Росс. Акад. Наук Докл.
Том 332 (1993), № 5

Russian Acad. Sci. Dokl. Math.
Vol. 48 (1994), No. 2

ON THE LAW OF REFLECTION FOR HIGHER-ORDER ELLIPTIC EQUATIONS

UDC 517.955

T. V. SAVINA

1. Formulation of the problem. The law of reflection

In 1870 Schwarz [1] introduced a symmetry principle for harmonic functions, which consists in the following.

Let U be a domain in the space $\mathbb{R}^2$ divided into two parts U_1 and U_2 by a real-analytic curve Γ, and let $u(x, y)$ be a solution of the Laplace equation $\Delta u = 0$ that vanishes on Γ. Then there exists an anticonformal mapping $R: U \to U$, which permutes the domains U_1 and U_2, relative to which the function $u(x, y)$ is odd, i.e., for any point $(x_0, y_0) \in U$

$$u(x_0, y_0) = -u(R(x_0, y_0)). \tag{1}$$

It is obvious that if the point $(x_0, y_0) \in U_1$, then the "reflected" point $R(x_0, y_0) \in U_2$.

The books of Davis [2], Khavinson and Shapiro [3], and Shapiro [4] are devoted to further investigations of the Schwarz symmetry principle.

By a *reflection formula* we mean a formula expressing the value of a function $u(x, y)$ at an arbitrary point $(x_0, y_0) \in U_1$ in terms of its value at points in U_2.

It is clear that (1) is the simplest representative of reflection formulas expressing the value at a point $(x_0, y_0) \in U_1$ in terms of a point $R(x_0, y_0) \in U_2$. Unfortunately, the symmetry principle (1) in this form does not carry over to more general situations. Thus, if a function $u(x, y)$ equal to zero on Γ is a solution of the Helmholtz equation $(\Delta + k^2)u = 0$ in the plane, then the symmetry principle holds only when Γ is a line segment, while for the Laplace equation in $\mathbb{R}^3$ it holds only when Γ is a part of either a plane or a sphere [5]. The possibility in principle of obtaining more general reflection formulas was demonstrated by Garabedian [6], and for the Helmholtz operator in the plane such a formula was obtained explicitly in [7].

The purpose of this note is to construct a reflection formula for *higher-order* elliptic equations. The problem is formulated as follows. Suppose a function $u(x, y)$ defined in a domain U, divided into two parts U_1 and U_2 by a real analytic curve Γ with equation $\varphi(x, y) = 0$, is a solution of the elliptic equation of order $2m$, $m > 1$,

$$Lu \equiv \left[\sum_{\alpha=0}^{2m} a_\alpha \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial y}\right)^{2m-\alpha} + \sum_{n=0}^{2m-1} \sum_{\alpha=0}^{n} a_{n\alpha}(x, y) \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial y}\right)^{n-\alpha}\right] u = 0, \tag{2}$$

having real-analytic coefficients, where the coefficients in the leading part are constants. Suppose also that $u(x, y)$ has a zero of order $m\Gamma$. It is required to express the values of $u(x, y)$ at points $(x_0, y_0) \in U_1$ in terms of its values in U_2.

1991 *Mathematics Subject Classification*. Primary 35J30; Secondary 35C15.

As the basic tool for constructing a reflection formula we use Green's formula (see, for example, [8])

$$u(x_0, y_0) = \int_\gamma \left\{ \sum_{j=0}^{2m-1} \widehat{B}_j u(x, y) \widehat{C}_j G(x, y, x_0, y_0)\, dy - \sum_{j=0}^{2m-1} \widehat{H}_j u(x, y) \widehat{P}_j G(x, y, x_0, y_0)\, dx \right\}, \tag{3}$$

where γ is a contour surrounding the point (x_0, y_0); $\widehat{B}_j$, $\widehat{C}_j$, $\widehat{H}_j$, and $\widehat{P}_j$ are differential operators of order $\leq 2m-1$, and $G(x, y, x_0, y_0)$ is the fundamental solution of equation (2).

2. Schwarz functions and construction of reflected points

In contrast to formula (1), where to each point of the domain U_1 there corresponds exactly one reflected point, for an equation of order $2m$ there are m^2 such points: $R_{jk}(x_0, y_0)$, $j, k = 1, \dots, m$.

