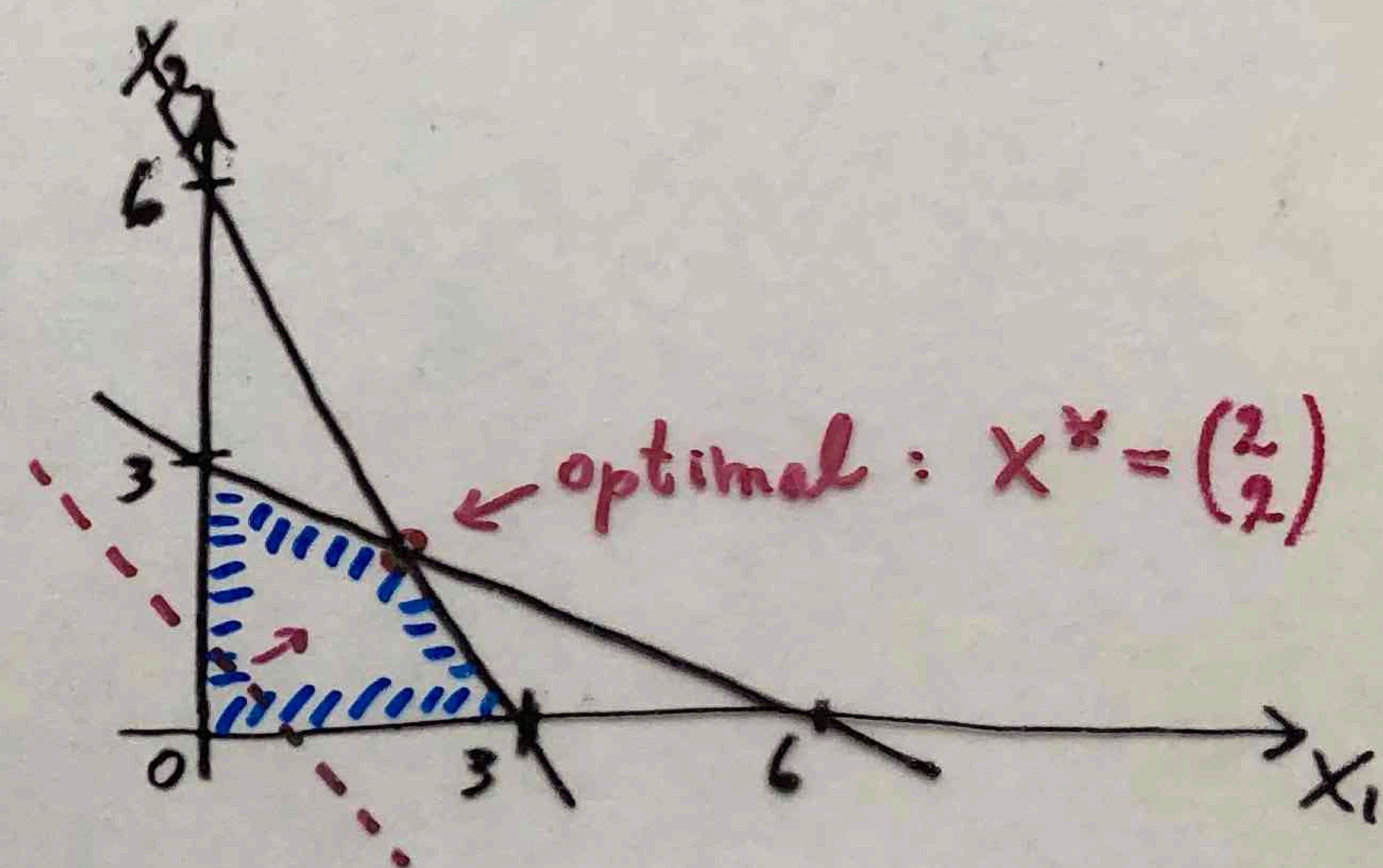
Towards the Simplex Method

- "Simplex" is in top ten algorithms having "the greatest influence on the development and practice of science and engineering in the 20^{th} century". (Journal of Computing in Science & Engineering)

