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A Reflection Formula for the Helmholtz Equation with the Neumann Condition

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Abstract—A reflection formula is proposed for the two-dimensional Helmholtz equation subject to the Neumann condition. In contrast to the classical Schwarz symmetry principle, the formula has a nonlocal (integral) form.

1. STATEMENT OF THE PROBLEM

The well-known Schwarz symmetry principle [1] is as follows.

Let $\Gamma \subset \mathbb{R}^2$ be a real analytic curve and P' be a point on it. Then, there exist a neighborhood U of P' partitioned by Γ into subdomains U_1 and U_2 and a unique anticonformal mapping $R: U \longrightarrow U$ that maps every point of Γ onto itself and U_1 and U_2 onto one another. Under the mapping, any harmonic function vanishing on Γ (Dirichlet condition) is odd; i.e.,

$$u(x_0, y_0) = -u(R(x_0, y_0)) \tag{1.1}$$

at any $(x_0, y_0) \in U$.

Under the Neumann condition ($\partial u/\partial n = 0$), the corresponding symmetry formula is

$$u(x_0, y_0) = u(R(x_0, y_0)) \tag{1.2}$$

(even continuation).

Obviously, if $(x_0, y_0) \in U_1$, then the corresponding "reflected" point $R(x_0, y_0)$ belongs to U_2.

The Schwarz symmetry principle was also discussed in [2–7]. Much attention was given to the actual construction of R. This can be done as follows. Consider a domain W in $\mathbb{C}^2$ and analytically extend the equation $f(x, y) = 0$ of Γ to this domain so that $W \cap \mathbb{R}^2 = U$. In the characteristic (under the Laplace operator) variables $z = x + iy$ and $\zeta = x - iy$, the equation of the complexified curve $\Gamma_\mathbb{C}$ is written as

$$f\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) = 0.$$

If $\operatorname{grad} f(x, y) \neq 0$ on Γ, then the equation of $\Gamma_\mathbb{C}$ (in a neighborhood of a real point) is solvable for both z and ζ. Let the corresponding solutions be $\zeta = S(z)$ and $z = \tilde{S}(\zeta)$. Then, R is defined by the formula

$$R(x, y) = R(z) = \overline{S(z)}. \tag{1.3}$$

The function $S(z)$ is called the Schwarz function of Γ (see, e.g., [2]).

Unfortunately, formulas (1.1) and (1.2) do not hold in such a simple form for spaces of higher dimensions or for more general equations [3]. The possibility of generalizing formula (1.1) was demonstrated in [4]. In [5, 6], such a formula was derived in explicit form for a two-dimensional elliptic equation with constant coefficients of the highest order terms. Moreover, this formula was shown to be true in the large when the equation is of the second order.

The goal of this study is to extend formula (1.2) to the case of the Helmholtz equation.

Statement of the problem. Let U be a given domain in $\mathbb{R}^2$ that is partitioned into subdomains U_1 and U_2 by a real analytic curve Γ described by the equation $f(x, y) = 0$, $df(x, y) \neq 0$. Let $u(x, y)$ be a solution to the Helmholtz equation

$$\Delta_{x,y} u + k^2 u = 0$$

subject to the condition

$$\partial u/\partial n|_\Gamma = 0.$$

It is necessary to express the values of the function $u(x, y)$ at $(x_0, y_0) \in U_1$ in terms of its values in U_2 (i.e., to construct a reflection formula).

Suppose for simplicity that Γ is an algebraic curve. In this case, $S(z)$ and $\tilde{S}(\zeta)$ are analytic functions on the entire plane $\mathbb{C}$ and have algebraic singularities only. Obviously, the mapping R defined by (1.3) is regular everywhere except for the intersection of the singularities of $S(z)$ with $\mathbb{R}^2$.

Theorem 1. *Under the assumptions stated above,*

$$u(x_0, y_0) = u(R(x_0, y_0)) + \frac{1}{2i}\int_{\Gamma}^{R(x_0, y_0)} V(x, y, x_0, y_0)\omega(u(x, y)) - u(x, y)\omega(V(x, y, x_0, y_0)), \quad (1.4)$$

where the integral is calculated along any curve joining an arbitrary point on Γ *with the point* $R(x_0, y_0)$, R *is given by* (1.3), $\omega(\cdot) = \frac{\partial}{\partial y}dx - \frac{\partial}{\partial x}dy$, *and* $V(x, y, x_0, y_0)$ *is defined as*

$$V(x, y, x_0, y_0) = V_1(x, y, x_0, y_0) - V_2(x, y, x_0, y_0)$$

so that V_i $(i = 1, 2)$ *are solutions to certain special problems.*

We prove the theorem in the following three steps:

(i) the original problem is reduced to a special problem;

(ii) the existence of a solution to this problem is proved, and a solution is constructed;

(iii) the properties of the solution constructed are analyzed, and the final formula is derived.