To describe the reflections R_{jk} we consider a domain W in the space $\mathbb{C}^2$ into which the equation of the curve Γ extends analytically, $W \cap \mathbb{R}^2 = U$. In W we consider a complex curve $\Gamma_\mathbb{C}$ whose equation $\varphi(x, y) = 0$ is an analytic continuation of the equation of the original curve Γ. Under the assumption that the characteristics of equation (2) in the domain W are simple, in this domain from each point $(x_0, y_0) \in U_1$ there issue $2m$ distinct characteristics of equation (2) which combine into m complex-conjugate pairs. Each of these characteristics intersects the analytic continuation of Γ. From the points of intersection there also issue $2m$ characteristics, some of which intersect the real plane at points of U_2. These points are called *reflected points*. More precisely, we introduce m pairs of characteristic variables

$$z_j = x + \lambda_j y, \quad z_{\bar{j}} = x + \bar{\lambda}_j y, \qquad j, \bar{j} = 1, \dots, m,$$

where λ_j and $\bar{\lambda}_j$ are complex-conjugate numbers, which are the roots of the characteristic equation $\sum_{\alpha=0}^{m} a_\alpha p^{2m-\alpha} = 0$. We remark that the variables z_j and $z_{\bar{j}}$ for $x, y \in \mathbb{R}$ are complex conjugates. Of course, for $x, y \in \mathbb{C}$ this property is not satisfied; in order to indicate that characteristic variables belong to a single pair the bar is placed not over the letter but over the index.

The equation of the complexified curve $\Gamma_\mathbb{C}$ can be rewritten in characteristic variables $\varphi(x, y) = \overline{\Phi}(z_k, z_{\bar{j}}) = 0$. If $d\varphi(x, y) \neq 0$ on Γ, then this equation can be solved for both variables; the corresponding solutions we denote by $z_k = S_{z_k z_{\bar{j}}}(z_{\bar{j}})$ and $z_{\bar{j}} = S_{z_{\bar{j}} z_k}(z_k)$. The functions $S_{z_k z_{\bar{j}}}(z_{\bar{j}})$ and $S_{z_{\bar{j}} z_k}(z_k)$ are called *Schwarz functions*. The coordinates of the reflected points are determined from the relations

$$R_{jk} : x + \lambda_k y = \overline{S_{z_k z_{\bar{j}}}(x_0 + \lambda_j y_0)}, \qquad j, k = 1, \dots, m. \tag{4}$$

3. The main result

For simplicity of formulations we assume that $U = \mathbb{R}^2$ and Γ is an algebraic curve (this means that $\varphi(x, y)$ is a polynomial in x and y with real coefficients). Under these assumptions the Schwarz functions are analytic functions in the entire plane $\mathbb{C}$ and possess singularities only of algebraic type.

Suppose $u(x, y)$ is an arbitrary solution of (2) which has a zero of order m on Γ. The following theorem, which is our main result, then holds.

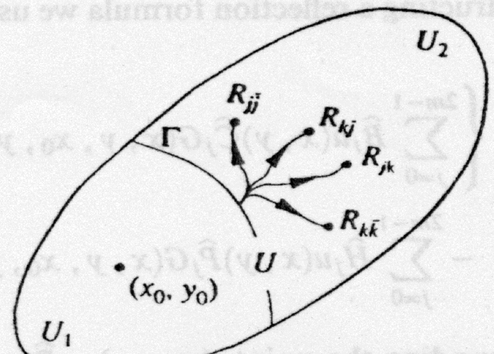

FIGURE 1

Theorem. *For points* (x_0, y_0) *located sufficiently close to the curve* Γ *the following reflection formula holds*:

$$
\begin{aligned}
u(x_0, y_0) = &- \sum_{k,j=1}^{m} c_{jk}(x_0, y_0) u(R_{jk}(x_0, y_0)) \\
&+ \sum_{j,k=1}^{m} 2\pi i \int_{\Gamma}^{R_{jk}(x_0, y_0)} \left\{ \sum_{l=0}^{2m-1} \widehat{B}_l u(x, y) \widehat{C}_l \left[\frac{\partial}{\partial \xi} (\tilde{g}_{jk}(x, y, x_0, \xi, y_0) \right.\right. \\
&\left.\left. - \tilde{g}_{jk}(x, y, x_0, \xi, y_0)) \right]\right|_{\xi=0} dy \\
&- \sum_{l=0}^{2m-1} \widehat{H}_l u(x, y) \widehat{P}_l \left[\frac{\partial}{\partial \xi} (\tilde{g}_{jk}(x, y, x_0, \xi, y_0) \right. \\
&\left.\left. - \tilde{g}_{j\bar{k}}(x, y, x_0, \xi, y_0)) \right] |_{\xi=0} dx \right\},
\end{aligned}
$$

