subject to

$$x_1 + x_2 \qquad = 7$$
$$3x_1 + x_2 + x_3 \geq 10$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

4.6-13. Label each of the following statements as true or false, and then justify your answer.

(a) When a linear programming model has an equality constraint, an artificial variable is introduced into this constraint in order to start the simplex method with an obvious initial basic solution that is feasible for the original model.

(b) When an artificial problem is created by introducing artificial variables and using the Big M method, if all artificial variables in an optimal solution for the artificial problem are equal to zero, then the real problem has no feasible solutions.

(c) The two-phase method is commonly used in practice because it usually requires fewer iterations to reach an optimal solution than the Big M method does.

4.6-14. Consider the following problem.

$$\text{Maximize} \quad Z = x_1 + 4x_2 + 2x_3,$$

subject to

$$4x_1 - x_2 + x_3 \le 5$$
$$-x_1 - x_2 + 2x_3 \le 10$$

and

$$x_2 \ge 0, \qquad x_3 \ge 0$$

(no nonnegativity constraint for x_1).

(a) Reformulate this problem so all variables have nonnegativity constraints.

D,I (b) Work through the simplex method step by step to solve the problem.

C (c) Use a software package based on the simplex method to solve the problem.

4.6-15.* Consider the following problem.

$$\text{Maximize} \quad Z = -x_1 + 4x_2,$$

subject to

$$-3x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 4$$
$$x_2 \ge -3$$

(no lower bound constraint for x_1).

(a) Solve this problem graphically.

(b) Reformulate this problem so that it has only two functional constraints and all variables have nonnegativity constraints.

D,I (c) Work through the simplex method step by step to solve the problem.

4.6-16. Consider the following problem.

$$\text{Maximize} \quad Z = -x_1 + 2x_2 + x_3,$$

subject to

$$3x_2 + x_3 \le 120$$
$$x_1 - x_2 - 4x_3 \le 80$$
$$-3x_1 + x_2 + 2x_3 \le 100$$

(no nonnegativity constraints).

(a) Reformulate this problem so that all variables have nonnegativity constraints.

D,I (b) Work through the simplex method step by step to solve the problem.

C (c) Use a computer package based on the simplex method to solve the problem.

4.6-17. This chapter has described the simplex method as applied to linear programming problems where the objective function is to be maximized. Section 4.6 then described how to convert a minimization problem to an equivalent maximization problem for applying the simplex method. Another option with minimization problems is to make a few modifications in the instructions for the simplex method given in the chapter in order to apply the algorithm directly.

(a) Describe what these modifications would need to be.

(b) Using the Big M method, apply the modified algorithm developed in part (a) to solve the following problem directly by hand. (Do not use your OR Courseware.)

$$\text{Minimize} \quad Z = 3x_1 + 8x_2 + 5x_3,$$

subject to

$$3x_2 + 4x_3 \ge 70$$
$$3x_1 + 5x_2 + 2x_3 \ge 70$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

4.6-18. Consider the following problem.

$$\text{Maximize} \quad Z = -2x_1 + x_2 - 4x_3 + 3x_4,$$

subject to

$$x_1 + x_2 + 3x_3 + 2x_4 \le 4$$
$$x_1 \qquad - x_3 + x_4 \ge -1$$
$$2x_1 + x_2 \qquad \le 2$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 2$$

and

$$x_2 \ge 0, \qquad x_3 \ge 0, \qquad x_4 \ge 0$$

(no nonnegativity constraint for x_1).

(a) Reformulate this problem to fit our standard form for a linear programming model presented in Sec. 3.2.

(b) Using the Big M method, construct the complete first simplex tableau for the simplex method and identify the corresponding initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.

(c) Using the two-phase method, construct row 0 of the first simplex tableau for phase 1.

C (d) Use a computer package based on the simplex method to solve the problem.

I **4.6-19.** Consider the following problem.

Maximize $Z = 4x_1 + 5x_2 + 3x_3$,

subject to

$$x_1 + x_2 + 2x_3 \geq 20$$
$$15x_1 + 6x_2 - 5x_3 \leq 50$$
$$x_1 + 3x_2 + 5x_3 \leq 30$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

Work through the simplex method step by step to demonstrate that this problem does not possess any feasible solutions.