2. REDUCTION OF THE ORIGINAL PROBLEM TO A PROBLEM WITH PRESCRIBED LOCATIONS OF SINGULARITIES

As a starting point, we use the well-known Green's formula, which expresses the value of the solution to the Helmholtz equation at any fixed point $(x_0, y_0) \in U_1 \subset \mathbb{R}^2$ in terms of the values of the solution on a contour γ encompassing this point:

$$u(x_0, y_0) = \int_\gamma G(x, y, x_0, y_0)\omega(u(x, y)) - u(x, y)\omega(G(x, y, x_0, y_0)), \quad (2.1)$$

where $\omega(\cdot) = \frac{\partial}{\partial y}dx - \frac{\partial}{\partial x}dy$ and $G(x, y, x_0, y_0)$ is any fundamental solution. Note that $G(x, y, x_0, y_0)$ has a logarithmic singularity at (x_0, y_0), and its analytic continuation has a logarithmic singularity on the complex characteristics l_1 and l_2 through this point:

$$l_1 = \{\psi_1(x, y) = (x - x_0) + i(y - y_0) = 0\},$$

$$l_2 = \{\psi_2(x, y) = (x - x_0) - i(y - y_0) = 0\}.$$

In particular, $G(x, y, x_0, y_0)$ can be represented as a restriction to $\mathbb{R}^2$ of the sum of two functions:

$$G(x, y, x_0, y_0) = G_1(x, y, x_0, y_0) + G_2(x, y, x_0, y_0), \quad (2.2)$$

where

$$G_1(x, y, x_0, y_0) = -\frac{1}{4\pi}\sum_{j=0}^{\infty}\frac{(-k^2\psi_1\psi_2)^j}{4^j(j!)^2}(\ln\psi_1 - C_j),$$

$$G_2(x, y, x_0, y_0) = -\frac{1}{4\pi}\sum_{j=0}^{\infty}\frac{(-k^2\psi_1\psi_2)^j}{4^j(j!)^2}(\ln\psi_2 - C_j), \quad (2.3)$$

$$C_0 = 0, \quad C_j = \sum_{l=1}^{j}\frac{1}{l}, \quad j = 1, 2, \dots . \quad (2.4)$$

Obviously, each of the functions defined by (2.3) has a singularity only on one of the characteristics.

Thus, formula (2.1) expresses the values of an arbitrary solution to the Helmholtz equation at a point $(x_0, y_0) \in U_1$ in terms of the values of this solution on a contour γ lying within U_1. Our goal will be achieved when we deform γ from U_1 into U_2. This is done in two steps. First, γ is transformed into a contour γ' lying on the complexification $\Gamma_{\mathbb{C}}$ of Γ. Next, the fundamental solution in the integrand of (2.1) is replaced by a function $\tilde{G}(x, y, x_0, y_0)$ whose singularities are located in such a way that γ' can be deformed from $\Gamma_{\mathbb{C}}$ into U_2 and satisfies the condition $\omega(\tilde{G}) = \omega(G)$ on $\Gamma_{\mathbb{C}}$.

Note that the value of integral (2.1) does not change if γ is deformed into a homological contour (the integrand in (2.1) is a closed form). Assuming that (x_0, y_0) is sufficiently close to Γ, we can deform the contour γ into $\gamma' \subset \Gamma_{\mathbb{C}}$ by the method described in [5].