where the $c_{jk}(x_0, y_0)$ *are coefficients depending on* Γ *and* $\sum_{k,j=1}^{m} c_{jk}(x_0, y_0) = 1$, *the* R_{jk} *are the mappings introduced in* (4), *the functions* $\tilde{g}_{jk}$ *and* $\tilde{g}_{j\bar{k}}$ *are defined below* (*see problem* (5)), $\widehat{B}_l$, $\widehat{C}_l$, $\widehat{H}_l$, *and* $\widehat{P}_l$ *are differential operators* (*see* (3)), *and the integrals are evaluated over any curves joining an arbitrary fixed point on the curve* Γ *with the points* R_{jk} (x_0, y_0) (*see Figure* 1).

4. THE FUNCTIONS $\tilde{g}_{jk}(x, y, x_0, \xi, y_0)$

We proceed to a description of the functions $\tilde{g}_{jk}$. We do this with the help of auxiliary functions $g_j(x, y, x_0, y_0)$. We have the following lemma.

Lemma. *The fundamental solution of equation* (2) (*at least in a neighborhood of the point* (x_0, y_0)) *can be represented in the form*

$$
\begin{aligned}
G(x, y, x_0, y_0) = K_0 \sum_{j=1}^{m} \{ &g_j(x, y, x_0, y_0) \ln(x - x_0 + \lambda_j(y - y_0)) \\
&+ g_j(x, y, x_0, y_0) \ln(x - x_0 + \bar{\lambda}_j(y - y_0)) \} + \cdots,
\end{aligned}
$$

where the dots denote the regular part of the fundamental solution, K_0 *is a known constant, and* g_j *and* g_j *are regular solutions of the adjoint equation* $L^* g_k = 0$,

$k = 1, \ldots, 2m$, having zeros of order $2m-2$ on the characteristics defined by the equation $x - x_0 + \lambda_j(y - y_0) = 0$ or $x - x_0 + \bar{\lambda}_j(y - y_0) = 0$ respectively.

The functions $\tilde{g}_{j\bar{k}}$ for any $j = 1, \ldots, m$ are now determined as solutions of the family of problems with parameter ξ

$$L^* \tilde{g}_{j\bar{k}}(x, y, x_0, \xi, y_0) = 0, \qquad k = 1, \ldots, m;$$
$$\tilde{g}_{j\bar{k}} = 0 \pmod{2m-1}$$

on the characteristic given by the equation

$$S_{z_j z_k}(x + \bar{\lambda}_k y) - (x_0 + \lambda_j y_0) = \xi; \tag{5}$$
$$\sum_{k=1}^{m} \tilde{g}_{jk} + \int_{x - x_0 + \lambda_j(y - y_0)}^{\xi} K_0 g_j(x, y, x_0, \eta, y_0)\, d\eta = 0 \pmod{m}$$

on the curve $\Gamma_{\mathbb{C}}$ defined by

$$x + \lambda_j y - S_{z_j z_k}(x + \bar{\lambda}_k y) = 0.$$

Solutions of problem (5) exist in $\mathbb{C}^4$ at least when the point (x_0, y_0) is located sufficiently close to Γ.

I am deeply grateful to V. E. Shatalov for systematic consultations while carrying out the work, and to B. Yu. Sternin for constant encouragement and support.

Bibliography

1. H. A. Schwarz, Monatsber. Königl. Preuss. Akad. Wiss. Berlin **1870**, 767–795.
2. Philip J. Davis, *The Schwarz function and its applications*, Carus Math. Monographs, vol. 17, Math. Assoc. Amer., Buffalo, NY, 1979.
3. D. Khavinson and H. S. Shapiro, *The Schwarz potential in* $\mathbf{R}^n$ *and Cauchy's problem for the Laplace equation*, Royal Inst. Technology, Stockholm, 1989.
4. Harold S. Shapiro, *The Schwarz function and its generalizations to higher dimensions*, Univ. of Arkansas-Fayetteville Lecture Notes in Math. Sci., vol. 9, Wiley, New York, 1987.
5. D. Khavinson and H. S. Shapiro, J. Analyse Math. **54** (1990), 60–76.
6. P. Garabedian, J. Math. Mech. **9** (1960), 241–271.
7. T. V. Savina, B. Yu. Sternin, and V. E. Shatalov, Dokl. Akad. Nauk SSSR **322** (1992), 48–51; English transl. in Soviet Math. Dokl. **45** (1992).
8. J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod, Paris, 1968.

Moscow State University

Received 19/FEB/93

Translated by J. R. SCHULENBERGER