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On a minimal element for a family of bodies producing the same external gravitational field

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A minimal element, mother body, for a family of bodies producing the same external gravitational field is the body in the family, whose support has Lebesgue measure zero and satisfies some additional requirements. The finite algorithm of constructing mother bodies in $\mathbb{R}^2$ is suggested. The local structure of mother bodies near singular points of continued logarithmic potential is investigated in generic positions.

Keywords: Analytic continuation; Mother body; Logarithmic potential; Balayage; Schwarz function

AMS Subject Classifications: 32Dxx; 31Axx

1. Introduction

The problem of analytic continuation of solutions to elliptic differential equations dates back to Schwartz, Herglotz and Poincaré. One of the classical statements of this problem is called "balayage inwards" and can be stated, for example, as follows. *Given a body* Ω *with a known mass distribution. Find a smaller body* Ω_1 *generating the same gravitational field outside* Ω [15].

This mathematical problem is known in geophysical literature as a problem of constructing families of bodies producing the same external gravitational field and related to inverse problem of geoprospecting. That is why this problem was studied by both mathematicians and geophysicists (see Novikov [1], Sretenskii [2], Zidarov [3,4], Strakhov and Filatov [5], Tsyrulskii [6], Kounchev [7,8], Gustafsson [9], Gustafsson and Sakai [10,11], Shapiro [15] and others).

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To describe the problem, suppose that an external gravitational field generated by some "heavy body" (i.e., contractible compact set in $\mathbb{R}^2$ provided with a mass distribution) is known. One wants to find the shape of this body and the mass distribution in it. Such a problem arises, for example, in geoprospecting if one measures a gravitational field on the Earth's surface and wants to find the mass distribution (inside the Earth) generating the field. Obviously, the solution of this problem is not unique. From Newton's time, it is known that a sphere with constant mass density generates the same gravitational field as the point mass of the same magnitude placed in the centre of the sphere. So, there exist different heavy bodies producing the same external gravitational field. Such heavy bodies are called *graviequivalent* ones, and one can consider a family of graviequivalent bodies. Using the Poincaré sweeping method, one can show that every family of graviequivalent bodies contain infinitely many elements. Therefore, this is a good idea to try to characterize each family by finding a minimal (in some sense) element in it, like a point mass characterizes a family of concentric balls. Such an attempt was first done by Zidarov [3] in 1968, who called this minimal element "*mother body*". Initially this notion was rather heuristic, but later on, more rigorous definitions of the mother body have been given by various authors, e.g., [7–9,12,13]. The most complete description of this notion is given in [9,11] and has function theoretic formulation. It seems, however, that it is possible to give more simple and geometrically clear definition of a mother body in terms of singularities (and corresponding cuts) of a multi-valued continued gravitational potential. In this article, we present such a definition for a two-dimensional case (see Definition 2 below). We also suggest a finite algorithm of constructing mother bodies in the sense of the given definition. The proposed algorithm allows one, in particular, to clarify whether the searched mother body exists. Moreover, we investigate the local structure of the mother body, at least in a generic position.

Note that the problem of constructing mother bodies is not always solvable, and the solution is not always unique. For example, the unit disc D in two-dimensional plane $\mathbb{R}^2_{(x,y)}$ with mass density $f(x,y) = \exp(x)$ does not determine any mother body at all. This is a consequence of the fact that the continuation of the field generated by such a body inside the unit disc has an essential singularity at the origin. (Connection between a mother body and singularities of continuation of the corresponding gravitational field will be explained in section 2.) The non-uniqueness of a mother body can be illustrated by the well-known Zidarov's example, who considered a square of constant mass density with deleted quarter and found two different mother bodies (see figure 1). More recent discussion devoted to polyhedron in two dimensions can be found in [11].

2. A mother body and a continuation of a gravitational field

Consider a heavy body concentrated in the domain $\Omega \subset \mathbb{R}^2_{(x,y)}$ and having the mass density $f(x,y) \geq 0$. Suppose that the function $f(x,y)$ can be continued up to an entire function in the complex plane $\mathbb{C}^2_{(x,y)}$, and the domain Ω has an algebraic boundary, that is,

$$\Gamma = \partial\Omega = \{(x,y) | P(x,y) = 0\}, \tag{1}$$

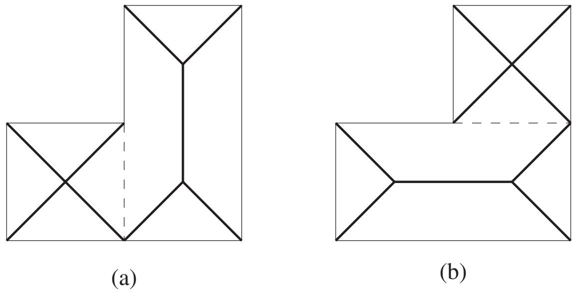

Figure 1. Non-uniqueness of a mother body.

where $P(x, y)$ is a polynomial with real coefficients.

The definition of the mother body used in this article is the following:

Definition 1 A mother body for a given heavy body Ω is a distribution with positive integrable mass density generating the same external gravitational field, whose support consists of a finite set of curvelinear segments or/and points contained in Ω and does not bound a two-dimensional subdomain of Ω.

Suppose that M is a mother body (in the sense of this definition) corresponding to Ω. It means that M consists of a finite set of segments of curves and, possibly, points. Note, that the set of curves included into M forms a planar graph, which is a tree, and, therefore, has "hanged" vertices (vertices of degree 1).

Since the body M generates the same (as Ω) external field, this field with potential $V(x, y)$ can be continued into $\mathbb{R}^2_{(x,y)} \setminus M$ as a harmonic function. As we shall see below, this harmonic continuation is a multi-valued function. So, the mother body M can be considered as a set of cuts selecting a single-valued branch of (multi-valued) continuation $\tilde{V}$ of the potential V. (During this analysis we neglect the point components of the mother body M since they are simply univalent singularities of the continuation. These singularities must be at most of the logarithmic type.)

Further, any hanged vertex of the set M is a singular point of the continuation $\tilde{V}$. Actually, if some hanged vertex $(x_0, y_0) \in M$ is a regular point of the continuation $\tilde{V}$, then the mass density on the curve originating from (x_0, y_0) vanishes identically (we recall that the mass density on the curve equals the jump of the normal derivative $\partial\tilde{V}/\partial n$ of the potential $\tilde{V}$).