Note that the first term in (2.1) vanishes on $\Gamma_{\mathbb{C}}$; therefore, formula (2.1) takes the form

$$u(x_0, y_0) = -\int_{\gamma'} u(x, y)\omega(G(x, y, x_0, y_0)). \tag{2.5}$$

In view of (2.2), formula (2.5) is rewritten as

$$u(x_0, y_0) = -\int_{\gamma'} u(x, y)\omega(G_1(x, y, x_0, y_0)) - \int_{\gamma'} u(x, y)\omega(G_2(x, y, x_0, y_0)). \tag{2.6}$$

To deform γ' into U_2, one must replace G_1 and G_2 by functions $\tilde{G}_1$ and $\tilde{G}_2$ whose singularities lie on the "reflected" characteristics $\tilde{l}_1$ and $\tilde{l}_2$, which cross $\mathbb{R}^2$ at points of U_2. Moreover, it is obvious that the relations $\omega(\tilde{G}_i) = \omega(G_i)$ must be satisfied on $\Gamma_{\mathbb{C}}$ $(i = 1, 2)$. Note that γ' is not closed on the Riemann surfaces of the terms on the right of (2.6). Therefore, $\tilde{\gamma} \subset U_2$ is also an open contour for an arbitrary function: the boundary of $\tilde{\gamma}$ belongs to Γ.

Thus, Green's formula (2.1) can be rewritten as

$$\begin{aligned} u(x_0, y_0) = {} & \int_{\tilde{\gamma}} [\tilde{G}_1(x, y, x_0, y_0)\omega(u(x, y)) - u(x, y)\omega(\tilde{G}_1(x, y, x_0, y_0))] \\ & + \int_{\tilde{\gamma}} [\tilde{G}_2(x, y, x_0, y_0)\omega(u(x, y)) - u(x, y)\omega(\tilde{G}_2(x, y, x_0, y_0))], \end{aligned} \tag{2.7}$$

where $\tilde{\gamma}$ is a contour encompassing $R(x_0, y_0)$, and $\tilde{G}_1$ and $\tilde{G}_2$ are solutions to the problems

$$\begin{aligned} \Delta_{x, y}\tilde{G}_i(x, y, x_0, y_0) + k^2\tilde{G}_i(x, y, x_0, y_0) &= 0, \\ \omega(\tilde{G}_i(x, y, x_0, y_0)) &= \omega(G_i(x, y, x_0, y_0)) \quad \text{on} \quad \Gamma_{\mathbb{C}}. \end{aligned} \tag{2.8}$$

Here, $\tilde{G}_i(x, y, x_0, y_0)$ has singularities only on the reflected characteristic $\tilde{l}_i$.

Following Garabedian, we call the function $\tilde{G} = \tilde{G}_1 + \tilde{G}_2$ a reflected fundamental solution. (Note that a reflected fundamental solution is generally not a fundamental solution.)

3. REFLECTED FUNDAMENTAL SOLUTION

In this section, we solve problem (2.8), to which the construction of a reflection formula was reduced in the preceding section. For convenience, we change to the characteristic variables $z = x + iy$ and $\zeta = x - iy$, in which problem (2.8) is written as

$$\begin{aligned} \frac{\partial^2 \tilde{G}_i}{\partial z \partial \zeta} + \frac{k^2}{4}\tilde{G}_i &= 0, \quad i = 1, 2, \\ \omega^*(\tilde{G}_i) &= \omega^*(G_i) \quad \text{on} \quad \Gamma_{\mathbb{C}}, \end{aligned} \tag{3.1}$$

$\tilde{G}_i$ has singularities only on the reflected characteristic $\tilde{l}_i = \{\tilde{\psi}_i = 0\}$.

Here,

$$\omega^*(\tilde{G}_j) = i\left(\frac{\partial \tilde{G}_j}{\partial z}dz - \frac{\partial \tilde{G}_j}{\partial \zeta}d\zeta\right),$$
$$\tilde{\psi}_1(\zeta) = \tilde{S}(\zeta) - z_0, \quad \psi_2(z) = S(z) - \zeta_0, \tag{3.2}$$

where $S(z)$ is the Schwarz function.

Solutions to (3.1) are sought in the form (see [8])

$$\tilde{G}_i(z, \zeta, z_0, \zeta_0) = \frac{1}{4\pi}\sum_{j=0}^{\infty} b_j^i(z, \zeta, z_0, \zeta_0) f_j(\tilde{\psi}_i), \tag{3.3}$$

where

$$f_j(\xi) = \begin{cases} (-1)^{-j-1}(-j-1)!\xi^j, & j \le -1, \\ \dfrac{\xi^j}{j!}(\ln\xi - C_j), & j = 0, 1, \dots, \end{cases} \tag{3.4}$$

and C_j are the constants given by (2.4).