4.7-1. Refer to Fig. 4.10 and the resulting *allowable range to stay feasible* for the respective right-hand sides of the Wyndor Glass Co. problem given in Sec. 3.1. Use graphical analysis to demonstrate that each given allowable range is correct.

4.7-2. Reconsider the model in Prob. 4.1-5. Interpret the right-hand side of the respective functional constraints as the amount available of the respective resources.
(a) Use graphical analysis as in Fig. 4.8 to determine the shadow prices for the respective resources.
(b) Use graphical analysis to perform sensitivity analysis on this model. In particular, check each parameter of the model to determine whether it is a *sensitive* parameter (a parameter whose value cannot be changed without changing the optimal solution) by examining the graph that identifies the optimal solution.
(c) Use graphical analysis as in Fig. 4.9 to determine the allowable range for each c_j value (coefficient of x_j in the objective function) over which the current optimal solution will remain optimal.
(d) Changing just one b_i value (the right-hand side of functional constraint i) will shift the corresponding constraint boundary. If the current optimal CPF solution lies on this constraint boundary, this CPF solution also will shift. Use graphical analysis to determine the allowable range for each b_i value over which this CPF solution will remain feasible.
C **(e)** Verify your answers in parts (a), (c), and (d) by using a computer package based on the simplex method to solve the problem and then to generate sensitivity analysis information.

4.7-3. Repeat Prob. 4.7-2 for the model in Prob. 4.1-6.

4.7-4. You are given the following linear programming problem.

Maximize $Z = 4x_1 + 2x_2$,

subject to

$2x_1$	≤ 16	(resource 1)
$x_1 + 3x_2$	≤ 17	(resource 2)
x_2	≤ 5	(resource 3)

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

(a) Solve this problem graphically.
(b) Use graphical analysis to find the shadow prices for the resources.
(c) Determine how many additional units of resource 1 would be needed to increase the optimal value of Z by 15.

4.7-5. Consider the following problem.

Maximize $Z = x_1 - 7x_2 + 3x_3$,

subject to

$2x_1 + x_2 - x_3 \leq 4$		(resource 1)
$4x_1 - 3x_2 \leq 2$		(resource 2)
$-3x_1 + 2x_2 + x_3 \leq 3$		(resource 3)

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **(a)** Work through the simplex method step by step to solve the problem.
(b) Identify the shadow prices for the three resources and describe their significance.
C **(c)** Use a software package based on the simplex method to solve the problem and then to generate sensitivity information. Use this information to identify the shadow price for each resource, the allowable range to stay optimal for each objective function coefficient, and the allowable range to stay feasible for each right-hand side.

4.7-6.* Consider the following problem.

Maximize $Z = 2x_1 - 2x_2 + 3x_3$,

subject to

$-x_1 + x_2 + x_3 \leq 4$		(resource 1)
$2x_1 - x_2 + x_3 \leq 2$		(resource 2)
$x_1 + x_2 + 3x_3 \leq 12$		(resource 3)

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

D,I **(a)** Work through the simplex method step by step to solve the problem.
(b) Identify the shadow prices for the three resources and describe their significance.
C **(c)** Use a software package based on the simplex method to solve the problem and then to generate sensitivity information. Use this information to identify the shadow price for each resource, the allowable range to stay optimal for each objective function coefficient and the allowable range to stay feasible for each right-hand side.

4.7-7. Consider the following problem.

Maximize $Z = 2x_1 + 4x_2 - x_3$,

subject to

$$
\begin{array}{ll}
3x_2 - x_3 \le 30 & \text{(resource 1)} \\
2x_1 - x_2 + x_3 \le 10 & \text{(resource 2)} \\
4x_1 + 2x_2 - 2x_3 \le 40 & \text{(resource 3)}
\end{array}
$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

D,I **(a)** Work through the simplex method step by step to solve the problem.
(b) Identify the shadow prices for the three resources and describe their significance.
C **(c)** Use a software package based on the simplex method to solve the problem and then to generate sensitivity information. Use this information to identify the shadow price for each resource, the allowable range to stay optimal for each objective function coefficient, and the allowable range to stay feasible for each right-hand side.

4.7-8. Consider the following problem.

