

Homework 4 Solutions

Problem 1. Solving IP.

Consider the following IP problem.

$$\begin{array}{ll} \text{Maximize} & Z = 11x_1 + 4x_2 \\ \text{s.t.} & 5x_1 + 2x_2 \leq 16 \\ & 2x_1 - x_2 \leq 4 \\ & -x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \text{ integer} \end{array}$$

a) Solve the integer program graphically.

If we draw an isoprofit line (e.g., $11x_1 + 4x_2 = 11$) and move it parallelly towards the direction of increasing the objective function, the last feasible integer point it will touch will be (2, 3). Thus, the optimal integer solution is (2, 3) with value 34.

b) Solve the LP relaxation graphically. Round this solution to the *nearest* integer solution and check whether it is feasible. Then enumerate *all* the rounded solutions by rounding this solution for the LP relaxation in all possible ways (i.e., by rounding each noninteger value both up and down). For each rounded solution, check for feasibility and, if feasible, calculate Z . Are any of these feasible rounded solutions optimal for the IP problem?

The feasible region of the LP-relaxation has the following CPF solutions: (0,0), (0,2), (2,3), (8/3, 4/3) and (2,0). The last point an isoprofit line touches the feasible region is (8/3, 4/3) with value 104/3. This is the optimal solution to the LP-relaxation.

The nearest integer solution to (8/3, 4/3) is (3,1) which is not feasible (constraints $5x_1 + 2x_2 \leq 16$ and $2x_1 - x_2 \leq 4$ are violated).

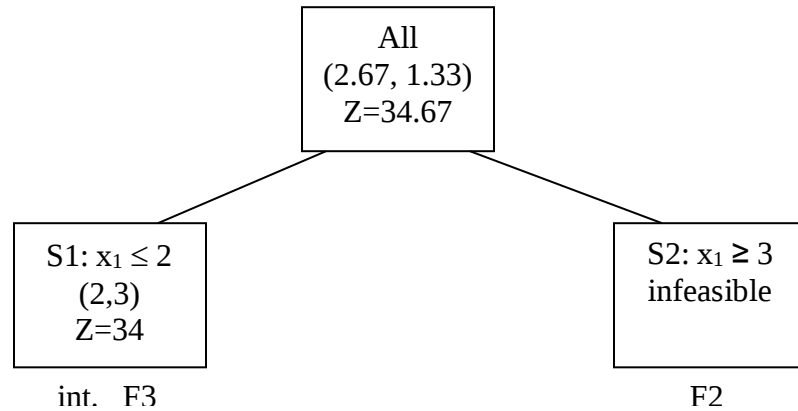
All rounded solutions with required information:

- (3,1) infeasible ($5x_1 + 2x_2 \leq 16$ and $2x_1 - x_2 \leq 4$ are violated)
- (3,2) infeasible ($5x_1 + 2x_2 \leq 16$ is violated)
- (2,1) feasible, $Z=26$
- (2,2) feasible, $Z=30$

None of these rounded solutions is optimal for IP.

- c) Use the branch-and-bound method to solve the IP. You can solve each subproblem in whatever way you find most convenient (graphical, simplex, etc.).

We have the following solution tree.



Subproblem S1 gives the optimal solution, (2,3) with value 34.

- d) Find a cutting plane which makes all CPF solutions of the LP-relaxation integral. What can you say about the new optimal solution of the LP-relaxation?

A new constraint $x_1 \leq 2$ would cut off the old LP-relaxation solution $(8/3, 4/3)$ but it wouldn't delete any integer point. Thus, $x_1 \leq 2$ is a cutting plane. The CPF solutions of the new LP-relaxation would be $(0,0)$, $(0,2)$, $(2,3)$ and $(2,0)$. Thus, the new optimal solution of the LP-relaxation is an integer point, $(2,3)$ which is also an optimal solution for IP.

Problem 2. Branch-and-bound.

You are given the following (incomplete) solution tree which is obtained by applying the branch-and-bound method to an integer program (maximization problem). The subproblems were solved in the increasing order of their indexes. Subproblem S8 is infeasible; the solutions of subproblems S4 and S10 are integral; all other subproblems have fractional solutions. Answer the following questions.

- a) What is the current incumbent? What is its value?

The current incumbent is the optimal solution of S10. Its value is 42.

- b) Give as tight as possible lower and upper bounds on the IP optimal value.

The tightest lower bound is 42, the value of the current incumbent.

The tightest upper bound is 46.1, the value of S6.

- c) Which nodes are fathomed? Why?

S7 is fathomed because its value is less than the value of the incumbent when S7 was solved: $37.3 < 38$ (criterion F1).