Let us illustrate the calculation of the coefficients in (3.3) using $\tilde{G}_2$ as an example. Substituting (3.3) into (3.1) and setting the coefficients of f_j equal to zero ((3.3) can be viewed as an expansion with respect to smoothness), we obtain the following recursion relations for the coefficients b_j:

$$b_0 = -1,$$
$$\frac{\partial b_{j+1}}{\partial \zeta}S'(z) = -\frac{\partial^2 b_j}{\partial \zeta \partial z} - \frac{k^2}{4}b_j, \tag{3.5}$$
$$b_{j+1}S'(z)|_{\zeta = S(z)} = S'(z)\frac{\partial b_j}{\partial \zeta} - \left.\frac{\partial b_j}{\partial z}\right|_{\zeta = S(z)} + \frac{(-k^2)^j(z-z_0)^{j-1}}{4^{j+1}(j+1)!}[4j(j+1) + S'(z)k^2(z-z_0)^2].$$

Thus, the formal series is constructed. Let us now analyze its convergence.

Lemma 1. *Series* (3.3) *is convergent in the neighborhood of* Γ.

Proof. Consider an auxiliary family of problems depending on a parameter η:

$$\frac{\partial^2 v(z, \zeta, \eta)}{\partial z \partial \zeta} + \frac{k^2}{4}(z, \zeta, \eta) = 0,$$
$$\left(S'(z)\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial z}\right)v(z, S(z), \eta) = S'(z) + \Phi(z, \eta), \tag{3.6}$$
$$v(\tilde{S}(\zeta_0 - \eta), \zeta, \eta) = 0,$$

where

$$\Phi(z, \eta) = -\sum_{j=0}^{\infty}[4j(j+1) + S'(z)k^2(z-z_0)^2]\frac{(-k^2)^j(z-z_0)^{j-1}[S(z) - \zeta_0 + \eta]^{j+1}}{4^{j+1}[(j+1)!]^2}.$$

The Taylor expansion of the solution to problem (3.6) (if it exists) is given by

$$v(z, \zeta, \eta) = \sum_{j=0}^{\infty} b_j(z, \zeta)\frac{[S(z) - \zeta_0 + \eta]^{j+1}}{(j+1)!}; \tag{3.7}$$

here, the coefficients b_j are equal to those in series (3.3), because b_j satisfy relations (3.5).

Now, let us prove that problem (3.6) has a holomorphic solution in the neighborhood of Γ. The solution is sought in the form

$$v(z, \zeta, \eta) = \int\limits_{\tilde{S}(\zeta_0 - \eta)}^{z} \mu(\beta, \eta)[\zeta + S(z) - 2S(\beta)]d\beta - [S(z) - \zeta_0 + \eta] - \int\limits_{\tilde{S}(\zeta_0 - \eta)}^{z} \Phi(\alpha, \eta)d\alpha.$$

For the density μ, we obtain the Volterra integral equation

$$\mu(z, \eta) + \frac{k^2}{4}\int\limits_{\tilde{S}(\zeta_0 - \eta)}^{z} \mu(\beta, \eta)[\zeta + S(z) - 2S(\beta)]d\beta = \tilde{\Phi}(z, \eta), \qquad (3.8)$$

where

$$\tilde{\Phi}(z, \eta) = \frac{k^2}{4}\left(S(z) - \zeta_0 + \eta + \int\limits_{\tilde{S}(\zeta_0 - \eta)}^{z} \Phi(\alpha, \eta)d\alpha\right).$$

Equations of form (3.8) in a complex domain were analyzed in [9, 10]. The results obtained in [9, 10] imply that a unique solution to (3.8) exists in a neighborhood of Γ. Moreover, the function $v(z, \zeta, \eta)$ is analytic in this neighborhood; hence, it can be represented as a Taylor series about $\eta = -[S(z) + \zeta_0]$, which coincides with expansion (3.7). The convergence of (3.7) implies the convergence of (3.3) at the points z where $|S(z)| \le r - \delta$. Here, r is the radius of convergence of (3.7) and $\delta > 0$.

4. PROPERTIES OF THE REFLECTED FUNDAMENTAL SOLUTION

In this section, we analyze the branching of the reflected fundamental solution around its singularities. Note that the classical fundamental solution of the Helmholtz operator does not pass to another sheet of the Riemann surface after its argument describes a closed contour around two branch lines. Here, we show that this property does not hold for the reflected fundamental solution even if the contour lies in $\mathbb{R}^2$.

