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ON THE REFLECTION LAW FOR THE HELMHOLTZ EQUATION

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1. Symmetry and reflection principles

In this note we consider the "reflection principle" for the Helmholtz equation in the two-dimensional case.

The formulation of the problem is as follows. Let U be a domain in the space $\mathbf{R}^2$ separated into two parts U_1 and U_2 by a real-analytic curve Γ with the equation $\varphi(x, y) = 0$, and suppose $u(x, y)$ is a solution of the homogeneous Helmholtz equation $\Delta u + k^2 u = 0$ vanishing on Γ.

It is required to express the values of u at points $(x_0, y_0) \in U_1$ in terms of its values in U_2 (to find a "reflection formula").

We remark that an analogous problem for the Laplace equation $\Delta u = 0$ was posed and solved back at the end of the nineteenth and the beginning of the twentieth centuries. In 1870 Schwarz [1] introduced a symmetry principle for harmonic functions that, under our assumptions, consists in the following (see Figure 1). There exists an anticonformal mapping $R: U \to U$ permuting the domains U_1 and U_2 relative to which any harmonic function vanishing on Γ is odd. (More precisely, the mapping R possibly acts in a somewhat smaller domain.)

The reader can find a discussion of the Schwarz symmetry principle, for example, in [2]–[4] and elsewhere. To describe the mapping R we consider a domain W in the space $\mathbf{C}^2$ to which the equation $\varphi(x, y) = 0$ of the curve Γ can be continued analytically such that $W \cap \mathbf{R}^2 = U$. After the change of variables $z = x + iy$, $\zeta = x - iy$ the equation of the complexified curve $\Gamma_{\mathbf{C}}$ can be rewritten in the form

$$\varphi\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) = 0$$

(we remark that the equation of the "real space" $\mathbf{R}^2$ in the coordinates (z, ζ) is $z = \overline{\zeta}$). If $\operatorname{grad}(x, y)\varphi(x, y) \neq 0$ on Γ, then the equation of $\Gamma_{\mathbf{C}}$ is solvable both for z and for ζ; the corresponding solutions we denote by $\zeta = S(z)$ and $z = \widetilde{S}(\zeta)$. The function $S(z)$ is called the *Schwarz function* of the curve Γ (see, for example, [2]). In these terms the mapping R mentioned above is given by $R(x_0, y_0) = R(z_0) = \overline{S(z_0)}$.

E. Study [5] gave an elegant geometric interpretation of the mapping R. Namely, it is easy to verify that $\widetilde{S}(\overline{z}_0) = \overline{S(z_0)}$, so that the points $(z_0, \overline{z}_0)$, $(z_0 S(z_0))$, $(\overline{z}_0, \widetilde{S}(\overline{z}_0))$, and $(R(z_0), \overline{R(z_0)})$ form a so-called "Study rectangle" with sides parallel to the "coordinate axes" Oz and $O\zeta$, two diagonal vertices of which correspond to the points (x_0, y_0) and $R(x_0, y_0)$, while the other two lie on Γ_C. A schematic representation of the situation is shown in Figure 2. We note also that the sides a, b, c, and d of the Study rectangle are characteristics of the Laplace operator.

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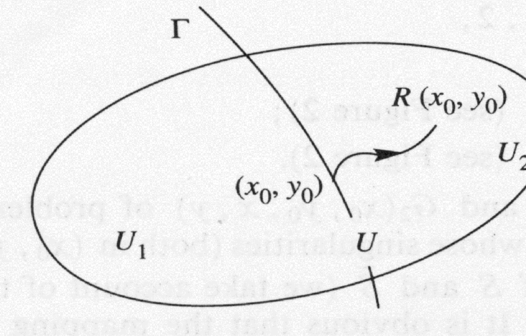