Maximize $Z = 5x_1 + 4x_2 - x_3 + 3x_4$,

subject to

$$
\begin{array}{ll}
3x_1 + 2x_2 - 3x_3 + x_4 \le 24 & \text{(resource 1)} \\
3x_1 + 3x_2 + x_3 + 3x_4 \le 36 & \text{(resource 2)}
\end{array}
$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0, \qquad x_4 \ge 0.$$

D,I **(a)** Work through the simplex method step by step to solve the problem.
(b) Identify the shadow prices for the two resources and describe their significance.
C **(c)** Use a software package based on the simplex method to solve the problem and then to generate sensitivity information. Use this information to identify the shadow price for each resource, the allowable range to stay optimal for each objective function coefficient, and the allowable range to stay feasible for each right-hand side.

4.9-1. Use the interior-point algorithm in your OR Courseware to solve the model in Prob. 4.1-4. Choose $\alpha = 0.5$ from the Option menu, use $(x_1, x_2) = (0.1, 0.4)$ as the initial trial solution, and run 15 iterations. Draw a graph of the feasible region, and then plot the trajectory of the trial solutions through this feasible region.

4.9-2. Repeat Prob. 4.9-1 for the model in Prob. 4.1-5.

4.9-3. Repeat Prob. 4.9-1 for the model in Prob. 4.1-6.

CASE 4.1   FABRICS AND FALL FASHIONS

From the tenth floor of her office building, Katherine Rally watches the swarms of New Yorkers fight their way through the streets infested with yellow cabs and the sidewalks littered with hot dog stands. On this sweltering July day, she pays particular attention to the fashions worn by the various women and wonders what they will choose to wear in the fall. Her thoughts are not simply random musings; they are critical to her work since she owns and manages TrendLines, an elite women's clothing company.

Today is an especially important day because she must meet with Ted Lawson, the production manager, to decide upon next month's production plan for the fall line. Specifically, she must determine the quantity of each clothing item she should produce given the plant's production capacity, limited resources, and demand forecasts. Accurate planning for next month's production is critical to fall sales since the items produced next month will appear in stores during September, and women generally buy the majority of the fall fashions when they first appear in September.

She turns back to her sprawling glass desk and looks at the numerous papers covering it. Her eyes roam across the clothing patterns designed almost six months ago,

the lists of materials requirements for each pattern, and the lists of demand forecasts for each pattern determined by customer surveys at fashion shows. She remembers the hectic and sometimes nightmarish days of designing the fall line and presenting it at fashion shows in New York, Milan, and Paris. Ultimately, she paid her team of six designers a total of $860,000 for their work on her fall line. With the cost of hiring runway models, hair stylists, and makeup artists, sewing and fitting clothes, building the set, choreographing and rehearsing the show, and renting the conference hall, each of the three fashion shows cost her an additional $2,700,000.

She studies the clothing patterns and material requirements. Her fall line consists of both professional and casual fashions. She determined the prices for each clothing item by taking into account the quality and cost of material, the cost of labor and machining, the demand for the item, and the prestige of the TrendLines brand name.

The fall professional fashions include:

Clothing Item	Materials Requirements	Price	Labor and Machine Cost
Tailored wool slacks	3 yards of wool	$300	$160
	2 yards of acetate for lining		
Cashmere sweater	1.5 yards of cashmere	$450	$150
Silk blouse	1.5 yards of silk	$180	$100
Silk camisole	0.5 yard of silk	$120	$ 60
Tailored skirt	2 yards of rayon	$270	$120
	1.5 yards of acetate for lining		
Wool blazer	2.5 yards of wool	$320	$140
	1.5 yards of acetate for lining		

The fall casual fashions include:

Clothing Item	Materials Requirements	Price	Labor and Machine Cost
Velvet pants	3 yards of velvet	$350	$175
	2 yards of acetate for lining		
Cotton sweater	1.5 yards of cotton	$130	$ 60
Cotton miniskirt	0.5 yard of cotton	$ 75	$ 40
Velvet shirt	1.5 yards of velvet	$200	$160
Button-down blouse	1.5 yards of rayon	$120	$ 90

She knows that for the next month, she has ordered 45,000 yards of wool, 28,000 yards of acetate, 9,000 yards of cashmere, 18,000 yards of silk, 30,000 yards of rayon, 20,000 yards of velvet, and 30,000 yards of cotton for production. The prices of the materials are listed on the next page.

Material	Price per yard
Wool	$ 9.00
Acetate	$ 1.50
Cashmere	$60.00
Silk	$13.00
Rayon	$ 2.25
Velvet	$12.00
Cotton	$ 2.50

Any material that is not used in production can be sent back to the textile wholesaler for a full refund, although scrap material cannot be sent back to the wholesaler.