Let us calculate the change in $\tilde{G}$ that results after its argument describes a circle of a small radius ρ around the point $(\tilde{S}(\bar{z}_0), S(z_0))$. To do this, we set $z = \tilde{S}(\zeta_0) + \rho\exp(i\varphi)$ and $\zeta = S(z_0) + \rho\exp(-i\varphi)$ in the formula

$$\tilde{G} = -\frac{1}{4\pi}\sum_{j=0}^{\infty} b_j^1\frac{[\tilde{S}(\zeta) - z_0]^j}{j!}[\ln(\tilde{S}(\zeta) - z_0) - C_j] - \frac{1}{4\pi}\sum_{j=0}^{\infty} b_j^2\frac{[S(z) - \zeta_0]^j}{j!}[\ln(S(z) - \zeta_0) - C_j] \qquad (4.1)$$

and represent the functions $S(z)$ and $\tilde{S}(\zeta)$ as Taylor series about the point $(\tilde{S}(\zeta_0), S(z_0))$:

$$S(z) = \zeta_0 + a_1\rho\exp(i\varphi) + o(\rho), \quad \tilde{S}(\zeta) = z_0 + \bar{a}_1\rho\exp(-i\varphi) + o(\rho).$$

The increment of (4.1) on the real plane $\zeta = \bar{z}$ is

$$2\pi i\left(-\frac{1}{4\pi}\right)\sum_{j=0}^{\infty}\left(b_j^1\frac{(\tilde{S}(\zeta) - z_0)^j}{j!} - b_j^2\frac{(S(z) - \zeta_0)^j}{j!}\right)$$

$$= \frac{ik^2}{8}\{\bar{a}_1\rho\exp(-i\varphi)(a_1 z_0 - a_1\tilde{S}(\bar{z}_0) - S(z_0) + \bar{z}_0) \qquad (4.2)$$

$$- a_1\rho\exp(i\varphi)(\bar{a}_1\bar{z}_0 - \bar{a}_1 S(z_0) - \tilde{S}(\bar{z}_0) + z_0) + \tilde{O}(\rho)\}.$$

This expression allows one to find out when the reflected fundamental solution does not pass to another sheet of the Riemann surface. This occurs when $k = 0$, which corresponds to the Laplace equation and when $a_j = \bar{a}_j = 0$ for $j \ge 2$ and $S(z) = a_1 z$, which means that Γ is a straight line segment. In other cases, the reflected fundamental solution passes to another sheet of the Riemann surface after its argument describes a closed curve around a singularity of $\tilde{G}$.

5. REFLECTION FORMULA AND REMARKS

Thus, the reflection formula (2.7) is written in terms of z and ζ as

$$u(x_0, y_0) = 4\int_{\tilde{\gamma}} \tilde{G}\omega^*(u) - u\omega^*(\tilde{G}), \tag{5.1}$$

where $\tilde{G} = \tilde{G}_1 + \tilde{G}_2$ is the reflected fundamental solution, $\omega^*(\cdot)$ is defined by (3.2), and $\tilde{\gamma}$ is the contour shown in the figure. Talking into account the structure of the reflected fundamental solution, we can bring this formula to its final form (1.4). Indeed, let us represent $\tilde{G}_i$ as (see (3.3), (3.4))

$$\tilde{G}_i = -\frac{1}{4\pi}(V_i \ln\tilde{\psi}_i - \tilde{V}_i), \tag{5.2}$$

where

$$V_i = \sum_{j=0}^{\infty} b_j^i \frac{(\tilde{\psi}_i)^j}{j!}, \quad \tilde{V}_i = -\sum_{j=0}^{\infty} b_j^i \frac{(\tilde{\psi}_i)^j}{j!} C_j, \quad i = 1, 2.$$

Obviously, the integrals containing $\tilde{V}_i$ vanish when (5.2) is substituted into (5.1) and the length of the arc PM tends to zero (see figure). The integrals containing logarithms can be transformed as in [5]. As a result, we obtain the final formula

$$u(x_0, y_0) = u(R(x_0, y_0)) + \frac{1}{2i}\int_{Q}^{R(x_0, y_0)} (V_1 - V_2)\omega^*(u) - u\omega^*(V_1 - V_2). \tag{5.3}$$

Remark 1. Since the integrand in (5.3) is a closed form and all of its terms vanish on $\Gamma_{\mathbb{C}}$, it follows that Q can be any point lying on Γ.

