THE SCHWARZ REFLECTION PRINCIPLE FOR POLYHARMONIC FUNCTIONS IN $\mathbb{R}^2$

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Abstract. A reflection formula for polyharmonic functions in $\mathbb{R}^2$ is suggested. The obtained formula generalizes the celebrated Schwarz reflection principle for harmonic functions to polyharmonic functions. We also offer modification of the obtained formula to the case of nonhomogeneous data on a reflecting curve.

1. Introduction

In this paper we give a generalization of the well known Schwarz reflection principle for harmonic functions to polyharmonic functions, where, a function $u(x, y)$ of class $C^{2p}(U)$ is said to be polyharmonic function of order p if it is a solution of the equation $\Delta^p u = 0$, where U is a domain in $\mathbb{R}^2$, p is a positive integer and Δ^p denotes the $p-th$ iterate of the Laplacian. It is well known that if u is polyharmonic function in U, then it is real analytic throughout U.

The Schwarz reflection principle for harmonic functions can be stated as follows.

Let $\Gamma \subset \mathbb{R}^2$ be a non-singular real analytic curve and $P' \in \Gamma$. Then, there exists a neighborhood U of P' and an anticonformal mapping $R : U \to U$ which is identity on Γ, permutes the components U_1, U_2 of $U \setminus \Gamma$ and relative to which any harmonic function $u(x, y)$ defined near Γ and vanishing on Γ is odd; i.e.,

$$u(x_0, y_0) = -u(R(x_0, y_0)) \tag{1.1}$$

for any point (x_0, y_0) sufficiently close to Γ. Note that if the point $(x_0, y_0) \in U_1$, then the "reflected" point $R(x_0, y_0) \in U_2$.

The Schwarz reflection principle has been studied by several researchers (see [1] – [17] and references there). In particular, the construction of the mapping R has been considered, e.g., in [1]. To describe the mapping R we consider a complex domain V in the space $\mathbb{C}^2$ to which the function f defining the curve Γ can be continued analytically such that $V \cap \mathbb{R}^2 = U$. Using the change of variables $z = x + iy$, $w = x - iy$, the equation of the complexified curve $\Gamma_{\mathbb{C}}$ can be rewritten in the form

$$f\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) = 0. \tag{1.2}$$

If $grad\, f(x, y) \neq 0$ on Γ, (1.2) can be solved with respect to z or w; the corresponding solutions we denote by $w = S(z)$ and $z = \widetilde{S}(w)$. The function $S(z)$ is called the *Schwarz function* of the curve Γ [1]. In these terms, the mapping R mentioned above is given by

$$R(x_0, y_0) = R(z_0) = \overline{S(z_0)}. \tag{1.3}$$

Key words and phrases. Reflection principle, polyharmonic functions.

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Observe that the mapping R depends only on the curve Γ and is defined only near Γ but may have conjugate-analytic continuation to a larger domain.

Formula (1.1) has been generalized to cover several other situations. For the case when Γ is a line, H. Poritsky [2] proved that a biharmonic function $u(x,y)$, i.e., a solution u of the biharmonic equation $\Delta^2_{x,y}u = 0$, defined for $y \geq 0$ and satisfying the conditions

$$u(x,0) = \frac{\partial u}{\partial y}(x,0) = 0$$

can be continued across the x-axis using the formula

$$u(x_0,y_0) = -u(R(x_0,y_0)) - 2y_0\frac{\partial u}{\partial y}(R(x_0,y_0)) - y_0^2\Delta_{x,y}u(R(x_0,y_0)), \tag{1.4}$$

where $R(x_0,y_0) = (x_0,-y_0)$ and $\Delta_{x,y} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. He also applied this formula to problems of planar elasticity. An analogous formula has been obtained by R.J. Duffin [3] for three-dimensional case. Duffin also considered spherical boundaries and applied his result to study viscous flows, among other things. A. Huber [4] has generalized formula (1.4) for polyharmonic functions of the form $u(\overline{x},y)$, where $\overline{x}$ denotes n-dimensional vector, having vanishing (Dirichlet) data on the hyperplane $y = 0$. He showed that such u satisfies the reflection law

$$u(\overline{x}_0,-y_0) = \sum_{m=0}^{p-1}\frac{(-y_0)^{p+m}}{(m!)^2}\,\Delta^m_{x,y}\Big(\frac{u(\overline{x}_0,y_0)}{y_0^{p-m}}\Big), \tag{1.5}$$

where p is the order of polyharmonicity of u. For a circular boundary on which a biharmonic function $u(x,y)$ satisfies the conditions

$$u = \frac{\partial u}{\partial r} = 0 \quad \text{for} \quad x^2+y^2=\rho^2,$$

J. Bramble [5] has shown that analogous to (1.4) u can be continued using the formula

$$\begin{aligned} u(x_0,y_0) =& -u(R(x_0,y_0)) \\ & -\frac{r_0^2-\rho^2}{r_0^2}\Big(r_0\frac{\partial u}{\partial r}u(R(x_0,y_0)+\frac{1}{4}(r_0^2-\rho^2)\Delta_{x,y}u(R(x_0,y_0))\Big), \end{aligned} \tag{1.6}$$

where $r_0 = \sqrt{x_0^2+y_0^2}$ and ρ is the radius of the circle. Papers by F. John [6] and L. Nystedt [7] are devoted to further studies of reflection of solutions of linear partial differential equations with various linear conditions on a hyperplane.