Finally, each cut included into a mother body M has to satisfy the following three conditions:

(1) The cut must be *admissible* in the sense that the limit values of the potential on both sides of the cut must coincide with each other. This follows from the absence of gravitational dipoles.
(2) The cut must be *positive*, i.e., the sum of normal derivatives of the potential on the sides of the cut (in the directions of the corresponding inner normals) must be a positive function. Indeed, this sum is equal to mass density on the cut, which must be positive from a physical point of view.
(3) All the cuts must be *contained in the domain*, $\operatorname{supp} M \subset \Omega$.

Of course, as it was mentioned above, for a mother body to exist, one should require in addition that all singularities of continuation $\tilde{V}$ of the potential V have not more than logarithmic growth. We shall suppose in the sequel that this latter requirement is fulfilled for all the considered problems.

Thus, we can reformulate the definition of mother bodies, which is equivalent to Definition 1.

Let Ω be a contractible compact set in $\mathbb{R}^2_{(x,y)}$ having an algebraic boundary, and $f(x,y)$ be a positive function vanishing outside Ω and continuable up to an entire function in $\mathbb{C}^2_{(x,y)}$.

Definition 2 A mother body for a given heavy body (Ω, f) is a union of singularities of the continued potential $\tilde{V}$ and a system of cuts subject to conditions (1)–(3) above.

The algorithm of constructing mother bodies in the sense of this definition will be considered in section 4 below. In the rest of this section and in the next one we discuss properties of continued potential $\tilde{V}$.

We recall that a heavy body (Ω, f) induces a gravitational field on $\mathbb{R}^2_{(x,y)}$ with potential

$$V(x,y) = -\frac{1}{2\pi}\int_{\Omega} \ln\sqrt{(x-x_0)^2+(y-y_0)^2}\, f(x_0,y_0)\, dx_0 \wedge dy_0 \tag{2}$$

(we suppose that the domain Ω is oriented with the positive orientation of the space $\mathbb{R}^2_{(x,y)}$), which is a solution of the Poisson's equation

$$\Delta V = -f \text{ in } \Omega$$

and, since f vanishes outside Ω, is a harmonic function in $\mathbb{R}^2_{(x,y)} \setminus \Omega$.

To describe the continuation of V to the domain initially occupied by masses, it is useful to complexify the problem, i.e., to consider the function $V(x,y)$ in the complex space $\mathbb{C}^2_{(x,y)}$. The variables (x_0, y_0) become complex as well, therefore the integral (2) is considered as an integral over a chain in the complex space $\mathbb{C}^2_{(x_0,y_0)}$ determined by the real domain Ω. After such a complexification, we perform the following transformation of variables

$$z = x + iy, \qquad \zeta = x - iy \tag{3}$$

in integral (2). As a result we obtain[1]

$$V(z,\zeta) = \frac{1}{8\pi i}\int_{\Omega} [\ln(z - z_0) + \ln(\zeta - \zeta_0)] f(z_0,\zeta_0)\, dz_0 \wedge d\zeta_0, \tag{4}$$

where the branches of the logarithms are chosen in such a way that formulas (2) and (4) are equivalent. Note, that the variables (z,ζ) are characteristic variables of the Laplace's operator in $\mathbb{C}^2_{(x,y)}$.

[1] From now on we do not distinguish the function $V(x,y)$ and its continuation $\tilde{V}(x,y)$ as well as functions of (x,y) and functions of (z,ζ).

Clearly, integral (4) hardly can be computed for an arbitrary function $f(z_0, \zeta_0)$. However, all required information can be obtained from singular parts of the derivatives $\partial V/\partial z$ and $\partial V/\partial \zeta$. These singular parts can be computed in terms of the so-called Schwarz function (see [14–16]). We recall that the *Schwarz function* $S(z)$ is defined as a solution of the equation

$$P(z, \zeta) = 0 \tag{5}$$

with respect to ζ, where (5) is a complexified equation of Γ in the characteristic variables (z, ζ) (see equation (1) above). We also need the inverse of the Schwarz function, $\widetilde{S}(\zeta)$, that is defined as a solution of (5) with respect to ζ. Since the polynomial P has real coefficients in the variables (x, y), one can easily verify that the following relation holds:

$$\widetilde{S}(\zeta) = \overline{S(\bar{\zeta})}.$$

Differentiating integral (4) with respect to z, we obtain

$$\frac{\partial V}{\partial z}(z, \zeta) = \frac{1}{8\pi i}\int_{\Omega}\frac{f(z_0, \zeta_0)\, dz_0 \wedge d\zeta_0}{z - z_0} = \frac{1}{8\pi i}\int_{\Omega} d\left\{\frac{F_1(z_0, \zeta_0)\, dz_0}{z - z_0}\right\}, \tag{6}$$

where $F_1(z_0, \zeta_0)$ is any function satisfying the following condition:

$$\frac{\partial F_1(z_0, \zeta_0)}{\partial \zeta_0} = f(z_0, \zeta_0).$$

Using the Stokes formula, we arrive at the relation

$$\frac{\partial V}{\partial z}(z, \zeta) = \frac{1}{8\pi i}\int_{\Gamma}\frac{F_1(z_0, \zeta_0)\, dz_0}{z - z_0}, \tag{7}$$

where Γ is considered as a one-dimensional homology class in the complex-analytic surface $\Gamma_{\mathbb{C}}$ defined by (5). Solving (5) for ζ and substituting it into integral (7), one obtains

$$\frac{\partial V}{\partial z}(z, \zeta) = \frac{1}{8\pi i}\int_{\Gamma}\frac{F_1(z_0, S(z_0))\, dz_0}{z - z_0}, \tag{8}$$

where Γ is again the contour in the complex plane $\mathbb{C}_z$ coinciding with the boundary of the domain Ω (here we identify the plane $\mathbb{C}_z$ with the real plane $\mathbb{R}^2_{(x,y)}$ with the help of the relation $z = x + iy$).

