Math 4630/5630

Homework 2 Solution

1. Define the following decision variables. Let
   x_t = number of calculators produced in time period t during regular hours, t=1,2,...,6
   y_t = number of calculators produced in time period t during overtime hours, t=1,2,...,6
   w_t = number of calculators stored at the beginning of time period t, t=1,2,...,7
   All our variables are required to be nonnegative integers.

   Then (total cost) = (regular production cost) + (overtime cost) + (inventory cost)
   $$= \sum_{t=1}^{6} 10x_t + \sum_{t=1}^{6} 12y_t + \sum_{t=1}^{7} 0.5w_t$$

   We have the following obvious constraints:

   - No inventory on hand now:
     $$w_1 = 0$$
   - No need to keep inventory after June:
     $$w_7 = 0$$
   - Regular shift production is limited to 100 units per month:
     $$x_t \le 100, \text{ for t=1,...,6}$$
   - Overtime production is limited to 75 units per month:
     $$y_t \le 75, \text{ for t=1,...,6}$$

   We also need balance constraints to relate the production levels with inventory levels:
   $$w_t + x_t + y_t = D_t + w_{t+1}, \text{ for t=1,2,...,6}$$
   D_t denotes the demand for month t. Note that the balance constraints provide that the demand is satisfied since w_{t+1} is a nonnegative number.

   Summarizing, we have the following IP:
   $$\begin{aligned}
   \text{Minimize} \quad & \sum_{t=1}^{6} 10x_t + \sum_{t=1}^{6} 12y_t + \sum_{t=1}^{7} 0.5w_t && (1)\\
   \text{subject to} \quad & w_t + x_t + y_t = D_t + w_{t+1}, \text{ for } t = 1, 2, ..., 6 && (2)\\
   & x_t \le 100, \text{ for } t = 1, ..., 6 && (3)\\
   & y_t \le 75, \text{ for } t = 1, ..., 6 && (4)\\
   & w_1 = 0 && (5)\\
   & w_7 = 0 && (6)\\
   & \text{all } x_t, y_t, w_t \in Z_+ && (7)
   \end{aligned}$$

   Note that constraint (6) is redundant since the objective function will force w_7 to be zero in an optimal solution anyway.

2. The correct order is:

   (f) $11(\log n)^2$
   (e) $n^{1/3} \log n$
   (b) $\frac{n}{\log n}$
   (g) $5n^2 + 2 \log n$
   (d) $n \log n + n^9$
   (a) 1.2^n
   (c) $n^{1/2} + \frac{1}{8} n!$

3. (a) The relevant LP is:

$$\begin{array}{lrcl} \max & 12x_1 + 18x_2 & & \text{(revenue)} \\ \text{s. t.} & 6x_1 + 6x_2 & \leq & 24 \text{ (skilled labor)} \\ & 3x_1 + 6x_2 & \leq & 18 \text{ (unskilled labor)} \\ & 10x_1 + 4x_2 & \leq & 20 \text{ (raw material)} \\ & x_2 & \geq & 2 \text{ (marketing consideration)} \\ & x_1, x_2 & \geq & 0 \end{array}$$

   (b) The feasible region is the quadrangle with the vertices at $(0, 2)$, $(0, 3)$, $(1, 2.5)$ and $(1.2, 2)$. The optimal solution is $(1, 2.5)$ with optimal value 57.

4. (a) TRUE. Consider the isoprofit line through $(0, 2)$, the isoprofit line through $(3, 3)$, and the isoprofit line through $(6, 3)$. Recall that these lines are all parallel to each other. Since $(3, 3)$ produces a larger value of the objective function than $(0, 2)$ and $(6, 3)$, it follows that the isoprofit line through $(3, 3)$ cannot coincide with the isoprofit line through $(0, 2)$ or $(6, 3)$, and moreover, it can be obtained by shifting (while maintaining the slope) the isoprofit line through $(0, 2)$ or $(6, 3)$ in the direction of increasing z. This means that $(3, 3)$ corresponds to the isoprofit line with maximum z.

   An example of objective function: $Z = 6x_2 - x_1$.

   (b) TRUE. Since $(3, 3)$ is an optimal solution, it is clear from the graph that multiple optimal solutions will only exist if the isoprofit line through $(3, 3)$ coincides with either one of the boundaries (of the feasible region) containing $(3, 3)$. Hence, either $(0, 2)$ or $(6, 3)$ are also optimal.

   Examples of objective functions: $Z = 3x_2 - x_1$ (makes $(0, 2)$ optimal), $Z = x_2$ (makes $(6, 3)$ optimal).

   (c) FALSE. Consider the objective function $Z = -x_1 - x_2$.

5. The standard form of an LP looks like

$$\begin{array}{rcl} \max c^T x & & \\ \text{subject to } Ax & \leq & b \\ x & \geq & 0 \end{array}$$

   (a) FALSE. If you take $Z = x_2$ as the objective function in problem 4, then any point on the line segment connecting $(3, 3)$ and $(6, 3)$ will be an optimal solution. But only the endpoints $(3, 3)$ and $(6, 3)$ are CPFs.

(b) FALSE. This expression is an *upper* bound on the number of basic solutions, so there are at *most* this many CPF solutions (remember that CPF solutions are the same as basic feasible solutions).

(c) TRUE. Suppose we have an optimal solution x^* which is not a CPF solution. Recall that any LP problem in standard form that has an optimal solution has a CPF solution which is also optimal. Let x^{**} be an optimal CPF solution to the same problem. Note that any point in the line segment joining x^* and x^{**} will also be optimal.

(d) FALSE. Consider an unbounded LP problem.