Remark 2. It follows from (2.8) and (5.2) that the functions V_i can be interpreted as solutions to the problems

$$\begin{gathered} \frac{\partial^2 V_i}{\partial z \partial \zeta} + \frac{k^2}{4} V_i = 0, \quad i = 1, 2, \\ \omega^*(V_i) = \omega^*(V_0) \text{ on } \Gamma_{\mathbb{C}}, \\ V_i = -1 \text{ on the characteristic } \tilde{l}_i, \end{gathered} \tag{5.4}$$

where $V_0(z, \zeta, z_0, \zeta_0)$ is the Riemann function for the Helmholtz operator. It can be shown (see, e.g., [9, 10]) that solutions to (5.4) exist in the entire space $\mathbb{C}^4$ and are set-valued analytic functions whose singularities with respect to (z, ζ) and (z_0, ζ_0) are identical to the singularities of S and $\tilde{S}$. Thus, the right-hand side of (5.3), defined originally in the small, is actually defined in the large. By the uniqueness theorem for analytic functions, this entails the validity of (5.3) in the large.

Remark 3. For the Laplace equation ($k = 0$), the integrand in (5.3) vanishes, and we have the classical result

$$u(x_0, y_0) = u(R(x_0, y_0)).$$

Remark 4. It easy to see that, if Γ is a straight line segment, then $V_1 = V_2$ and the integral on the right-hand side of (5.3) vanishes.

6. REFLECTION FORMULA FOR A NONHOMOGENEOUS CONDITION ON Γ

In the preceding sections, we obtained a reflection formula for the solution to the Helmholtz equation with normal derivative vanishing on Γ (homogeneous Neumann condition). In this section, formula (5.3) is extended to the case of a nonhomogeneous Neumann condition. A formula corresponding to the Dirichlet condition was given in [7, Chapter 6].

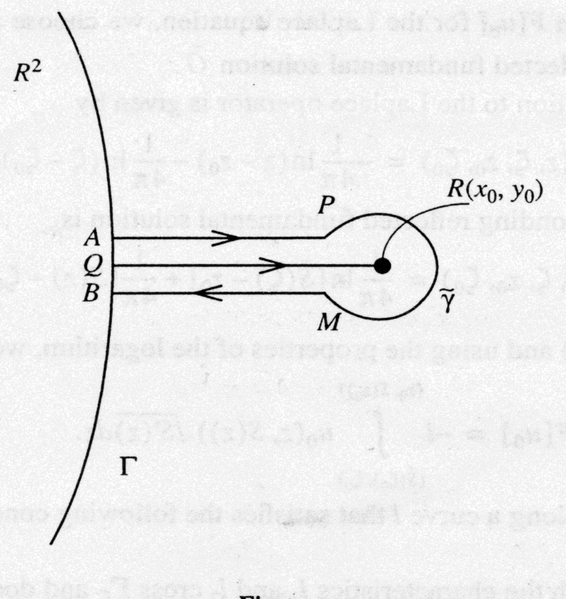

Figure.

Let $u(x, y)$ be a solution to the Helmholtz equation

$$(\Delta_{x,y} + k^2)u(x, y) = 0$$

subject to the condition

$$\left.\frac{\partial u(x, y)}{\partial n}\right|_{\Gamma} = u_0(x, y)|_{\Gamma}, \tag{6.1}$$

where $u_0(x, y)$ is a given function that is holomorphic at least in the neighborhood of Γ. Again, we represent the value of $u(x, y)$ at a point $(x_0, y_0) \in U_1$ as

$$u(x_0, y_0) = \int_{\gamma'} G\omega^*(u) - u\omega^*(G), \tag{6.2}$$

where $\gamma' \subset \Gamma_C$. However, since $u(x, y)$ satisfies the nonhomogeneous condition (6.1), we cannot merely substitute $\tilde{G}$ for G in (6.2) (the first term in (6.2) does not vanish on Γ_C). It is necessary to add an additional term depending on u_0:

$$u(x_0, y_0) = \int_{\gamma'} [\tilde{G}\omega^*(u) - u\omega^*(\tilde{G})] + F[u_0], \tag{6.3}$$