We emphasize that the relation (8) was obtained for values of z lying in the complement of the domain Ω in the complex plane $\mathbb{C}_z$ (or, what is the same, in the real plane $\mathbb{R}^2_{(x,y)}$). To obtain the continuation of this function inside Ω, we change this contour

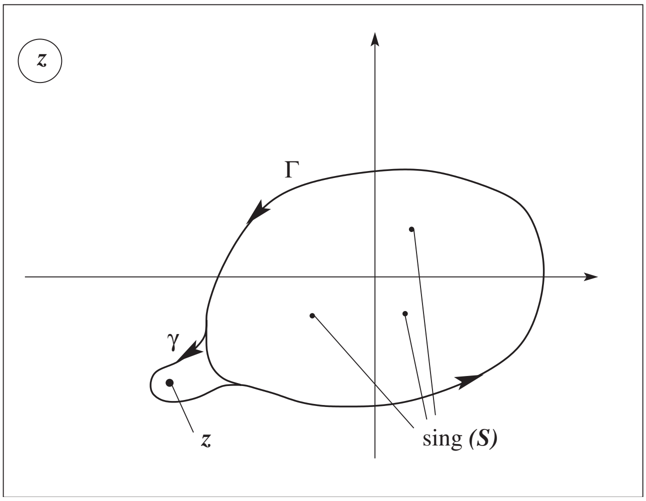

Figure 2. Changing the contour of integration.

to the contour γ encircling both the domain Ω and the point z, but containing no singular points of function $S(z_0)$ lying outside Ω (see figure 2).

The residue theorem shows that

$$\frac{\partial V}{\partial z}(z,\zeta) = \frac{1}{8\pi i}\int\limits_{\gamma} \frac{F_1(z_0, S(z_0))\,dz_0}{z - z_0} + \frac{1}{4}F_1(z, S(z)) = \frac{1}{4}F_1(z, S(z)) + \Phi_1(z),$$

where $\Phi_1(z)$ is holomorphic in the domain Ω. This expression describes all the singularities of continuation $\partial V/\partial z$[2] inside the domain Ω.

Similarly, one can obtain an expression of the derivative $\partial V/\partial \zeta$,

$$\frac{\partial V}{\partial \zeta}(z,\zeta) = \frac{1}{4}F_2\Big(\widetilde{S}(\zeta), \zeta\Big) + \Phi_2(\zeta),$$

where $\Phi_2(\zeta)$ is a regular function in Ω and $F_2(z,\zeta)$ is a function satisfying the relation

$$\frac{\partial F_2}{\partial z}(z,\zeta) = f(z,\zeta). \tag{9}$$

In what follows it will be convenient to fix the choice of the function $F_2(z,\zeta)$ in the following way:

$$F_2(z,\zeta) = \overline{F_1\big(\bar{\zeta}, \bar{z}\big)}. \tag{10}$$

[2] The derivative $\partial V/\partial z$ plays a crucial role in the investigation of singularities of continuation of potential V inside the domain Ω. This derivative is known in the Russian geophysical literature as a complex vector of gravitational field (see, e.g., [6]).

One can verify that this function satisfies (9) for any function f, which is real-valued as far as (x, y) are real. Thus, up to a regular function in Ω, one has

$$V(z,\bar{z}) = \frac{1}{4}\int F_1(z, S(z))\,dz + \frac{1}{4}\int F_2\left(\widetilde{S}(\zeta), \zeta\right) d\zeta\Big|_{\zeta=\bar{z}} = \frac{1}{4}\mathrm{Re}\int F_1(z, S(z))\,dz.$$

We conclude this section with the following remark:

Remark 1 The location of singularities of the continued potential is uniquely determined by the Schwarz function $S(z)$, i.e., by the boundary $\partial\Omega$, while (as it is shown below) the geometry of admissible cuts depends on the mass density $f(x, y)$ as well.

3. Investigation of admissible cuts and local structure of mother bodies

As we already mentioned, a mother body consists of singularities, sing(S), of continuation V and a set of cuts determining a single-valued branch of V. Each cut included into the set must be admissible, i.e., the limit values of this branch on both sides of the cut must coincide with each other. In other words, an admissible cut is the set of zeros of the variation of the potential V along some element, l, of a fundamental group

$$\pi_1\left(\Omega\backslash\mathrm{sing}(S), z^0\right)$$

of domain $\Omega\backslash\mathrm{sing}(S)$ with a base point z^0 lying on the curve Γ. The loop l determining the element of the fundamental group is shown in figure 3.

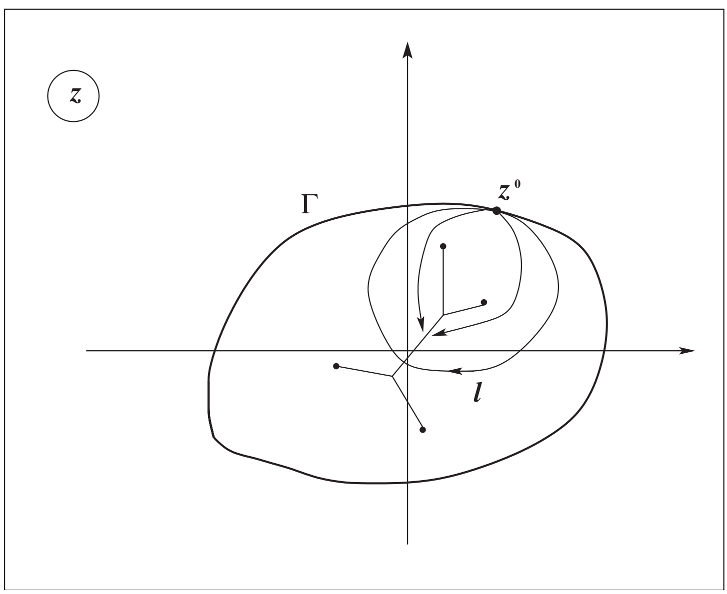

Figure 3. Definition of an element of a fundamental group.

This leads us to the following definition:

Definition 3 A cut c is called admissible with respect to the element

$$l \in \pi_1(\Omega \setminus \mathrm{sing}(S), z^0)$$

if and only if $\mathrm{var}_l\, V|_c = 0$.

Remark 2 All considerations in this section will be carried out on the real plane $\mathbb{R}^2_{(x,y)}$. We shall use here the characteristic coordinates (z, ζ), so that the equation of the real space is $\zeta = \bar{z}$.