Continuation of polyharmonic functions in two variables across analytic curves has been considered by J. Sloss [8] and R. Kraft [9]. Using different methods of H. Lewi [10], they obtained a number of boundary conditions that guarantee the existence of a continuation, but they did not carry out any explicit formulas giving such continuation.

The purpose of this paper is to obtain a reflection formula for polyharmonic functions across real analytic curves in $\mathbb{R}^2$ and to investigate properties of the mapping induced by the formula (see the next two sections). By a reflection formula we mean a formula expressing the value of a function $u(x,y)$ at an arbitrary point $(x_0,y_0)\in U_1$ in terms of its values at points in U_2. Note that though all the formulas mentioned above are point-to-point, this situation seems quite rare for solutions of partial differential equations. In particular, for solutions of the Helmholtz equation $(\Delta_{x,y}+k^2)u(x,y) = 0$ vanishing on a curve Γ, point-to-point reflection in the sense of the Schwarz reflection principle holds only when Γ is a line, while for harmonic functions in $\mathbb{R}^3$ it holds only when Γ is either a plane or a sphere [11], [12]. The paper by P. Ebenfelt and D. Khavinson [12] is devoted to further study of point-to-point reflection for harmonic functions. There, it was shown that point-to-point reflection in the sense of the Schwarz reflection principle is very rare in $\mathbb{R}^n$ when $n > 3$ is even, and that it never holds when $n \geq 3$ is odd, unless Γ is a sphere or a hyperplane. Reflection properties of solutions of the Helmholtz equation have also been considered in [13], [14] and [15].

2. Reflection formula for biharmonic functions

In this section we consider partial case of reflection formula for polyharmonic functions — reflection formula for biharmonic functions.

Suppose $u(x,y)$, defined in a sufficiently small neighborhood U of a non-singular real analytic curve Γ defined by the equation $f(x,y) = 0$, is a solution of the problem,

$$\begin{cases} \Delta^2_{x,y} u(x,y) = 0 \text{ near } \Gamma \\ u(x,y)_{|\Gamma} = 0 \pmod 2, \end{cases} \tag{2.1}$$

where, we use the notation $u(x,y)_{|\Gamma} = 0 \pmod 2$ if u and its derivatives of order less than 2 vanish on Γ. Let U_1, U_2 denote components of $U \setminus \Gamma$. Our aim is to express the value of $u(x,y)$ at an arbitrary point $P(x_0,y_0) \in U_1$ in terms of its values in U_2.

For simplicity, we assume Γ is an algebraic curve. Under this assumption, the Schwarz function and its inverse are analytic in the whole plane $\mathbb{C}$ except for finitely many algebraic singularities.

Theorem 2.1. *Under the assumptions formulated above, the following reflection formula holds:*

(2.2)

$$\begin{aligned} u(P) = -u(Q) &- \Big(x_0 - \frac{S(x_0+iy_0) + \widetilde{S}(x_0-iy_0)}{2}\Big)\frac{\partial u}{\partial x}(Q) \\ &- \Big(y_0 + \frac{S(x_0+iy_0) - \widetilde{S}(x_0-iy_0)}{2i}\Big)\frac{\partial u}{\partial y}(Q) - \frac{1}{4}\big(x_0^2 + y_0^2 - S(x_0+iy_0)(x_0+iy_0) \\ &- \widetilde{S}(x_0-iy_0)(x_0-iy_0) + S(x_0+iy_0)\widetilde{S}(x_0-iy_0)\big)\Delta_{x,y}u(Q), \end{aligned}$$

where $P = (x_0, y_0)$ *and* $Q = R(P)$.

Proof. To prove this theorem we use the idea suggested by Garabedian [16], to start from Green's formula, expressing the value of a solution of an arbitrary linear p.d.e. at a point P via the values of this solution on a contour $\gamma \subset U_1$ surrounding the point P. The corresponding formula for biharmonic functions is

$$\begin{aligned} u(P) = \int_\gamma &\left(G\frac{\partial \Delta u}{\partial y} - \Delta u \frac{\partial G}{\partial y} + \Delta G \frac{\partial u}{\partial y} - u\frac{\partial \Delta G}{\partial y}\right) dx \\ &- \left(G\frac{\partial \Delta u}{\partial x} - \Delta u \frac{\partial G}{\partial x} + \Delta G \frac{\partial u}{\partial x} - u\frac{\partial \Delta G}{\partial x}\right) dy, \end{aligned} \tag{2.3}$$

where $\Delta = \Delta_{x,y}$ and $G = G(x, y, x_0, y_0)$ is an arbitrary fundamental solution of the bi-Laplacian. The most suitable one for what follows is