- Recall the example:

$$\begin{aligned} \max\ & x_1 + x_2 \\ \text{s.t. } & 2x_1 + 4x_2 \le 12 \\ & 2x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned}$$



- Intersection points of constraint boundaries are important; characterize them.
- Definition: Let $Ax \le b$ be a p by n system of linear inequalities (can include $x \ge 0$). Then a sol-n $\bar{x}$ is called a Corner Point Feasible Solution (CPF) if
  - i) $A\bar{x} \le b$ ($\bar{x}$ is feasible)
  - ii) n or more inequality constraints are satisfied at equality.
- Examples: $\binom{0}{0}, \binom{3}{0}, \binom{0}{3}, \binom{2}{2}$ are CPF sol-ns.

- Def-n: Extreme point of $\{x : Ax \leq b\} = S$ is a point in the set that does not lie on any line segment joining two other points of S.
- Extreme points and CPF's are the same in LP.

- Def-n: Two CPF's are adjacent to each other if they share $n-1$ constraint boundaries.

  Ex.: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ are adjacent in our ex.
- Def-n: The line segment connecting two adjacent CPF's is called an edge.

Some solution concepts for Simplex

- If an LP has an optimal sol-n then there is a CPF which is optimal.

  Thus, look for an optimal CPF.
- If a CPF has no adjacent CPF solutions that are better (in terms of obj. f-n) then it must be an optimal sol-n.

  otherwise move to a better adjacent CPF.

The structure of Simplex

Initialization

↓

Optimality test

If no ↓ Find better CPF → Optimality test

If yes (optimal) Stop

How to realize this algebraically?

Idea: Introduce another description of CPF which is easy to manipulate algebraically.

- Add slack variables x_s :

$$\left\{\begin{matrix} Ax \leq b \\ x \geq 0 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} Ax + x_s = b \\ x \geq 0,\ x_s \geq 0 \end{matrix}\right\} \rightarrow \text{augmented system}$$

Advantage: Equality system, easy to treat

Disadvantage: # of variables increased by m

Example:

$$\begin{matrix} 2x_1 + x_2 \leq 6 \\ 2x_1 + 4x_2 \leq 12 \\ x_1, x_2 \geq 0 \end{matrix} \longrightarrow \begin{matrix} 2x_1 + x_2 + x_3 = 6 \\ 2x_1 + 4x_2 + x_4 = 12 \\ x_1, x_2, x_3, x_4 \geq 0 \end{matrix}$$

original variables: $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ slack variables: $\bar{x}_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$

$$\begin{Bmatrix} Ax \leq b \\ x \geq 0 \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} Ax + x_s = b \\ x \geq 0,\ x_s \geq 0 \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} [A\ I]\begin{bmatrix} x \\ x_s \end{bmatrix} = b \\ \begin{bmatrix} x \\ x_s \end{bmatrix} \geq 0 \end{Bmatrix}$$

- Want to solve $[A\ I]\begin{bmatrix} x \\ x_s \end{bmatrix} = b$.
- A has site $m \times n$ $\rightarrow$ $[A\ I]$ has size $m \times (n+m)$
  $\rightarrow$ we have n degrees of freedom $\rightarrow$ set n variables to 0 and solve the rest

Example:

$$\begin{aligned} 2x_1 + x_2 + x_3 \quad\quad &= 6 \\ 2x_1 + 4x_2 \quad\quad + x_4 &= 12 \end{aligned}$$

$m=2,\ n+m=4 \rightarrow 2$ degrees of freedom
$\rightarrow$ set 2 variables to 0

▼ set $x_1 = x_2 = 0$; solve $\begin{aligned} x_3 &= 6 \\ x_4 &= 12 \end{aligned}$;

get solution $\begin{bmatrix} x \\ x_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 12 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