Using the results obtained in the previous section, we can describe the set of admissible cuts as the set of integral curves of a certain vector field in the plane $\mathbb{C}_z$. To do this, we represent any vector tangent to the real space as a complex number c. Then, in terms of z and ζ the derivative D_c of the function $V(z, \zeta)$ along the vector c is given by

$$D_c V(z, \bar{z}) = c\,\frac{\partial V}{\partial z}(z, \bar{z}) + \bar{c}\,\frac{\partial V}{\partial \zeta}(z, \bar{z}) \equiv \frac{1}{4}\left[cF_1(z, S(z)) + \bar{c}F_2\left(\widetilde{S}(\bar{z}), \bar{z}\right)\right]$$

modulo regular in the domain Ω functions. Taking into account (10), the latter formula can be rewritten as

$$D_c V(z, \bar{z}) \equiv \frac{1}{2}\mathrm{Re}\,[cF_1(z, S(z))]. \tag{11}$$

Let l be, as above, some element of the fundamental group $\pi_1(\Omega \setminus \mathrm{sing}\,(S), z_0)$. Then, taking the variation of the right- and left-hand sides of relation (11), we arrive at the equality:

$$D_c \mathrm{var}_l\, V(z, \bar{z}) = \frac{1}{2}\mathrm{Re}\,[c\,\mathrm{var}_l\, F_1(z, S(z))]$$

(regular terms vanish under the action of the operator var_l). Equating to zero the right-hand side of this relation, we obtain the equation for the vector field c:

$$\mathrm{Re}\,[c\,\mathrm{var}_l\, F_1(z, S(z))] = 0. \tag{12}$$

Integral curves of this vector field are the lines, on which function $V(z, \bar{z})$ is a constant. So, any admissible cut is an integral curve of the vector field c. Note also that singular points of this vector field are singular points of the Schwarz function $S(z)$ and points, at which function $F_1(z, S(z))$ vanishes.

Now we must derive an "initial conditions" for integral curves of the vector field c, so that they are admissible cuts (that is, zero levels of the function $F_1(z, S(z))$). To do this, it is necessary to investigate a local structure of admissible cuts in neighborhoods of singularities of the Schwarz function $S(z)$, because admissible cuts have the singular points of $S(z)$ as their origins. We shall carry out such an investigation in the generic position. In other words, we suppose that the singularity, z_0, is brought from a finite regular characteristic point of the manifold $\Gamma_{\mathbb{C}}$, and the tangency between this singular point and the corresponding characteristic ray is quadratic. Under such requirements the function $S(z)$ at point z_0 has the singularity of the square root type:

$$S(z) = \sqrt{z - z_0}\, S_1(z) + S_2(z), \tag{13}$$

where $S_1(z)$ and $S_2(z)$ are regular functions of z in a neighborhood of the point z_0.

Denoting by $\zeta_0 = S(z_0) = S_2(z_0)$ the value of the Schwarz function at the singular point z_0 and expanding the function $F_1(z, \zeta)$ into the Taylor series at point (z_0, ζ_0), one has:

$$F_1(z, \zeta) = \sum_{j \geq 0,\ k \geq 0} b_{jk}(z - z_0)^j(\zeta - \zeta_0)^k. \tag{14}$$

Substituting (13) into (14) and expanding the functions $S_1(z)$ and $S_2(z)$ into the Taylor series at the point z_0, up to the terms regular near z_0, one obtains:

$$F_1(z, S(z)) \equiv \sqrt{z - z_0} \sum_{j=0}^{\infty} c_j (z - z_0)^j.$$

Using the relation $\partial F_1(z, \zeta)/\partial \zeta = f(z, \zeta)$, it is easy to verify that

$$c_0 = f(z_0, \zeta_0) S_1(z_0).$$

Note that (z_0, ζ_0) is the characteristic point of the surface $\Gamma_{\mathbb{C}}$ generating the singularity z_0 of the Schwarz function $S(z)$, so that the singularity of the function $\partial V/\partial z$ (as well as $\partial V/\partial \zeta$) is determined by values of the function f at characteristic points of $\Gamma_{\mathbb{C}}$. As we shall see below, the number c_0 determines the behavior of admissible cuts in a neighborhood of the point z_0. Hence, the set of admissible cuts depends on the mass density $f(x, y)$ (unlike the set of singularities of the potential $V(x, y)$, which is determined by the geometry of the domain Ω only).

Now we have

$$V(z, \bar{z}) = \frac{1}{2} \mathrm{Re} \int F_1(z, S(z))\, dz \equiv \frac{1}{2} \mathrm{Re} \left[\frac{2}{3} c_0 (z - z_0)^{3/2} (1 + \psi(z)) \right],$$

where $\psi(z)$ is a regular near z_0 function vanishing at this point. Taking the variation of both sides of the latter relation, we arrive at the formula

$$\mathrm{var}_l\, V(z, \bar{z}) = \frac{1}{2} \mathrm{var}_l\, \mathrm{Re} \left[\frac{2}{3} c_0 (z - z_0)^{3/2} (1 + \psi(z)) \right].$$

Let us introduce the polar coordinates in a neighborhood of the point z_0:

$$z = z_0 + \rho e^{i\varphi}.$$

Then the equation of admissible cuts near this point looks

$$\mathrm{var}_l\, V(z, \bar{z}) = \frac{2R}{3} \rho^{3/2} \mathrm{Re} \left[e^{\left(\frac{3i\varphi}{2} + i\theta\right)} \left(1 + \psi\left(z_0 + \rho e^{i\varphi}\right)\right) \right] = 0,$$

where R and θ are determined by the relation $c_0 = Re^{i\theta}$, or

$$\cos\left(\frac{3\varphi}{2}+\theta\right)\mathrm{Re}\left(1+\psi\left(z_0+\rho e^{i\varphi}\right)\right)-\sin\left(\frac{3\varphi}{2}+\theta\right)\mathrm{Im}\left(1+\psi\left(z_0+\rho e^{i\varphi}\right)\right)=0. \qquad (15)$$

Suppose that ρ is small, and consider (15) as an equation for φ. Since the function $\psi\left(z_0+\rho e^{i\varphi}\right)$ is of order $O(\rho)$, the principal term in this equation gives

$$\cos\left(\frac{3\varphi}{2}+\theta\right)=0, \text{ or } \varphi=\varphi_k=\frac{\pi}{3}-\frac{2\theta}{3}+\frac{2\pi k}{3}, \; k=0,1,2. \qquad (16)$$

The derivative of the left-hand side of equation (15) with respect to φ does not vanish for $\rho=0$ at each point φ_k, therefore this equation has the unique smooth solution $\varphi=\varphi_k(\rho)$ near $\rho=0$ such that $\varphi_k(0)=\varphi_k$. Each of these solutions determine an admissible (with respect to a small loop l encircling the point z_0) cut near the point z_0. So, the initial condition for an admissible cut near the singular points is:

$$\lim_{\rho\to 0}\varphi(\rho)=\varphi_k,$$

where φ_k are given by formula (16). The above consideration can be summarized in the form of the following theorem.