$$G = -\frac{1}{16\pi}((x-x_0)^2 + (y-y_0)^2)\ln((x-x_0)^2 + (y-y_0)^2).$$

It is obvious that G is analytic function in $\mathbb{R}^2$ except at the point $P(x_0, y_0)$. Its continuation to the complex space has logarithmic singularities on the complex characterstics passing through this point, i.e., on $K_P := \{(x-x_0)^2 + (y-y_0)^2 = 0\}$. In characteristic coordinates G can be rewritten as

$$\begin{aligned} G(z, w, z_0, w_0) &= -\frac{1}{16\pi}(G_1(z, w, z_0, w_0) + G_2(z, w, z_0, w_0)), \quad \text{where} \\ G_1 = (z-z_0)(w-w_0)\ln(z-z_0), &\quad G_2 = (z-z_0)(w-w_0)\ln(w-w_0). \end{aligned} \tag{2.4}$$

Our goal will be achieved if we can deform the contour γ from the domain U_1 to the domain U_2. Note that since the integrand in (2.3) is a closed form, the value of the integral does not change while we deform the contour γ homotopically. We deform it first to the complexified curve $\Gamma_{\mathbb{C}}$. This deformation is possible if the point P lies so close to the curve Γ that there exists a connected domain $\Omega \subset \Gamma_{\mathbb{C}}$ such that

(i) Ω contains both points of intersections of the characterstic lines passing through the point P and,

(ii) Ω can be univalently projected onto a plane domain (for details, see [15]).

Taking into account conditions (2.1), formula (2.3) can be rewritten in the form

$$u(P) = \int_{\gamma\prime} \left(G\frac{\partial \Delta u}{\partial y} - \Delta u \frac{\partial G}{\partial y}\right) dx - \left(G\frac{\partial \Delta u}{\partial x} - \Delta u \frac{\partial G}{\partial x}\right) dy, \tag{2.5}$$

where contour $\gamma\prime \subset \Omega$ is homotopic to γ in $\mathbb{C}^2 \setminus \{(x-x_0)^2 + (y-y_0)^2 = 0\} =: \mathbb{C}^2 \setminus K_P$. To deform the contour $\gamma\prime$ from $\Gamma_{\mathbb{C}}$ to the real domain U_2 we can replace the fundamental solution by the so called *reflected fundamental solution* $\widetilde{G}$ [16], which must be a biharmonic function satisfying on $\Gamma_{\mathbb{C}}$ the condition $G - \widetilde{G} = 0$ (mod 2) and having singularities only on the characteristic lines intersecting the real space at point $Q = R(P)$ in the domain U_2 and intersecting $\Gamma_{\mathbb{C}}$ at $K_P \cap \Gamma_{\mathbb{C}}$. If we find such a function, we will be able to deform contour to the domain U_2 and the value of the integral does not change. It is easy to verify that the following function satisfies the conditions mentioned above:

$$\begin{aligned} \widetilde{G}(z, w, z_0, w_0) &= -\frac{1}{16\pi}(\widetilde{G}_1(z, w, z_0, w_0) + \widetilde{G}_2(z, w, z_0, w_0)) \text{ where,} \\ \widetilde{G}_1 &= (z-z_0)(w-w_0)\ln(\widetilde{S}(w) - z_0) + (z - \widetilde{S}(w))(w-w_0), \\ \widetilde{G}_2 &= (z-z_0)(w-w_0)\ln(S(z) - w_0) + (w - S(z))(z-z_0). \end{aligned} \tag{2.6}$$

With this change, we can deform the contour $\gamma\prime$ from the complexified curve $\Gamma_{\mathbb{C}}$ to the real domain U_2 [15]. As a result, we obtain

$$
\begin{aligned}
u(P) = \int\limits_{\widetilde{\gamma}} & \left(\widetilde{G}\frac{\partial \Delta u}{\partial y} - \Delta u \frac{\partial \widetilde{G}}{\partial y} + \Delta\widetilde{G}\frac{\partial u}{\partial y} - u\frac{\partial \Delta \widetilde{G}}{\partial y}\right) dx \\
& - \left(\widetilde{G}\frac{\partial \Delta u}{\partial x} - \Delta u \frac{\partial \widetilde{G}}{\partial x} + \Delta\widetilde{G}\frac{\partial u}{\partial x} - u\frac{\partial \Delta \widetilde{G}}{\partial x}\right) dy,
\end{aligned}
\tag{2.7}
$$

where $\widetilde{\gamma} \subset U_2$ is a contour that surrounds the point Q and has endpoints on the curve Γ. Formula (2.7) in characteristic variables has the form,

$$
\begin{aligned}
u(P) = 4i \int\limits_{\widetilde{\gamma}} & \left(\widetilde{G}\frac{\partial^3 u}{\partial z^2 \partial w} + \frac{\partial^2 \widetilde{G}}{\partial z \partial w}\frac{\partial u}{\partial z} - u\frac{\partial^3 \widetilde{G}}{\partial z^2 \partial w} - \frac{\partial^2 u}{\partial z \partial w}\frac{\partial \widetilde{G}}{\partial z}\right) dz \\
& - \left(\widetilde{G}\frac{\partial^3 u}{\partial z \partial w^2} + \frac{\partial^2 \widetilde{G}}{\partial z \partial w}\frac{\partial u}{\partial w} - u\frac{\partial^3 \widetilde{G}}{\partial z \partial w^2} - \frac{\partial^2 u}{\partial z \partial w}\frac{\partial \widetilde{G}}{\partial w}\right) dw.
\end{aligned}
\tag{2.8}
$$