She knows that the production of both the silk blouse and cotton sweater leaves leftover scraps of material. Specifically, for the production of one silk blouse or one cotton sweater, 2 yards of silk and cotton, respectively, are needed. From these 2 yards, 1.5 yards are used for the silk blouse or the cotton sweater and 0.5 yard is left as scrap material. She does not want to waste the material, so she plans to use the rectangular scrap of silk or cotton to produce a silk camisole or cotton miniskirt, respectively. Therefore, whenever a silk blouse is produced, a silk camisole is also produced. Likewise, whenever a cotton sweater is produced, a cotton miniskirt is also produced. Note that it is possible to produce a silk camisole without producing a silk blouse and a cotton miniskirt without producing a cotton sweater.

The demand forecasts indicate that some items have limited demand. Specifically, because the velvet pants and velvet shirts are fashion fads, TrendLines has forecasted that it can sell only 5,500 pairs of velvet pants and 6,000 velvet shirts. TrendLines does not want to produce more than the forecasted demand because once the pants and shirts go out of style, the company cannot sell them. TrendLines can produce less than the forecasted demand, however, since the company is not required to meet the demand. The cashmere sweater also has limited demand because it is quite expensive, and TrendLines knows it can sell at most 4,000 cashmere sweaters. The silk blouses and camisoles have limited demand because many women think silk is too hard to care for, and TrendLines projects that it can sell at most 12,000 silk blouses and 15,000 silk camisoles.

The demand forecasts also indicate that the wool slacks, tailored skirts, and wool blazers have a great demand because they are basic items needed in every professional wardrobe. Specifically, the demand for wool slacks is 7,000 pairs of slacks, and the demand for wool blazers is 5,000 blazers. Katherine wants to meet at least 60 percent of the demand for these two items in order to maintain her loyal customer base and not lose business in the future. Although the demand for tailored skirts could not be estimated, Katherine feels she should make at least 2,800 of them.

(a) Ted is trying to convince Katherine not to produce any velvet shirts since the demand for this fashion fad is quite low. He argues that this fashion fad alone accounts for $500,000 of the fixed design and other costs. The net contribution (price of clothing item − materials cost − labor cost) from selling the fashion fad should cover these fixed costs. Each velvet shirt generates a net contribution of $22. He argues that given the net contribution,

even satisfying the maximum demand will not yield a profit. What do you think of Ted's argument?
(b) Formulate and solve a linear programming problem to maximize profit given the production, resource, and demand constraints.

Before she makes her final decision, Katherine plans to explore the following questions independently except where otherwise indicated.

(c) The textile wholesaler informs Katherine that the velvet cannot be sent back because the demand forecasts show that the demand for velvet will decrease in the future. Katherine can therefore get no refund for the velvet. How does this fact change the production plan?
(d) What is an intuitive economic explanation for the difference between the solutions found in parts (b) and (c)?
(e) The sewing staff encounters difficulties sewing the arms and lining into the wool blazers since the blazer pattern has an awkward shape and the heavy wool material is difficult to cut and sew. The increased labor time to sew a wool blazer increases the labor and machine cost for each blazer by $80. Given this new cost, how many of each clothing item should Trend-Lines produce to maximize profit?
(f) The textile wholesaler informs Katherine that since another textile customer canceled his order, she can obtain an extra 10,000 yards of acetate. How many of each clothing item should TrendLines now produce to maximize profit?
(g) TrendLines assumes that it can sell every item that was not sold during September and October in a big sale in November at 60 percent of the original price. Therefore, it can sell all items in unlimited quantity during the November sale. (The previously mentioned upper limits on demand concern only the sales during September and October.) What should the new production plan be to maximize profit?

CASE 4.2   NEW FRONTIERS

Rob Richman, president of AmeriBank, takes off his glasses, rubs his eyes in exhaustion, and squints at the clock in his study. It reads 3 A.M. For the last several hours, Rob has been poring over AmeriBank's financial statements from the last three quarters of operation. AmeriBank, a medium-sized bank with branches throughout the United States, is headed for dire economic straits. The bank, which provides transaction, savings, and investment and loan services, has been experiencing a steady decline in its net income over the past year, and trends show that the decline will continue. The bank is simply losing customers to nonbank and foreign bank competitors.