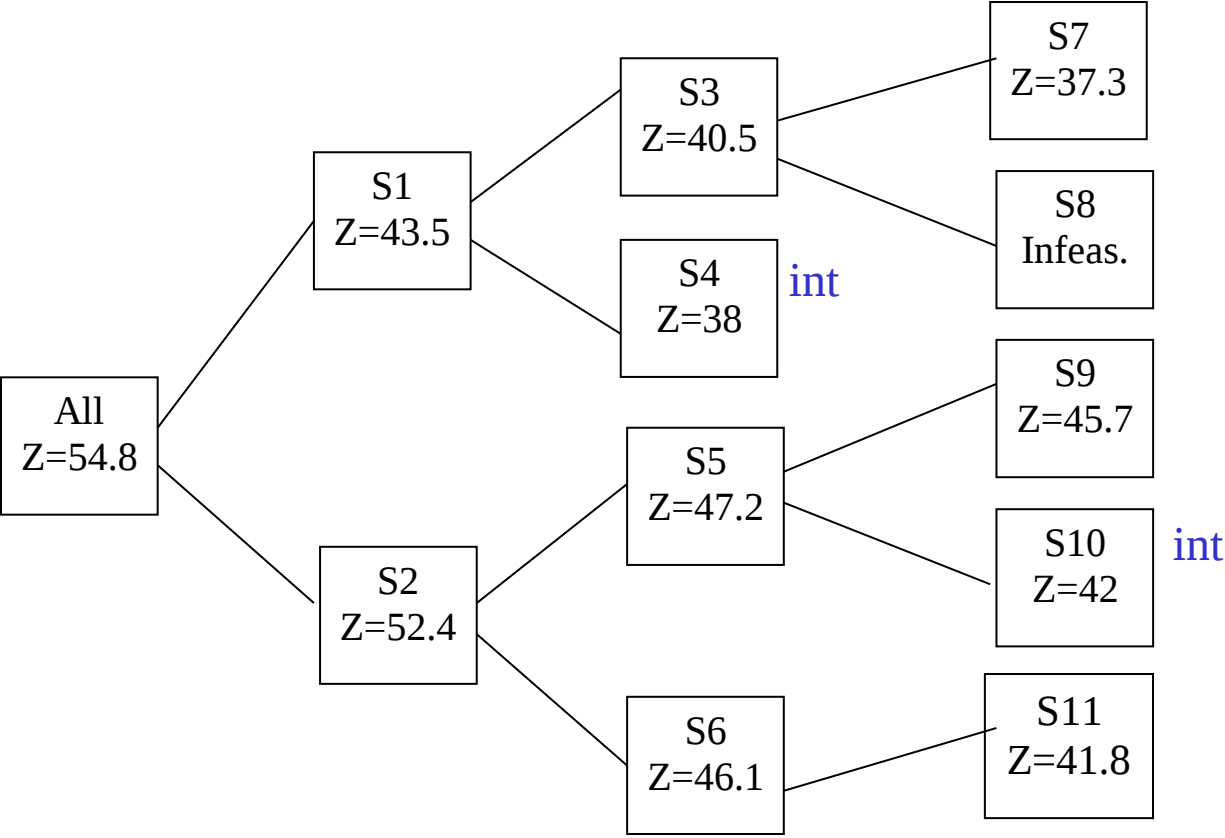
S11 is fathomed because its value is less than the value of the current incumbent: $41.8 < 42$ (criterion F1).

S8 is fathomed because it is infeasible (criterion F2).

S4 and S10 are fathomed because their solutions are integral (criterion F3).

- d) Suppose before starting the branch-and-bound we have applied a fast heuristic to the problem. The heuristic has returned an integer solution with $Z=41$. How would this information change the solution tree?

S3 would be fathomed because its value would be less than the value of the current incumbent: $40.5 < 41$ (criterion F1). As a result, we wouldn't branch on S3 and wouldn't have S7 and S8 in the tree.



Problem 3. Cutting planes for knapsack problem.

You are given the following knapsack problem:

item	1	2	3	4	5
size	31	28	45	18	25
benefit	43	35	50	24	30

The knapsack size is 60. Multiple copies of the same item can be taken.

- a) Give an integer programming formulation for this problem.

Let x_i be the number of copies of item i to take in the knapsack ($i=1, 2, 3, 4, 5$).
Then the IP model is:

$$\begin{aligned} \text{Maximize } Z &= 43x_1 + 35x_2 + 50x_3 + 24x_4 + 30x_5 \\ \text{s.t. } &31x_1 + 28x_2 + 45x_3 + 18x_4 + 25x_5 \leq 60 \end{aligned}$$

- b) Recall that for $k \geq 1$ and $S = \{ i \mid w_i > W / k \}$, we have the following cutting plane:

$$\sum_{i \in S} x_i \leq k - 1$$

Add cutting planes for $k=2, 3, 4$ to the formulation of part (a).

The new constraints are

$$\begin{aligned} x_1 + x_3 &\leq 1 && (\text{for } k=2) \\ x_1 + x_2 + x_3 + x_5 &\leq 2 && (\text{for } k=3) \\ x_1 + x_2 + x_3 + x_4 + x_5 &\leq 3 && (\text{for } k=4) \end{aligned}$$