If we substitute (2.6) into (2.8) and move one endpoint of the contour $\widetilde{\gamma}$ along the curve Γ to the other endpoint, integral terms containing products of the function u and regular part of the function $\widetilde{G}$ and their derivatives vanish. Integral terms containing logarithms can be combined and written as,

$$
\begin{aligned}
& \int\limits_{\widetilde{\gamma}} (\ln(S(z) - w_0) + \ln(\widetilde{S}(w) - z_0))\Big\{\big((z - z_0)(w - w_0)\frac{\partial^3 u}{\partial z^2 \partial w} + \frac{\partial u}{\partial z} \\
& - \frac{\partial^2 u}{\partial z \partial w}(w - w_0)\big)dz - \big((z - z_0)(w - w_0)\frac{\partial^3 u}{\partial z \partial w^2} + \frac{\partial u}{\partial w} - \frac{\partial^2 u}{\partial z \partial w}(z - z_0)\big)dw\Big\},
\end{aligned}
\tag{2.9}
$$

where $\widetilde{\gamma}$ is the loop surrounding the point Q and having endpoints on the curve Γ. The first logarithm gets the increment $2\pi i$ along the loop, while the second $-(-2\pi i)$. Thus, compressing $\widetilde{\gamma}$ to a segment joining Q to Γ, we find that the integrand in (2.9) reduces to zero.

Thus, we obtain

$$
\begin{aligned}
u(P) = -\frac{i}{4\pi} \int\limits_{\widetilde{\gamma}} & \Big(\frac{(w - w_0)(\widetilde{S}(w))' u_z}{\widetilde{S}(w) - z_0} + \frac{(z - z_0)(S(z))' u_z}{S(z) - w_0} - \frac{2(S(z))' u}{S(z) - w_0} \\
& - \frac{(z - z_0)(S(z))'' u}{S(z) - w_0} + \frac{(z - z_0)((S(z))')^2 u}{(S(z) - w_0)^2} - \frac{(z - z_0)(w - w_0)(S(z))' u_{zw}}{S(z) - w_0}\Big)dz \\
& - \Big(\frac{(w - w_0)(\widetilde{S}(w))' u_w}{\widetilde{S}(w) - z_0} + \frac{(z - z_0)(S(z))' u_w}{S(z) - w_0} - \frac{2(\widetilde{S}(w))' u}{\widetilde{S}(w) - z_0} \\
& - \frac{(w - w_0)(\widetilde{S}(w))'' u}{\widetilde{S}(w) - z_0} + \frac{(w - w_0)((\widetilde{S}(w))')^2 u}{(\widetilde{S}(w) - z_0)^2} - \frac{(z - z_0)(w - w_0)(\widetilde{S}(w))' u_{zw}}{\widetilde{S}(w) - z_0}\Big)dw.
\end{aligned}
\tag{2.10}
$$

Calculating the residues we finally obtain,

$$\begin{aligned} u(P) = &- u(Q) - (z_0 - \widetilde{S}(w_0))\frac{\partial u}{\partial z}(Q) - (w_0 - S(z_0))\frac{\partial u}{\partial w}(Q) \\ &- (z_0 - \widetilde{S}(w_0))(w_0 - S(z_0))\frac{\partial^2 u}{\partial z \partial w}(Q). \end{aligned} \tag{2.11}$$

Formula (2.11) in variables x, y is equivalent to (2.2). Note that this formula gives continuation of a biharmonic function from the domain $U_1 \subset \mathbb{R}^2$ to the domain $U_2 \subset \mathbb{R}^2$ as a multi-valued function whose singularities coincide with one of the functions S or $\widetilde{S}$, where U_1, U_2 are components of $U \setminus \Gamma$. □

Remark 2.2. Formula (2.11) can be easily verified by expanding the function $u(z, w)$ in Taylor series at the point Q. Moreover, this method allows us to obtain a reflection formula for biharmonic functions having nonhomogeneous conditions on the curve Γ. To see this, let us expand the function $u(z, w)$ in Taylor series at the point Q:

$$\begin{aligned} u(z,w) = &+ u(Q) + \frac{\partial u}{\partial z}(Q)(z - \widetilde{S}(w_0)) + \frac{1}{2}\frac{\partial^2 u}{\partial z^2}(Q)(z - \widetilde{S}(w_0))^2 + \cdots \\ &+ \frac{\partial u}{\partial w}(Q)(w - S(z_0)) + \frac{1}{2}\frac{\partial^2 u}{\partial w^2}(Q)(w - S(z_0))^2 + \cdots \\ &+ \frac{\partial^2 u}{\partial z \partial w}(Q)(z - \widetilde{S}(w_0))(w - S(z_0)) \\ &+ \frac{1}{2}\frac{\partial^3 u}{\partial z \partial w^2}(Q)(z - \widetilde{S}(w_0))(w - S(z_0))^2 + \cdots \\ &+ \frac{1}{2}\frac{\partial^3 u}{\partial z^2 \partial w}(Q)(z - \widetilde{S}(w_0))^2(w - S(z_0)) + \cdots . \end{aligned} \tag{2.12}$$