THEOREM 1 *Each admissible cut is an integral curve of the vector field c determined by relation* (12). *Moreover, under the requirements of a generic position, in a neighborhood of each branch point of the Schwarz function there exist three directions of admissible cuts given by relation* (16). *These cuts are positive or negative depending on the sign of the derivative with respect to* φ *of the left-hand side of relation* (15) *at the point* $(\varphi=\varphi_k,\ \rho=0)$.

4. Algorithm of constructing mother bodies

Now we are in the position to formulate an algorithm for constructing a mother body for a given heavy body (Ω, f) assuming that the following is true:

(1) The boundary Γ of the domain Ω is an algebraic curve.
(2) The function $f(x,y)$ is continuable up to an entire function in the complex space $\mathbb{C}^2_{(x,y)}$.
(3) The singularities of the continuation of the potential inside the domain Ω are not more than that of logarithmic type.

The last requirement can be verified using the formulas for singular parts of the continuation of the potential obtained in section 2.

The suggested algorithm consists of the following four steps:

Step 1 Determining singularities of the Schwarz function lying inside the domain Ω (that is, the set of singularities of the continuation of the potential $V(x,y)$ into the

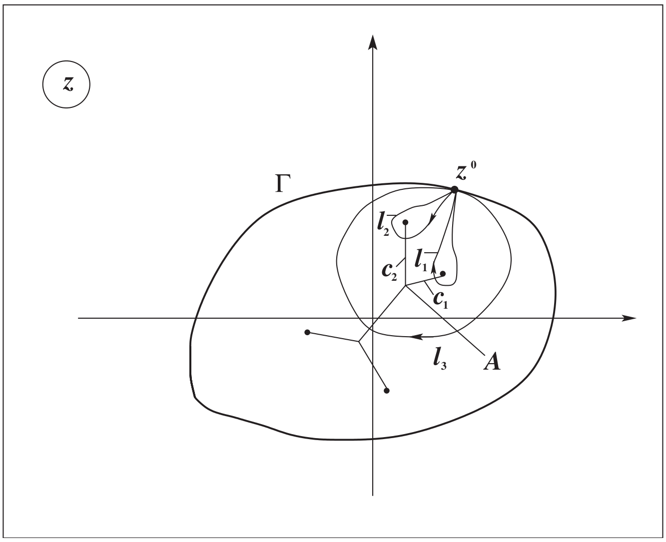

Figure 4. Non-elementary loop.

domain initially occupied by masses). This step simply requires solving algebraic equations describing the boundary in terms of the characteristic variables (z, ζ).

Step 2 Constructing a set of admissible cuts, C_a, for each simple loop with a base point z^0, surrounding one or more of the branch points found in Step 1. To carry out this step one has to compute integral curves of the vector field c constructed above. The initial conditions for this integral curves are defined as follows:

For a loop l encircling a single singular point, (e.g., loops l_1 or l_2 in figure 4) the initial condition is given by the local structure of the set of admissible cuts in a neighborhood of this point.

For a loop surrounding two singular points (see loop l_3 in figure 4), the initial condition for the set of admissible cuts is given by the intersection, A, of admissible cuts (cuts c_1 and c_2 in figure 4) corresponding to each of these points.

For admissible cuts corresponding to a loop encircling n singular points, initial points can be found as intersections of admissible cuts corresponding to a loop l_1 encircling the first $n-1$ of these points and the cuts corresponding to the loop l_2 surrounding the last of them.

Obviously, the described process will be completed in a finite number of steps. Certainly, if there exist singular points of the vector field c different from singular points of the Schwarz function (that is, points of singularity of c determined by zeros of the function $F_1(z, S(z))$; see relation (12)), then one has to investigate the local structure of the vector field c near such a point. This can be done using the formula (12) defining the field c.

Step 3 Selecting a set of bounded admissible cuts $C_b \subset C_a$, such that $C_b \subset \Omega$. This step can be fulfilled simply by deleting those edges from the graph C_a constructed in the previous step, that intersect the complement of the domain Ω in the plane $\mathbb{R}^2_{(x,y)}$.

Step 4 Constructing a mother body for (Ω, f). This can be done by examining each maximal tree of the constructed graph C_b (or of each maximal forest, if this graph is a non-connected). If all cuts of that maximal tree are positive, then it determines a mother body for (Ω, f).

In the next section we employ this algorithm for constructing mother bodies for specific domains.

5. Examples

For simplicity, we suppose that the mass density of the heavy body is identically equal to 1 for all examples considered below. Thus, all mother bodies are uniquely determined by the geometry of the domain Ω.

5.1. *An ellipse*

Let

$$\Omega = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

be an ellipse with the half-axes a and b, $a > b$ (see figure 5). Then the Schwarz function is given by

$$S(z) = \frac{a^2 + b^2}{d^2} + \frac{2ab}{d^2} \sqrt{z^2 - d^2},$$

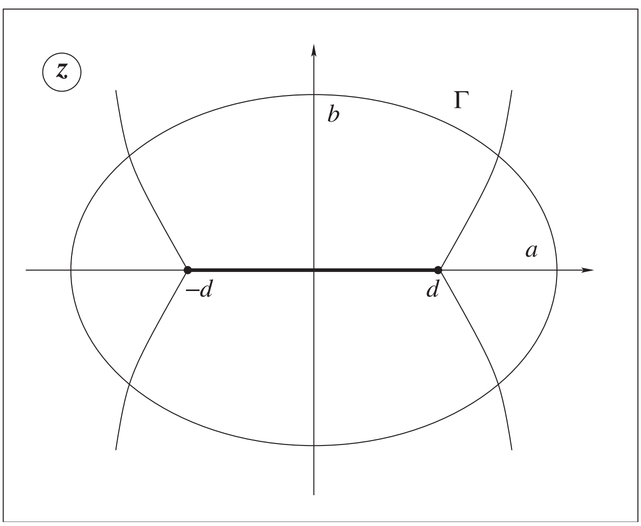

Figure 5. Admissible cuts for an ellipse.