FIGURE 1

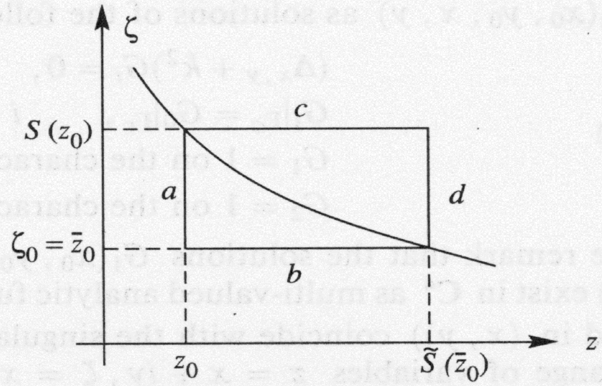

FIGURE 2

This situation demonstrates, in particular, *the necessity of passing to complex space in considering the symmetry principle.*

It would be very tempting to try to extend the principle just described to more general equations, and also to equations in spaces of higher dimension. Unfortunately, in the form given above the symmetry principle does not carry over to more general situations. Thus, in particular, for the Helmholtz equation in the plane the symmetry principle is valid only for the case where Γ is a line segment, while for the Laplace equation in $\mathbf{R}^3$ it is valid only when Γ is part of either a plane or a sphere(1) (see [7]). In the more general formulation presented at the beginning of the paper(2) this problem was studied by Lewy [8] for operators of the form $\Delta + a(x, y)\partial/\partial x + b(x, y)\partial/\partial y + c(x, y)$ in the plane and by Garabedian [9] for higher dimensions. In [8] Lewy proves only the possibility of obtaining a reflection formula and does not derive it, while in Garabedian's paper (in which there is the possibility in principle of obtaining an explicit formula) because of the second-degree character of the computations such a formula turns out to be extremely complicated.

Our purpose in this note is to obtain an explicit reflection formula for the Helmholtz equation in a form sufficiently simple and convenient for applications, and to clarify the connection of this formula with a certain special Cauchy-Goursat problem (precise formulations are given below).

2. THE MAIN RESULT

For simplicity of formulations we assume that $U = \mathbf{R}^2$ and Γ is an algebraic curve (this means that $\varphi(x, y)$ is polynomial in x and y with real coefficients). Under these assumptions $S(z)$ and $\widetilde{S}(\zeta)$ are analytic functions in the entire plane $\mathbf{C}$ possessing singularities only of algebraic type.

We denote by $G_0(x_0, y_0, x, y)$ the Riemann function for the Helmholtz operator. It is known (see, for example, [10]) that

$$G_0(x_0, y_0, x, y) = J_0(k\sqrt{(x - x_0)^2 + (y - y_0)^2}), \tag{1}$$

where J_0 is the Bessel function of order zero. This, in particular, shows that G_0 is an entire function of all its arguments. We define functions $G_1(x_0, y_0, x, y)$ and

(1)In this connection it is interesting to note that in [6], Chapter III, Apel'tsin nevertheless attempts to prove the validity of the symmetry principle for the Helmholtz equation in the case of an arbitrary boundary. Of course, this "proof" is wrong.

(2)We call this more general formulation the *reflection problem*, retaining the term "symmetry" for the narrower case discussed above. Thus, a symmetry is an operator induced (pointwise) by a mapping of domains, while a reflection is, generally speaking, an operator of a more general nature.

$G_2(x_0, y_0, x, y)$ as solutions of the following two Cauchy-Goursat problems in $\mathbf{C}^2$:

$$(2)\quad \begin{aligned} &(\Delta_{x,y}+k^2)G_i = 0, \qquad i=1,2,\\ &G_i|_{\Gamma_{\mathbf{C}}} = G_0|_{\Gamma_{\mathbf{C}}}, \qquad i=1,2;\\ &G_1 = 1 \text{ on the characteristic} \quad \text{(see Figure 2)};\\ &G_2 = 1 \text{ on the characteristic} \quad \text{(see Figure 2)}.\end{aligned}$$