Note that in (2.12), we used the condition

$$\frac{\partial^{4+i+j} u}{\partial z^{2+i} \partial w^{2+j}} = 0 \quad \text{for} \quad i, j = 0, 1, 2, \cdots .$$

Substituting the point $A = A(z_0, S(z_0))$ into (2.12), we obtain

$$u(A) - u(Q) = \frac{\partial u}{\partial z}(Q)(z_0 - \widetilde{S}(w_0)) + \frac{1}{2}\frac{\partial^2 u}{\partial z^2}(Q)(z_0 - \widetilde{S}(w_0))^2 + \cdots . \tag{2.13}$$

Similarly, substituting the point $B = B(\widetilde{S}(w_0), w_0)$ into (2.12), we obtain

$$u(B) - u(Q) = \frac{\partial u}{\partial w}(Q)(w_0 - S(z_0)) + \frac{1}{2}\frac{\partial^2 u}{\partial w^2}(Q)(w_0 - S(z_0))^2 + \cdots . \tag{2.14}$$

Differentiating (2.12) with respect to z at the point B, we obtain

$$\frac{\partial u}{\partial z}(B) - \frac{\partial u}{\partial z}(Q) - \frac{\partial^2 u}{\partial z \partial w}(Q)(w_0 - S(z_0)) = \frac{1}{2}\frac{\partial^3 u}{\partial z \partial w^2}(Q)(w_0 - S(z_0))^2 + \cdots . \tag{2.15}$$

And differentiating (2.12) with respect to w at the point A, we obtain

$$\frac{\partial u}{\partial w}(A) - \frac{\partial u}{\partial w}(Q) - \frac{\partial^2 u}{\partial z \partial w}(Q)(z_0 - \widetilde{S}(w_0)) = \frac{1}{2}\frac{\partial^3 u}{\partial z^2 \partial w}(Q)(z_0 - \widetilde{S}(z_0))^2 + \cdots . \tag{2.16}$$

Finally, using (2.12) at the point P and taking into account (2.13) - (2.16), we obtain that

(2.17)
$$u(P) = -u(Q) + u(A) + u(B) + \big(z_0 - \widetilde{S}(w_0)\big)\Big(\frac{\partial u}{\partial z}(B) - \frac{\partial u}{\partial z}(Q)\Big) + \big(w_0 - S(z_0)\big)\Big(\frac{\partial u}{\partial w}(A) - \frac{\partial u}{\partial w}(Q)\Big) - \big(z_0 - \widetilde{S}(w_0)\big)\big(w_0 - S(z_0)\big)\frac{\partial^2 u}{\partial z \partial w}(Q).$$

Note that A and B are points of intersection of the characterstic lines with the complexified curve $\Gamma_{\mathbb{C}}$. Therefore, formula (2.17) generalizes the well known nonhomogeneous formula for harmonic functions [17]:

$$u(P) + u(Q) = u(A) + u(B).$$

Thus, formula (2.17) allows us to construct a reflection formula for biharmonic functions satisfying on the curve Γ the following nonhomogeneous conditions:

$$u(x,y)_{|\Gamma} = g(x),$$
$$\frac{\partial u}{\partial y}(x,y)_{|\Gamma} = g_1(x),$$

where g and g_1 are holomorphic functions in a neighborhood of the curve Γ.

Remark 2.3. For the special case when Γ is a line with equation $f(x,y) \equiv ay + bx + c = 0$, formula (2.11) in (x,y) coordinates has a simpler form

$$u(P) = -u(Q) - \beta\big(2b\frac{\partial u}{\partial x}(Q) + 2a\frac{\partial u}{\partial y}(Q) + f(P)\Delta_{x,y}u(Q)\big),$$

where $\beta = f(P)/(a^2+b^2)$ is a known number. In particular, if $a = 1$ and $b = c = 0$, this formula coincides with formula (1.4) of H. Poritsky [2].

The corresponding nonhomogeneous formula (2.17) for the case of a line becomes

$$\begin{aligned} u(P) = &- u(Q) - \beta\big(2b\frac{\partial u}{\partial x}(Q) + 2a\frac{\partial u}{\partial y}(Q) + f(P)\Delta_{x,y}u(Q)\big) \\ &+ u(A) + u(B) + \beta(b+ia)(\frac{\partial u}{\partial x}(B) - i\frac{\partial u}{\partial y}(B)) \\ &+ \beta(b-ai)(\frac{\partial u}{\partial x}(A) + i\frac{\partial u}{\partial y}(A)). \end{aligned} \tag{2.18}$$

Remark 2.4. For the special case when Γ is a part of a circle with equation $x^2 + y^2 = \rho^2$, formula (2.11) reduces to formula (1.6) of J. Bramble [5].