where $d=\sqrt{a^2-b^2}$ is the half of the interfocal distance. In this simple case the complex field vector $\partial V/\partial z$ can be explicitly computed using the formula (8):

$$\frac{\partial V}{\partial z}(z,\zeta)=\frac{1}{8\pi i}\int_{\Gamma}\frac{S(z_0)\,dz_0}{z-z_0}=\frac{ab}{2d^2}\left(\sqrt{z^2-d^2}-z\right).$$

Similarly,

$$\frac{\partial V}{\partial \zeta}(z,\zeta)=\frac{ab}{2d^2}\left(\sqrt{\zeta^2-d^2}-\zeta\right).$$

Therefore, the potential V (up to the constant term) is given by:

$$V(z,\zeta)=\frac{ab}{d^2}\operatorname{Re}\left\{z\sqrt{z^2-d^2}-d^2\ln\left(z+\sqrt{z^2-d^2}\right)-z^2\right\}+\text{const.}$$

Since the function

$$\operatorname{var}_l\frac{\partial V}{\partial z}(z,\zeta)=\frac{ab}{d^2}\sqrt{z^2-d^2}$$

does not vanish at any point except the singular points of the Schwarz function, points $\pm d$, the vector field c determining the set of admissible cuts has singularities at these points only. Let us determine the directions of admissible cuts originating, say, from the point $z=d$. Using the formula (16) one obtains

$$\varphi_k=\frac{\pi}{3}+\frac{2\pi k}{3},\qquad k=0,1,2,$$

since $c_0=2ab/d^2$ and, hence, $\theta=0$. Similarly, the directions of admissible cuts at $z=-d$ are

$$\varphi_k=-\frac{\pi}{3}+\frac{2\pi k}{3},\qquad k=0,1,2.$$

One of the admissible cuts is the interfocal segment, while the rest of the four cuts go to infinity in the plane $\mathbb{R}^2_{(x,y)}$. Hence, the picture of the admissible cuts is such as it is drawn in figure 5. So, the finite graph mentioned in Step 3 of the algorithm formulated above consists of the interfocal segment, and the only thing left is to verify the positivity of this cut. The simple calculations lead us to the following expression for the mass density (of the mother body) having the interfocal segment as a support:

$$\sigma=\frac{ab}{d^2}\sqrt{d^2-x^2}.$$

Since the right-hand side of this expression is a positive function of x, *the interfocal segment is the unique mother body of the ellipse uniformly filled by masses.*

5.2. *A curve of the fourth order*

Here we consider a heavy body $(\Omega, 1)$ with the domain Ω given by

$$x^4 + y^4 \le 1$$

(see figure 6).

The corresponding Schwarz function is

$$S(z) = \sqrt{-3z^2 + 2\sqrt{2}\sqrt{z^4 + 1}}. \tag{17}$$

This function has singularities at eight points

$$z = e^{i(\pi/4 + \pi k/2)}, \qquad k = 0, 1, 2, 3,$$
$$z = 2\sqrt{2}e^{i\pi k/2}, \qquad k = 0, 1, 2, 3.$$

The last four of these points lie in the complement of the domain Ω and, hence, are of no interest. The four singular points lying inside Ω are plotted in figure 6.

Similar to the previous example, the vector field c determining the set of admissible cuts does not vanish at any point but for the singular points of the Schwarz function. Let us investigate the local structure of admissible cuts in a neighborhood of each branch point. For the point $z = e^{i\pi/4}$, due to equation (16), one has the following expression for the directions of the admissible cuts:

$$\varphi_k = \frac{7\pi}{12} + \frac{2\pi k}{3}, \quad k = 0, 1, 2.$$

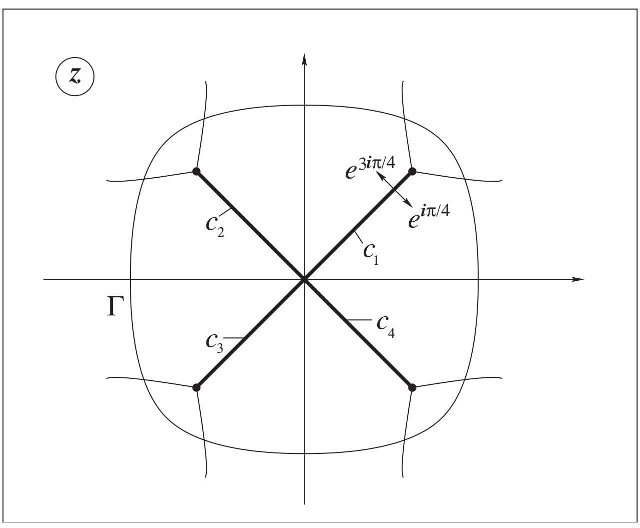

Figure 6. Admissible cuts for $x^4 + y^4 = 1$.

Here the argument θ of the number c_0 is equal to $-3\pi/8$. Thus, the picture of admissible cuts is such as it is drawn in figure 6. The fact that the straight line segment connecting points $e^{i\pi/4}$ and $e^{5i\pi/4}$, as well as the segment connecting $e^{-i\pi/4}$ and $e^{3i\pi/4}$, is an admissible cut follows from the symmetry. The rest of the cuts are not contained in the domain Ω and, hence, are of no interest. So, the graph mentioned in the Step 3 of the algorithm, consists of two straight line segments mentioned above. This graph is a tree, and the only thing rest is to verify the positivity of all the cuts included in this graph.

Verification of the last assertion is a little bit more complicated problem than it was in the previous example, and we shall point out the main steps of this verification. Clearly, due to the symmetry it is sufficient to carry out this verification for one of the four singular points of $S(z)$, lying inside Ω, say, for the point $z = e^{i\pi/4}$.