   AmeriBank is not alone in its struggle to stay out of the red. From his daily industry readings, Rob knows that many American banks have been suffering significant losses because of increasing competition from nonbank and foreign bank competitors offering services typically in the domain of American banks. Because the nonbank and foreign bank competitors specialize in particular services, they are able to better capture the market for those services by offering less expensive, more efficient, more convenient services. For example, large corporations now turn to foreign banks and commercial paper offerings for loans, and affluent Americans now turn to money-market funds for investment. Banks face the daunting challenge of distinguishing themselves from nonbank and foreign bank competitors.

Rob has concluded that one strategy for distinguishing AmeriBank from its competitors is to improve services that nonbank and foreign bank competitors do not readily provide: transaction services. He has decided that a more convenient transaction method must logically succeed the automatic teller machine, and he believes that electronic banking over the Internet allows this convenient transaction method. Over the Internet, customers are able to perform transactions on their desktop computers either at home or at work. The explosion of the Internet means that many potential customers understand and use the World Wide Web. He therefore feels that if AmeriBank offers Web banking (as the practice of Internet banking is commonly called), the bank will attract many new customers.

Before Rob undertakes the project to make Web banking possible, however, he needs to understand the market for Web banking and the services AmeriBank should provide over the Internet. For example, should the bank only allow customers to access account balances and historical transaction information over the Internet, or should the bank develop a strategy to allow customers to make deposits and withdrawals over the Internet? Should the bank try to recapture a portion of the investment market by continuously running stock prices and allowing customers to make stock transactions over the Internet for a minimal fee?

Because AmeriBank is not in the business of performing surveys, Rob has decided to outsource the survey project to a professional survey company. He has opened the project up for bidding by several survey companies and will award the project to the company which is willing to perform the survey for the least cost.

Sophisticated Surveys is one of three survey companies competing for the project. Rob provided each survey company with a list of survey requirements to ensure that AmeriBank receives the needed information for planning the Web banking project.

Because different age groups require different services, AmeriBank is interested in surveying four different age groups. The first group encompasses customers who are 18 to 25 years old. The bank assumes that this age group has limited yearly income and performs minimal transactions. The second group encompasses customers who are 26 to 40 years old. This age group has significant sources of income, performs many transactions, requires numerous loans for new houses and cars, and invests in various securities. The third group encompasses customers who are 41 to 50 years old. These customers typically have the same level of income and perform the same number of transactions as the second age group, but the bank assumes that these customers are less likely to use Web banking since they have not become as comfortable with the explosion of computers or the Internet. Finally, the fourth group encompasses customers who are 51 years of age and over. These customers commonly crave security and require continuous information on retirement funds. The banks believes that it is highly unlikely that customers in this age group will use Web banking, but the bank desires to learn the needs of this age group for the future. AmeriBank wants to interview 2,000 customers with at least 20 percent from the first age group, at least 27.5 percent from the second age group, at least 15 percent from the third age group, and at least 15 percent from the fourth age group.

Rob understands that the Internet is a recent phenomenon and that some customers may not have heard of the World Wide Web. He therefore wants to ensure that the sur-

vey includes a mix of customers who know the Internet well and those that have less exposure to the Internet. To ensure that AmeriBank obtains the correct mix, he wants to interview at least 15 percent of customers from the Silicon Valley where Internet use is high, at least 35 percent of customers from big cities where Internet use is medium, and at least 20 percent of customers from small towns where Internet use is low.

Sophisticated Surveys has performed an initial analysis of these survey requirements to determine the cost of surveying different populations. The costs per person surveyed are listed in the following table:

Region	Age Group			
	18 to 25	**26 to 40**	**41 to 50**	**51 and over**
Silicon Valley	$4.75	$6.50	$6.50	$5.00
Big cities	$5.25	$5.75	$6.25	$6.25
Small towns	$6.50	$7.50	$7.50	$7.25

Sophisticated Surveys explores the following options cumulatively.