Example 2.5. Let us consider the simplest example of applying nonhomogeneous formula for continuation of biharmonic functions. Let $u(x,y)$ be a biharmonic function defined in the upper half-plane and satisfy on the x-axis the following conditions

$$\begin{aligned} u(x,y)_{|_{y=0}} &= 1, \\ \frac{\partial u}{\partial y}(x,y)_{|_{y=0}} &= x. \end{aligned} \tag{2.19}$$

Note that if the point P has coordinates (x_0, y_0), then the reflected point $Q = Q(x_0, -y_0)$, $A = A(x_0 + iy_0, x_0 + iy_0)$ and $B = B(x_0 - iy_0, x_0 - iy_0)$. Thus, nonhomogeneous formula (2.18) for this case can be rewritten in the form

$$
\begin{aligned}
u(x_0,y_0) &= -u(x_0,-y_0) - 2y_0\frac{\partial u}{\partial y}(x_0,-y_0) - y_0^2\Delta u(x_0,-y_0)\\
&+ u(x_0+iy_0,x_0+iy_0) + u(x_0-iy_0,x_0-iy_0)\\
&+ \Big(\frac{\partial u}{\partial x}(x_0-iy_0,x_0-iy_0) - i\frac{\partial u}{\partial y}(x_0-iy_0,x_0-iy_0)\Big)iy_0\\
&- \Big(\frac{\partial u}{\partial x}(x_0+iy_0,x_0+iy_0) + i\frac{\partial u}{\partial y}(x_0+iy_0,x_0+iy_0)\Big)iy_0.
\end{aligned} \tag{2.20}
$$

Taking into account (2.19) we finally have,

$$
u(x_0,y_0) = -u(x_0,-y_0) - 2y_0\frac{\partial u}{\partial y}(x_0,-y_0) - y_0^2\Delta u(x_0,-y_0) + 2x_0y_0 + 2. \tag{2.21}
$$

Note that formula (2.20) generalizes Poritsky's reflection formula (1.4) to the case of nonhomogeneous conditions on the reflecting line.

3. Reflection formula for polyharmonic functions

In this section we generalize the reflection formula obtained in the previous section to polyharmonic functions.

Let $u(x,y)$, defined in a sufficiently small neighborhood U of a non-singular real analytic curve Γ defined by the equation $f(x,y)=0$, be a solution of the problem,

$$
\begin{cases}
\Delta^p_{x,y}u(x,y) = 0 \text{ near } \Gamma\\
u(x,y)_{|\Gamma} = 0 \pmod{p}.
\end{cases} \tag{3.1}
$$

Theorem 3.1. *Under the assumptions formulated above, there exists a point-to-point reflection formula which, in z,w coordinates, has the form,*

$$
\begin{aligned}
u(P) = -u(Q) - \sum_{m=1}^{p-1}\Big(&\frac{1}{(m!)^2}(z_0-\widetilde{S}(w_0))^m(w_0-S(z_0))^m\Delta^m_{z,w}u(Q)\\
&+\frac{1}{m!}(w_0-S(z_0))^m\sum_{n=0}^{m-1}\frac{1}{n!}(z_0-\widetilde{S}(w_0))^nD_w^{m-n}\circ\Delta^n_{z,w}u(Q)\\
&+\frac{1}{m!}(z_0-\widetilde{S}(w_0))^m\sum_{n=0}^{m-1}\frac{1}{n!}(w_0-S(z_0))^nD_z^{m-n}\circ\Delta^n_{z,w}u(Q)\Big),
\end{aligned} \tag{3.2}
$$

where, $\Delta_{z,w} = \frac{\partial^2}{\partial z\partial w}$, $D_z^\alpha = \frac{\partial^\alpha}{\partial z^\alpha}$ and $D_w^\alpha = \frac{\partial^\alpha}{\partial w^\alpha}$.

Proof. We will prove the theorem using the same idea as in the previous section. A fundamental solution for this case has the form,

$$
G = -\frac{1}{4^p\pi}\frac{((x-x_0)^2+(y-y_0)^2)^{p-1}}{(p-1)!^2}\ln((x-x_0)^2+(y-y_0)^2)
$$

or, in characteristic coordinates,

(3.3)

$$
G(z,w,z_0,w_0) = -\frac{1}{4^p\pi}(G_1(z,w,z_0,w_0)+G_2(z,w,z_0,w_0)), \quad \text{where,}
$$

$$
G_1 = \frac{(z-z_0)^{p-1}(w-w_0)^{p-1}}{(p-1)!^2}\ln(z-z_0),\ G_2 = \frac{(z-z_0)^{p-1}(w-w_0)^{p-1}}{(p-1)!^2}\ln(w-w_0).
$$