The expression for the sum of normal derivatives of the potential V on both sides of the cut c_1 (see figure 6) equals

$$D_{e^{-i\pi/4}}[V^-] + D_{e^{3i\pi/4}}[V^+],$$

where V^+ and V^- are values of the potential V on the upper and lower sides of the cut c_1, respectively. Here, as above, for any given complex number c we denote the derivative in the direction of the vector $\vec{c} \in \mathbb{R}^2$. Due to formula (11), this expression can be rewritten as

$$\frac{1}{2}\mathrm{Re}\left[e^{-i\pi/4}S(z)\right]^- + \frac{1}{2}\mathrm{Re}\left[e^{3i\pi/4}S(z)\right]^+. \tag{18}$$

Let us first calculate the first summand of the latter expression up to derivatives of functions regular near the point $e^{i\pi/4}$ (such terms do not contribute to the final expression). One has

$$\frac{1}{2}\mathrm{Re}\left[e^{-i\pi/4}S(z)\right]^- = \frac{1}{2}\mathrm{Re}\left[e^{-i\pi/4}\sqrt{-3z^2 + 2\sqrt{2}\sqrt{z^4+1}}\right]^-. \tag{19}$$

To determine the right branches of the square roots involved into the last expression, we perform the analytic continuation of the function (17) along the path l_1 shown in figure 7. At the origin A of this path $\zeta = S(z)$, $\zeta = z = 1$, and, therefore, both square roots must have positive real values. Now one can verify that the expression under the square root on the right in (17) changes along the path $\widetilde{l_1}$ shown in figure 8.

This path consists of the straight line segment $[1, 2\sqrt{2}]$ of the real axis and the segment of the ellipse

$$\frac{x^2}{8} + \frac{y^2}{9} = 1$$

lying in the fourth quarter of the plane $\mathbb{R}^2_{(x,y)}$. Extracting the square root of this expression and multiplying the result by $e^{-i\pi/4}$, one can see that the expression under the Re sign on the right of (19) changes along the path l_1' drawn in figure 7

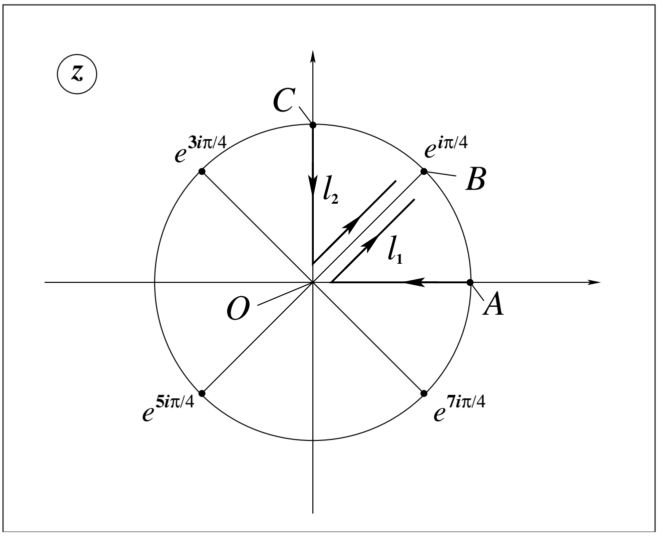

Figure 7. Paths of analytic continuation.

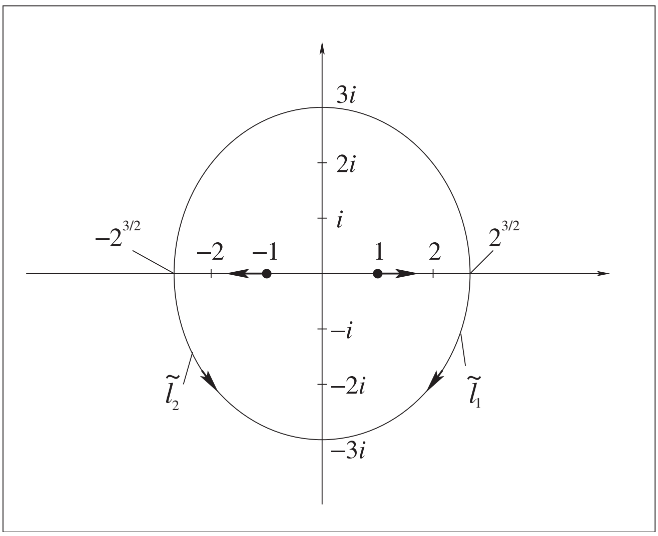

Figure 8. Expression under the square root sign.

when z changes along the cut c_1. Hence, the first summand in the expression (18) is positive.

Similarly, one can see that the expression under the square root in the right in (17) changes along the path $\widetilde{l_2}$ shown in figure 8 when the point z runs along the path l_2 (see figure 7). Hence, the expression under the Re sign in (18) changes along the path l_2' (figure 9), and the first summand in (18) is also positive.

So, *the union of the four cuts* c_j, $j = 1, 2, 3, 4$ *forms the* (*unique*) *mother body of* $(\Omega, 1)$.

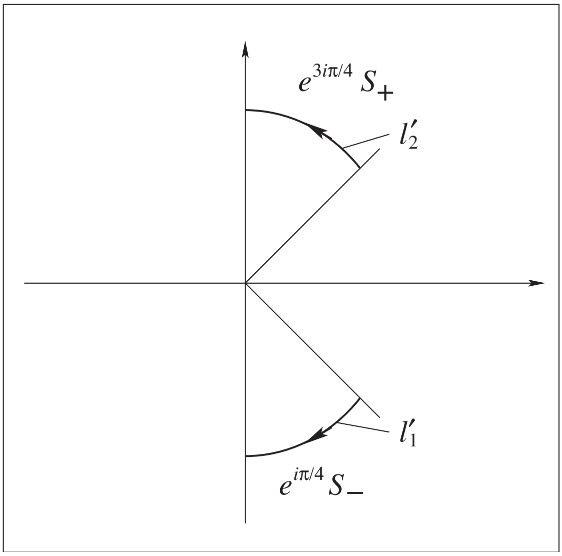

Figure 9. Jump of the normal derivative.

5.3. *One more curve of the fourth order: Cassini oval*

Here we consider a heavy body $(\Omega, 1)$, where the domain Ω is given by the equation

$$\left(x^2+y^2\right)^2 - 2b^2\left(x^2-y^2\right) = a^4 - b^4,$$

where a and b are positive constants (so-called Cassini oval). This curve consists of two closed curves for $a<b$, and one closed curve for $a>b$. We shall investigate the last case. This example is interesting from the following point of view. If we compute the Schwarz function of the Cassini oval, we shall obtain the expression

$$S(z) = \frac{\sqrt{b^2z^2 + a^4 - b^4}}{\sqrt{z^2-b^2}},$$

which has (inside the domain Ω) two singular points $z = \pm b$ with singularities of the type of the inverse square root (the domain and the singular points are shown in figure 10). This happens since these singularities are generated not by regular characteristic points of the surface $\Gamma_{\mathbb{C}}$ but by singular points of this surface.