▼ set $x_3 = x_4 = 0$; solve $\begin{aligned} 2x_1 + x_2 &= 6 \\ 2x_1 + 4x_2 &= 12 \end{aligned}$;

get solution $\begin{bmatrix} x \\ x_s \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ or $x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

was the optimal CPF

▼ set $x_1 = x_3 = 0$; solve $\begin{aligned} x_2 \quad &= 6 \\ 4x_2 + x_4 &= 12 \end{aligned}$;

get $\begin{bmatrix} x \\ x_s \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ -12 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

was not feasible

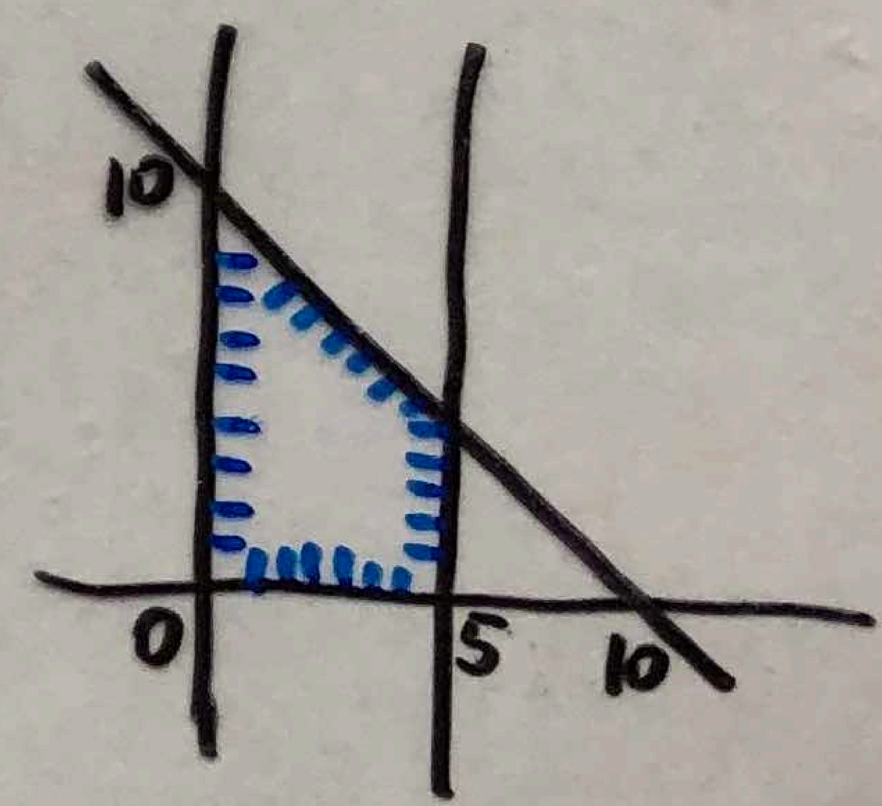
- When does this work? What if the resulting system is <u>singular</u>?

<u>Example:</u> $\left\{ \begin{aligned} x_1 \quad\quad &\leq 5 \\ x_1 + x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned} \right\} \rightarrow \left\{ \begin{aligned} x_1 \quad + x_3 &= 5 \\ x_1 + x_2 + x_4 &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned} \right\}$

▼ set $x_1 = x_3 = 0$;

Resulting system:

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$



singular → no sol-n

(parallel lines don't intersect)

Basic solutions

Def-n: Given a system of linear equalities
$Ax = b$, where $A \in \mathbb{R}^{p \times q}$, $q > p$,
a solution $\bar{x} \in \mathbb{R}^q$ is called <u>basic solution</u> if

i) $A\bar{x} = b$

ii) $q-p$ variables are set to 0
(denote these variables $\bar{x}_N$ and call them <u>nonbasic variables</u>)

iii) the columns corresponding to remaining p variables are <u>linearly independent</u>
(denote these variables $\bar{x}_B$ and call them <u>basic variables</u>).

- <u>Thus</u>, $\bar{x} = (\bar{x}_B, \bar{x}_N) = (\bar{x}_B, \bar{0})$
- In our case, the system was

$$[A \ I] \begin{bmatrix} x \\ x_s \end{bmatrix} = b$$

where $[A \ I] \in \mathbb{R}^{m \times (n+m)}$.

<u>Thus</u>, the number of nonbasic variables is
$(n+m) - m = n$;

the —"— basic —"—
m.