Green's formula for polyharmonic functions becomes,

$$u(P) = \sum_{k=0}^{p-1} \int_\gamma \omega(\Delta_{x,y}^k u) \cdot \Delta_{x,y}^{p-k-1} G - \Delta_{x,y}^k u \cdot \omega(\Delta_{x,y}^{p-k-1} G), \tag{3.4}$$

where p is the order of polyharmonicity of u and $\omega = \frac{\partial}{\partial y}dx - \frac{\partial}{\partial x}dy$. We will be able to deform the contour γ to the domain U_2 if we can construct the corresponding reflected fundamental solution $\widetilde{G}$. It must satisfy the following problem

$$\begin{cases} \Delta_{z,w}^p \widetilde{G} = 0, \\ \widetilde{G} - G = 0 \pmod p \quad \text{on } \Gamma_{\mathbb{C}}, \\ \widetilde{G} \text{ has singularities only on the characteristics } \widetilde{l}_j = \{\widetilde{\psi}_j = 0\},\ j = 1, 2\ , \end{cases} \tag{3.5}$$

where,

$$\widetilde{\psi}_1(w) = \widetilde{S}(w) - z_0, \quad \widetilde{\psi}_2(z) = S(z) - w_0.$$

Lemma 3.2. *The reflected fundamental solution $\widetilde{G}$ has the form*

$$\widetilde{G} = -\frac{1}{4^p\pi} \frac{(z-z_0)^{p-1}(w-w_0)^{p-1}}{(p-1)!^2} \ln(\widetilde{S}(w) - z_0)(S(z) - w_0)) + v(z, w, z_0, w_0), \tag{3.6}$$

where $v(z, w, z_0, w_0)$ is a p-harmonic function that is analytically continuable along any path free of singularities of the Schwarz function and its inverse.

Proof. We will seek $\widetilde{G}$ in the form

$$\widetilde{G}(z, w, z_0, w_0) = -\frac{1}{4^p\pi}(\widetilde{G}_1(z, w, z_0, w_0) + \widetilde{G}_2(z, w, z_0, w_0)),$$

where $\widetilde{G}_j$, $j = 1, 2$ are p-harmonic functions with singularities only on the characteristic complex lines $\widetilde{l}_j$ and satisfy the condition $\widetilde{G}_j - G_j = 0 \pmod p$ on the complexification $\Gamma_{\mathbb{C}}$. To prove the lemma it is sufficient to show that, for example, the function $\widetilde{G}_2$ has the form

$$\widetilde{G}_2 = \frac{(z-z_0)^{p-1}(w-w_0)^{p-1}}{(p-1)!^2} \ln(S(z) - w_0) + \sum_{k=1}^{p-1} \frac{(w - S(z))^k}{k!} \Phi_k(z, z_0, w_0), \tag{3.7}$$

where Φ_k's are functions that are analytically continuable along any path free of singularities of the Schwarz function. It is obvious that such function (3.7) is p-harmonic, since differentiating it p times with respect to w gives zero. Let us find the functions Φ_k from the condition

$$\frac{\partial^k \widetilde{G}_2}{\partial w^k}_{|_{w=S(z)}} = \frac{\partial^k G_2}{\partial w^k}, \quad k = 1, ..., p-1\,. \tag{3.8}$$

Differentiating function $\widetilde{G}_2$ $k-$times with respect to w gives

$$\frac{\partial^k \widetilde{G}_2}{\partial w^k} = \frac{(z-z_0)^{p-1}(w-w_0)^{p-k-1}}{(p-1)!(p-k-1)!}\ln(S(z)-w_0) + \Phi_k(z,z_0,w_0) + \sum_{m=k+1}^{p-1}\frac{(w-S(z))^{m-k}}{(m-k)!}\Phi_m(z,z_0,w_0), \tag{3.9}$$

and restricting this to $\Gamma_{\mathbb{C}}$ yields

$$\frac{\partial^k \widetilde{G}_2}{\partial w^k} = \frac{(z-z_0)^{p-1}(w-w_0)^{p-k-1}}{(p-1)!(p-k-1)!}\ln(w-w_0) + \Phi_k(z,z_0,w_0). \tag{3.10}$$

Differentiating G_2 (using Leibnitz rule), we obtain

$$\frac{\partial^k G_2}{\partial w^k} = \frac{(z-z_0)^{p-1}(w-w_0)^{p-k-1}}{(p-1)!(p-k-1)!}\ln(w-w_0) + \frac{(z-z_0)^{p-1}(w-w_0)^{p-k-1}}{(p-1)!}C_k, \tag{3.11}$$

where C_k is a known constant depending only on k and p. Comparing (3.10) and (3.11) we see that

$$\Phi_k = C_k\frac{(z-z_0)^{p-1}(S(z)-w_0)^{p-k-1}}{(p-1)!}.$$

This proves the lemma. □

Since we have constructed the reflected fundamental solution (3.6), which has singularities only on the characteristic lines $\widetilde{l}_j$ intersecting the real plane at $Q = R(P)$ in the domain U_2, we can deform the contour γ from the domain U_1 to a contour $\widetilde{\gamma}$ in U_2 surrounding the reflected point Q and having endpoints on the curve Γ. Therefore, using z, w variables, Green's formula (3.4) can be rewritten as

$$u(P) = 4^{p-1}\sum_{k=0}^{p-1}\int_{\widetilde{\gamma}}\omega^*(\Delta^k_{z,w}u)\cdot\Delta^{p-k-1}_{z,w}\widetilde{G} - \Delta^k_{z,w}u\cdot\omega^*(\Delta^{p-k-1}_{z,w}\widetilde{G}), \tag{3.12}$$

where $\omega^* = i(\frac{\partial}{\partial z}dz - \frac{\partial}{\partial w}dw)$.