We remark that the solutions $G_1(x_0, y_0, x, y)$ and $G_2(x_0, y_0, x, y)$ of problems (2) exist in $\mathbf{C}^4$ as multi-valued analytic functions whose singularities (both in (x_0, y_0) and in (x, y)) coincide with the singularities of S and $\widetilde{S}$ (we take account of the change of variables $z = x+iy$, $\zeta = x-iy$). It is obvious that the mapping R introduced above by the formula

$$(3)\qquad R(x_0, y_0) = R(z_0) = \overline{S(z_0)}$$

is regular off the intersection of the singularities of the function S with the real space $\mathbf{R}^2$. We introduce the function

$$(4)\qquad G(x_0, y_0, x, y) = G_1(x_0, y_0, x, y) - G_2(x_0, y_0, x, y).$$

Let $u(x, y)$ be an arbitrary solution of the Helmholtz equation

$$(5)\qquad (\Delta + k^2)u(x, y) = 0,$$

vanishing on the curve Γ. The following theorem holds and is the main theorem of this note.

Theorem 1. *Under the assumptions formulated above,*

(6)

$$\begin{aligned} u(x_0, y_0) &= -u(R(x_0, y_0))\\ &+\frac{1}{2i}\int_{\Gamma}^{R(x_0, y_0)}\left\{\left(u(x,y)\frac{\partial G}{\partial x}(x_0, y_0, x, y) - G(x_0, y_0, x, y)\frac{\partial u}{\partial x}(x,y)\right)dy\right.\\ &\left.-\left(u(x,y)\frac{\partial G}{\partial y}(x_0, y_0, x, y) - G(x_0, y_0, x, y)\frac{\partial u}{\partial y}(x,y)\right)dx\right\},\end{aligned}$$

where R is defined by (3), *G is defined by* (4), *and the integral is taken along any curve joining Γ with the point $R(x_0, y_0)$ (see Figure* 1).

Remark 1. It is not hard to verify that the form under the integral on the right side of (6) is closed. This fact is a consequence of (2) for the terms G_i of the function G and of (5) for the function u. Moreover, $G|_\Gamma = G_1|_\Gamma - G_2|_\Gamma = 0$ (because G_i is a solution of problem (2)) and $u|_\Gamma = 0$ (by the assumptions formulated above). Therefore, the integral on the right side of (6) does not depend on the path of integration.

The functions G_1 and G_2 (at least in a neighborhood of the curve Γ) can be computed explicitly as series. We illustrate this with the example of computing G_1. We have (in the coordinates (z, ζ))

$$(7)\qquad G_1(z_0, \zeta_0, z, \zeta) = \sum_{j=0}^{\infty} a_j(z_0, \zeta_0, z, \zeta)\frac{(\widetilde{S}(\zeta) - z_0)^j}{j!},$$

where $a_0 = 1$, while the functions a_j satisfy the following recurrent system of relations:

$$4\frac{\partial a_{j+1}}{\partial z}\frac{\partial \widetilde{S}(\zeta)}{\partial \zeta} = -4\frac{\partial^2 a_j}{\partial z\,\partial\zeta} - k^2 a_j, \qquad a_j\big|_{z=\widetilde{S}(\zeta)} = \frac{(-1)^j(\zeta-\zeta_0)^j k^{2j}}{4^j\cdot j!}$$

for $j = 0, 1, 2, \ldots$.

The series (7) converges for (z_0, ζ_0) and (z, ζ) sufficiently close to Γ_C. Outside a neighborhood of Γ_C in which (7) converges the functions G_1 and G_2 can be obtained as solutions of Volterra integral equations by a method close to that expounded in [10].

3. Corollaries

The classical symmetry formulas for the Laplace operator (for $k = 0$) and for the Helmholtz operator in the case where Γ is a straight line are special cases of (6). Direct computation shows that in both these cases $G = 0$, and therefore

$$u(x_0, y_0) = -u(R(x_0, y_0)).$$

However, for the case where Γ is a circle (and $k \neq 0$) it is easy to show that the integral term in (6) does not vanish (the theorem of Khavinson and Shapiro [3] mentioned above is an indirect corroboration of this).

Remark 2. Our results generalize to solutions of equations of the form

$$\Delta u + a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u = 0$$

with entire coefficients a, b, and c.

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