where

$$F[u_0] = \int_{\gamma'} (G - \tilde{G})u_0(z, S(z))\sqrt{S'(z)}dz \tag{6.4}$$

is a given function at any point (x_0, y_0). The first term in (6.3) can be transformed by the technique described in Section 5. Thus, the final reflection formula for the Helmholtz equation with a nonhomogeneous Neumann condition is given by

$$u(x_0, y_0) = u(R(x_0, y_0)) + \frac{1}{2i}\int_{\Gamma}^{R(x_0, y_0)} [V(x, y, x_0, y_0)\omega(u(x, y))$$

$$- u(x, y)\omega(V(x, y, x_0, y_0))] + F[u_0(x_0, y_0)].$$

To calculate the additional term $F[u_0]$ for the Laplace equation, we choose a fundamental solution G and construct the corresponding reflected fundamental solution $\tilde{G}$.

A suitable fundamental solution to the Laplace operator is given by

$$G(z, \zeta, z_0, \zeta_0) = -\frac{1}{4\pi}\ln(z - z_0) - \frac{1}{4\pi}\ln(\zeta - \zeta_0).$$

It is easy to see that the corresponding reflected fundamental solution is

$$\tilde{G}(z, \zeta, z_0, \zeta_0) = \frac{1}{4\pi}\ln[\tilde{S}(\zeta) - z_0] + \frac{1}{4\pi}[S(z) - \zeta_0].$$

Substituting G and $\tilde{G}$ into (6.4) and using the properties of the logarithm, we obtain

$$F[u_0] = -i\int_{(\tilde{S}(\zeta_0),\, \zeta_0)}^{(z_0,\, S(z_0))} u_0(z, S(z))\sqrt{S'(z)}dz. \tag{6.5}$$

Here, the integral is evaluated along a curve l that satisfies the following conditions:

(i) $l \subset \Gamma_C$;

(ii) l joins the points at which the characteristics l_1 and l_2 cross Γ_C and does not go around them.

Formula (6.5) can be verified by direct calculation. Indeed, it is well known that the solution to the Laplace equation

$$\partial^2 u/\partial z\partial\zeta = 0$$

can be represented as

$$u(z, \zeta) = g(z) + f(\zeta).$$

Hence, we have

$$u(z_0, \zeta_0) - u(\tilde{S}(\zeta_0)) = g(z_0) - g(\tilde{S}(\zeta_0)) + f(\zeta_0) - f(S(z_0)).$$

The Neumann condition on Γ_C is written in terms of (z, ζ) as

$$\left(\frac{\partial g}{\partial z} - S'(z)\frac{\partial f}{\partial \zeta}\right) = -i\sqrt{S'(z)}u_0(z, S(z)),$$

which yields

$$\frac{\partial g}{\partial z} = -iu_0\sqrt{S'} + S'\frac{\partial f}{\partial \zeta}.$$

Integrating this expression along Γ_C

$$\int_{(\tilde{S}(\zeta_0),\, \zeta_0)}^{(z_0,\, S(z_0))} \frac{\partial g}{\partial z}dz = -i\int_{(\tilde{S}(\zeta_0),\, \zeta_0)}^{(z_0,\, S(z_0))} u_0\sqrt{S'}dz + \int_{(\tilde{S}(\zeta_0),\, \zeta_0)}^{(z_0,\, S(z_0))} \frac{\partial f}{\partial \zeta}S'dz,$$

we obtain

$$g(z_0) - g(\tilde{S}(\zeta_0)) = -i\int_{(\tilde{S}(\zeta_0),\, \zeta_0)}^{(z_0,\, S(z_0))} u_0\sqrt{S'}dz + f(S(z_0)) - f(\zeta_0),$$

which finally yields

$$u(z_0, \zeta_0) = u(\tilde{S}(\zeta_0), S(z_0)) - i\int_{(\tilde{S}(\zeta_0),\, \zeta_0)}^{(z_0,\, S(z_0))} u_0(z, S(z))\sqrt{S'(z)}dz. \tag{6.6}$$

Let us present the simplest example of an application of (6.6). Suppose that a harmonic function $u(x, y)$ is defined on the upper half-plane and satisfies the condition

$$\partial u/\partial y|_{y=0} = c = \text{const}.$$

It is necessary to extend this function harmonically to the lower half-plane. In this case, we have $S(z) = z$, $\tilde{S}(\zeta) = \zeta$, and $u_0 = c$; and formula (6.6) takes the form

$$u(x_0, y_0) = u(x_0, -y_0) + 2cy_0.$$

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