Another important result from Lemma 3.2 is the fact that the reflected fundamental solution (3.6) does not ramify in the neighborhood of the reflected point $Q(\widetilde{S}(w_0), S(z_0))$. This is "not a trivial fact" since, for example, the reflected fundamental solution for the Helmholtz operator does not have this property [15]. According to this, if we substitute (3.6) into (3.12) and move one endpoint of the contour $\widetilde{\gamma}$ along the curve Γ to the other endpoint, terms containing products of the functions u, v and their derivatives vanish. Sum of integrals containing logarithms is equal to zero. The rest of terms have pole at the point Q and therefore, calculating the residues, we obtain a point-to-point reflection formula. However, direct transformation of (3.12) leads to cumbersome calculations, so knowing that point-to-point reflection formula exists, we can now use the Taylor series to obtain it. Moreover, we will also obtain it for nonhomogeneous conditions on the curve Γ. Indeed, let us expand the p-harmonic function $u(z, w)$ in Taylor series at the point

Q:

$$\begin{aligned} u(z,w) &= \sum_{m=0}^{p-1} \frac{1}{m!}(w - S(z_0))^m \sum_{n=m+1}^{\infty} \frac{1}{n!}(z - \widetilde{S}(w_0))^n (D_z^n(D_w^m u)(Q) \\ &+ \sum_{m=0}^{p-1} \frac{1}{m!}(z - \widetilde{S}(w_0))^m \sum_{n=m+1}^{\infty} \frac{1}{n!}(w - S(z_0))^n (D_w^n(D_z^m u)(Q) \\ &+ \sum_{m=0}^{p-1} \frac{1}{(m!)^2}(z - \widetilde{S}(w_0))^m (w - S(z_0))^m (D_z^m D_w^m u)(Q). \end{aligned} \tag{3.13}$$

Formula (3.13) implies:

$$\begin{aligned} & D_w^m u(A) - \sum_{n=0}^{m} \frac{1}{n!}(z_0 - \widetilde{S}(w_0))^n (D_z^n D_w^m u)(Q) = \\ & \sum_{n=m+1}^{\infty} \frac{1}{n!}(z_0 - \widetilde{S}(w_0))^n (D_z^n D_w^m u)(Q), \quad m = 0, ..., p-1 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} & D_z^m u(B) - \sum_{n=0}^{m} \frac{1}{n!}(w_0 - S(z_0))^n (D_w^n D_z^m u)(Q) = \\ & \sum_{n=m+1}^{\infty} \frac{1}{n!}(w_0 - S(z_0))^n (D_w^n D_z^m u)(Q), \quad m = 0, ..., p-1 \end{aligned} \tag{3.15}$$

where $A = A(z_0, S(z_0))$ and $B = B(\widetilde{S}(w_0), w_0)$.

Finally, replacing the infinite parts of the sum in (3.13) at the point P by the finite sums given by (3.14) and (3.15) we obtain,

$$\begin{aligned} u(P) &= -u(Q) + u(A) + u(B) \\ &- \sum_{m=1}^{p-1} \Big(\frac{1}{(m!)^2}(z_0 - \widetilde{S}(w_0))^m (w_0 - S(z_0))^m \Delta_{z,w}^m u(Q) \\ &+ \frac{1}{m!}(w_0 - S(z_0))^m \sum_{n=0}^{m-1} \frac{1}{n!}(z_0 - \widetilde{S}(w_0))^n D_w^{m-n} \circ \Delta_{z,w}^n u(Q) \\ &+ \frac{1}{m!}(z_0 - \widetilde{S}(w_0))^m \sum_{n=0}^{m-1} \frac{1}{n!}(w_0 - S(z_0))^n D_z^{m-n} \circ \Delta_{z,w}^n u(Q) \Big) \\ &+ \sum_{m=1}^{p-1} \Big(\frac{1}{m!}(w_0 - S(z_0))^m D_w^m u(A) + \frac{1}{m!}(z_0 - \widetilde{S}(w_0))^m D_z^m u(B) \Big), \end{aligned} \tag{3.16}$$

where $\Delta_{z,w} = \frac{\partial^2}{\partial z \partial w}$, $D_z^\alpha = \frac{\partial^\alpha}{\partial z^\alpha}$ and $D_w^\alpha = \frac{\partial^\alpha}{\partial w^\alpha}$.

Thus, we have obtained a reflection formula for polyharmonic functions with nonhomogeneous conditions on a curve Γ. Note that points A and B lie on the complexification $\Gamma_{\mathbb{C}}$, and therefore, if the function u satisfy (3.1) we have (3.2). □

Remark 3.3. Formula (3.2) for the case of a line with equation $y = 0$ reduces to Huber's formula (1.5) with $n = 1$.

Acknowledgements. We would like to thank Professor Dmitry Khavinson for suggesting this problem to us as well as for helpful discussions during this work.

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