Understanding the Math Behind Bridge

By Jim Schultz

"Nothing is more terrible than ignorance in action." -- Goethe

If you're the kind of bridge player who is satisfied with memorizing various probabilities like the probability of dropping a singleton king under your ace, this article it *not* for you. But if you're the kind of person who enjoys *understanding* how these probabilities are determined, then read on. The focus is on building a better understanding of bridge (and mathematics) rather than on making you a better bridge player, but don't be surprised if a deeper understanding does help you become a better player.

Bridge columns and articles often cite probabilities to determine the best line of play. For example, in a newspaper column in which South held ♦AJ75 and dummy held ♦Q10984, noted bridge expert Phillip Alder wrote, "West will have all four diamonds only 4.8 percent of the time. And when South starts with his diamond ace he will drop the singleton king 12.4 percent of the time." Have you ever wondered where these numbers come from? You will now see how to find such numbers, with all the necessary math beyond ordinary arithmetic, as well as many useful numbers in computing probabilities and hints on how to use popular calculators to find any others you might wish to know to better enjoy the game. If you have the ability to play bridge, chances are you have the ability to understand how these probabilities are determined.

Combinations, the Basis for Understanding

When playing bridge, you are constantly counting things, like how many trump are out. It should then be no surprise that counting is an essential part of determining your chances in bridge.

The main underlying idea is the Fundamental Principle of Counting, illustrated in this example:

Example 1. How many honors are there?
Solution: Each of the 4 suits contains 5 honors (A, K, Q, J, 10), so there are 4 x 5 = 20 honors.

This can be generalized in the following way:

Fundamental Principle of Counting. If a first thing can be done in *m* ways and if *for each of these ways* a second thing can be done in *n* ways, then the two things can be done in *m x n* ways.

Example 2. You are defending and partner leads the ace and then the king of spades. In how many orders can you choose to follow suit from a hand containing three spades: 9, 8, and 5?
Solution: There are 3 ways to follow to the first trick. For each of these 3 ways there are 2 ways to play your next spade. By the Fundamental Principle of Counting there are 3 x 2 = 6 ways for you to play.

Here is a complete list which confirms this: 98, 95, 89, 85, 59, 58. It is having 6 ways to play that gives you the flexibility to signal attitude and/or count.

Example 3. In the previous example, how many orders are there to play your three different spades?
Solution: From Example 2, there are 6 ways to play your first 2 spades. For each of these there is 1 way to play your remaining spade, so there are 3 x 2 x 1 = 6 ways. Note that once you play the first two spades you have only one choice in playing the third spade, so the answer is the same as in the previous example.

Examples 2 and 3 involved problems where order was important. Counting situations where *order is not important*, including the examples mentioned at the beginning, involve what are called **combinations**.

Example 4. How many doubletons are there in spades? (Note here that AK is considered the same as KA, since order doesn't matter.)
Solution: As before there are 13 ways to choose the first spade, and for each of these there are 12 ways to choose the second spade. But since doubleton ace-king and each other doubleton is counted twice, you need to divide by 2, so the answer is $\frac{13 x 12}{2}$ = 78.

(This is not unlike seeing only the legs of a number of people standing behind a curtain, when you could find the number of people by counting the legs and dividing by 2.)

It will be convenient to represent the number of combinations of 13 things taken 2 at a time by C(13, 2). This lets us express the computations of Example 4 as C(13, 2) = $\frac{13 x 12}{2}$. To obtain a more general rule which works when more numbers are involved, you can think of finding C(13, 2) this way:

Step 1: Using the Fundamental Principle of Counting, the number of ways to choose 2 cards from 13 *when order matters* is 13 x 12.

Step 2: For each choice the same 2 cards can be arranged in 2 x 1 ways (meaning each combination of two cards has been counted twice), so divide by 2 x 1:

C(13, 2) = $\frac{13 x 12}{2 x 1}$ = 78.

The next examples illustrate this more general approach when more than 2 cards are chosen.

Example 5. Find the number of combinations of 3 club honors.
Solution: There are 5 club honors (A, K, Q, J, 10) from which to choose 3.
Step 1: The number of ways to choose 3 cards from 5 when order matters is 5 x 4 x 3.
Step 2: Divide by 3 x 2 x 1, the number of ways of rearranging the same 3 cards:

C(5, 3) = $\frac{5 x 4 x 3}{3 x 2 x 1}$ = 10.

Here is a complete list of the 10 combinations: AKQ, AKJ,AK10, AQJ, AQ10, AJ10, KQJ, KQ10, KJ10, QJ10. Note that since you are not concerned with order, since AKQ, AQK, KAQ, KQA, QAK, and QKA are merely 6 different ways of writing the same combination.