(a) Formulate a linear programming model to minimize costs while meeting all survey constraints imposed by AmeriBank.
(b) If the profit margin for Sophisticated Surveys is 15 percent of cost, what bid will they submit?
(c) After submitting its bid, Sophisticated Surveys is informed that it has the lowest cost but that AmeriBank does not like the solution. Specifically, Rob feels that the selected survey population is not representative enough of the banking customer population. Rob wants at least 50 people of each age group surveyed in each region. What is the new bid made by Sophisticated Surveys?
(d) Rob feels that Sophisticated Survey oversampled the 18- to 25-year-old population and the Silicon Valley population. He imposes a new constraint that no more than 600 individuals can be surveyed from the 18- to 25-year-old population and no more than 650 individuals can be surveyed from the Silicon Valley population. What is the new bid?
(e) When Sophisticated Surveys calculated the cost of reaching and surveying particular individuals, the company thought that reaching individuals in young populations would be easiest. In a recently completed survey, however, Sophisticated Surveys learned that this assumption was wrong. The new costs for surveying the 18- to 25-year-old population are listed below.

Region survey cost per person

Region	Cost
Silicon Valley	$6.50
Big cities	$6.75
Small towns	$7.00

Given the new costs, what is the new bid?

(f) To ensure the desired sampling of individuals, Rob imposes even stricter requirements. He fixes the exact percentage of people that should be surveyed from each population. The requirements are listed below:

Population percentage of people surveyed

18 to 25	25%
26 to 40	35%
41 to 50	20%
51 and over	20%
Silicon Valley	20%
Big cities	50%
Small towns	30%

By how much would these new requirements increase the cost of surveying for Sophisticated Surveys? Given the 15 percent profit margin, what would Sophisticated Surveys bid?

CASE 4.3   ASSIGNING STUDENTS TO SCHOOLS

The Springfield school board has made the decision to close one of its middle schools (sixth, seventh, and eighth grades) at the end of this school year and reassign all of next year's middle school students to the three remaining middle schools. The school district provides bussing for all middle school students who must travel more than approximately a mile, so the school board wants a plan for reassigning the students that will minimize the total bussing cost. The annual cost per student of bussing from each of the six residential areas of the city to each of the schools is shown in the following table (along with other basic data for next year), where 0 indicates that bussing is not needed and a dash indicates an infeasible assignment.

Area	No. of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost per Student		
					School 1	School 2	School 3
1	450	32	38	30	$300	0	$700
2	600	37	28	35	—	$400	$500
3	550	30	32	38	$600	$300	$200
4	350	28	40	32	$200	$500	—
5	500	39	34	27	0	—	$400
6	450	34	28	38	$500	$300	0
				School capacity:	900	1,100	1,000

The school board also has imposed the restriction that each grade must constitute between 30 and 36 percent of each school's population. The above table shows the percentage of each area's middle school population for next year that falls into each of

the three grades. The school attendance zone boundaries can be drawn so as to split any given area among more than one school, but assume that the percentages shown in the table will continue to hold for any partial assignment of an area to a school.

You have been hired as an operations research consultant to assist the school board in determining how many students in each area should be assigned to each school.

(a) Formulate a linear programming model for this problem.
(b) Solve the model.
(c) What is your resulting recommendation to the school board?

After seeing your recommendation, the school board expresses concern about all the splitting of residential areas among multiple schools. They indicate that they "would like to keep each neighborhood together."

(d) Adjust your recommendation as well as you can to enable each area to be assigned to just one school. (Adding this restriction may force you to fudge on some other constraints.) How much does this increase the total bussing cost? (This line of analysis will be pursued more rigorously in Case 12.4.)

The school board is considering eliminating some bussing to reduce costs. Option 1 is to eliminate bussing only for students traveling 1 to 1.5 miles, where the cost per student is given in the table as $200. Option 2 is to also eliminate bussing for students traveling 1.5 to 2 miles, where the estimated cost per student is $300.

(e) Revise the model from part (*a*) to fit Option 1, and solve. Compare these results with those from part (*c*), including the reduction in total bussing cost.
(f) Repeat part (*e*) for Option 2.

The school board now needs to choose among the three alternative bussing plans (the current one or Option 1 or Option 2). One important factor is bussing costs. However, the school board also wants to place equal weight on a second factor: the inconvenience and safety problems caused by forcing students to travel by foot or bicycle a substantial distance (more than a mile, and especially more than 1.5 miles). Therefore, they want to choose a plan that provides the best trade-off between these two factors.

(g) Use your results from parts (*c*), (*e*), and (*f*) to summarize the key information related to these two factors that the school board needs to make this decision.
(h) Which decision do you think should be made? Why?

Note: This case will be continued in later chapters (Cases 6.3 and 12.4), so we suggest that you save your analysis, including your basic model.