Due to this fact, we cannot apply directly the result of the investigation of the local structure of the set of admissible cuts obtained in section 3. However, the computations similar to those in the mentioned section show that each of the two branch points possesses exactly two directions of admissible cuts emanating from these points, namely $\varphi = 0$ and $\varphi = \pi$. From symmetry, it immediately follows that the only admissible cuts for the heavy body are three segments $[-\infty, -b]$, $[-b, b]$ and $[b, +\infty]$ of the real axis. Since only one of these segments is contained in the domain Ω, the only possibility of constructing a mother body is to consider the

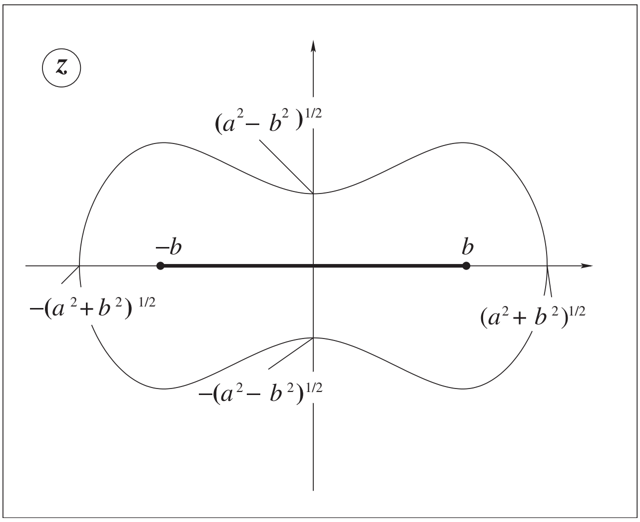

Figure 10. Cassini oval.

segment $[-b, b]$. Computation of the mass distribution, supported on this segment and generating the same gravitational field as the initial body, leads us to the expression

$$\sigma = \frac{\sqrt{b^2x^2 + a^4 - b^4}}{8\sqrt{b^2 - x^2}}.$$

Since this expression is positive and integrable function on the segment $[-b, b]$, *this segment is a unique mother body for the Cassini oval.*

5.4. *A rectangle*

This last example shows that even in the case when we cannot use the Schwarz function for the investigation of singularities, the above described algorithm of constructing a mother body is applicable. Consider a heavy body $(\Omega, 1)$, where the domain Ω is a rectangle

$$\begin{cases} 0 \le x \le a, \\ 0 \le y \le b \end{cases}$$

in the plane $\mathbb{R}^2_{(x,y)}$. Here a and b are positive constants and we suppose, to be definite, that $b < a$ (see figure 11).

Using relation (6) for the complex field vector, one has

$$\frac{\partial V}{\partial z} \equiv \frac{1}{8\pi i}\{-2z\ln(-z) - 2(a-z)\ln(a-z) + 2(a+ib-z)\ln(a+ib-z) - 2(ib-z)\ln(ib-z)\}$$

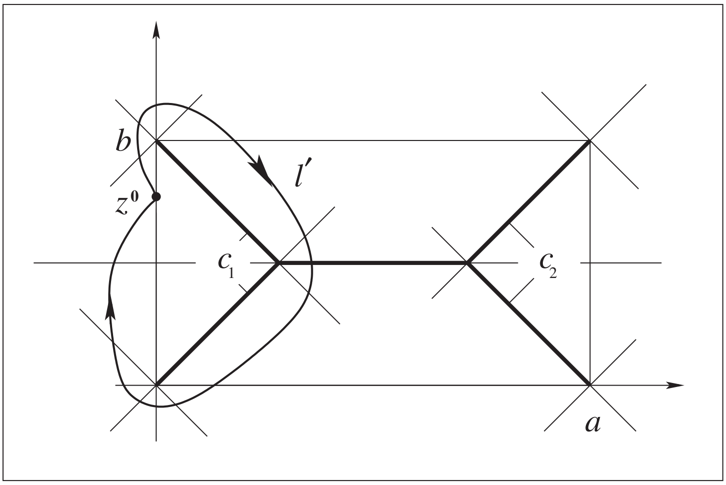

Figure 11. Admissible cuts for a rectangle.

and, consequently, up to the regular terms

$$\int \frac{\partial V}{\partial z}\, dz \equiv \frac{1}{8\pi i}\left\{-z^2 \ln(-z) + (a-z)^2 \ln(a-z) \right.$$
$$\left. - (a+ib-z)^2 \ln(a+ib-z) + (ib-z)^2 \ln(ib-z)\right\}.$$

Hence, one has

$$\mathrm{var}_l\left[\int \frac{\partial V}{\partial z}\, dz\right] = -\frac{1}{4} z^2,$$

where the variation is taken over the loop l encircling the point $z=0$. Consequently,

$$\mathrm{var}_l[V] = 2\mathrm{Re}\,\mathrm{var}_l\left[\int \frac{\partial V}{\partial z}\, dz\right] = -\frac{1}{2}\left(x^2 - y^2\right).$$

So, the point $z=0$ determines four directions of admissible cuts that emanated from these points. Moreover, it is obviously that the lines $x = \pm y$ are admissible cuts.

Similarly, investigation of all other vertices of the rectangle leads to six more admissible cuts, which are also straight lines that emanated from these vertices with the angles $\pm\pi/4$ (see figure 11). The constructed set of admissible cuts determines two open rectangles c_1 and c_2 with hanged vertices at those of the rectangle. However, these two cuts do not determine a single-valued branch of the potential V. One has to take into consideration the admissible cuts corresponding to the loop l' encircling the two left (or the two right, which leads a posterior to the same result) vertices of the rectangle.

The computation of the variation along this loop gives

$$\mathrm{var}_{l'}[V] = b\left(y - \frac{b}{2}\right)$$

and, hence, the admissible cut corresponding to this loop is the straight line $y = b/2$. Adding the segment of this straight line connecting the two angular points of the open rectangles c_1 and c_2, we arrive at the system of cuts (shown in figure 11 by thick lines) which determines a single-valued branch of the potential V. We leave to the reader the verification of the fact that this system consists of positive cuts (see also [3,4,9,11]), so that *the mother body of the rectangle occupied by the uniform mass distribution has the support shown in figure* 11.

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