Example 6. Find the number of 5-card heart holdings.
Solution: This involves finding C(13, 5).
Step 1: The number of ways to choose 5 cards from 13 when order matters is 13 x 12 x 11 x 10 x 9.
Step 2: Divide by 5 x 4 x 3 x 2 x 1, the number of ways of rearranging the same 5 cards:

$$C(13, 5) = \frac{13 \, x \, 12 \, x \, 11 \, x \, 10 \, x \, 9}{5 \, x \, 4 \, x \, 3 \, x \, 2 \, x \, 1} = 1287$$

A convenient shorthand for 5 x 4 x 3 x 2 x 1 is "five factorial", written 5!. Similarly, 13 x 12 x 11 x 10 x 9 looks like 13!, with all the factors from 8 on down cancelled out. This enables us to write 13 x 12 x 11 x 10 x 9 as 13! / 8!

This means that $C(13, 5) = \frac{13!}{5! \, 8!}$

The computations of Example 4 and 5 can be written as

$$C(13, 2) = \frac{13!}{2! \, 11!} \text{ and } C(5, 3) = \frac{5!}{3! \, 2!}$$

In general, $C(n, r) = \frac{n!}{r! \, (n-r)!}$

Example 7. How many bridge hands are possible?
Solution: A bridge hand consists of 13 cards chosen from 52, so

$$C(52, 13) = \frac{52!}{13! \, 39!} = \frac{52 \, x \, 51 \, x \, 50 \, x \, 49 \, x \, 48 \, x \, 47 \, x \, 46 \, x \, 45 \, x \, 44 \, x \, 43 \, x \, 42 \, x \, 41 \, x \, 40}{13 \, x \, 12 \, x \, 11 \, x \, 10 \, x \, 9 \, x \, 8 \, x \, 7 \, x \, 6 \, x \, 5 \, x \, 4 \, x \, 3 \, x \, 2 \, x \, 1} = 635{,}013{,}559{,}600$$

, or about 635 billion. (If you dealt one hand per minute 24 hours a day 7 days a week it would take over a million years!)

Here is where a good calculator comes in handy! For example, on many Texas Instruments graphing calculators MATH PRB 3 gives C(n, r), so these key strokes will compute C(52, 13):

52 MATH PRB 3 13 ENTER.

The answer may appear as 6.350135596E11, which is scientific notation for 635,013,559,600. The E11 at the end of the number means multiply by 10^{11}, or in other words "move the decimal point 11 places to the right".

For your reference, values of some of the most commonly occurring combinations generated by an Excel spreadsheet using the built-in function COMBIN, are given below. For example, 78, the value of C(13, 2) which appears across from 13 and under 2, was computed using the Excel formula =combin(13, 2).

							r							
n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1	1	1												
2	1	2	1											
3	1	3	3	1										
4	1	4	6	4	1									
5	1	5	10	10	5	1								
6	1	6	15	20	15	6	1							
7	1	7	21	35	35	21	7	1						
8	1	8	28	56	70	56	28	8	1					
9	1	9	36	84	126	126	84	36	9	1				
10	1	10	45	120	210	252	210	120	45	10	1			
11	1	11	55	165	330	462	462	330	165	55	11	1		
12	1	12	66	220	495	792	924	792	495	220	66	12	1	
13	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1

C(n, r) for n = 1 to 13

This pattern of numbers is known to mathematicians and many middle school students as **Pascal's triangle**. It has many interesting properties. Some of the most useful to this bridge theme are discussed next.

Example 8. How many ways are there to choose *none* of the 13 clubs so you have a void in clubs? Solution: There is only one way to do this: *Don't choose* the ace, *don't choose* the king, *don't choose* the queen, *etc.* For each card, there is only 1 choice that can be made: *not* to choose that card. Using the notation for combinations, C(13, 0) = 1.

Similarly C(52, 0) = 1, and in general, C(n, 0) = 1 for all values of n.

It is also clear that there is symmetry in the table. For example, C(5, 1) = C(5, 4). In terms of cards, this is because the number of ways to choose which 1 of 5 cards to include, represented by C(5, 1), is the same as the number of ways to choose which 1 of 5 cards *not* to include, represented by C(5, 4). In much the same way C(5, 2) = C(5, 3), because the number of ways to choose which 2 of 5 cards to include, represented by C(5, 2), is the same as the number of ways to choose which 2 of 5 cards *not* to include, represented by C(5, 3).

You may wish to look for other interesting properties, such as how to predict one row from the previous one.

Here are a few more combinations that will be useful: For example, if you are declarer you can see all but 26 cards, so the number of possible hands for East chosen from the remaining cards is C(26, 13), which from the chart is 10,400,600.

	r				
n	**0**	**13**	**26**	**39**	**52**
26	1	10,400,600	1		
39	1	8,122,425,444	8,122,425,444	1	
52	1	635,013,559,600	495,918,532,948,104	635,013,559,600	1

Example 9. How many hands have a void in clubs?
Solution. From Example 8 there is C(13, 0) = 1 way to choose the clubs. All 13 cards in the hand must be chosen from the other 39 cards, or C(39, 13) ways. Using the chart above or a calculator this is 8,122,425,444 ways. So the answer is 1 x 8,122,425,444 = 8,122,425,444.

Of course, the number of ways to have a void is the same for any suit, but it's not quite true that there is a total of 4 x 8,122,425,444 hands which are voids. This is because some hands may have 2 or even 3 voids, so the same hand will be counted more than once. Taking this into account, the actual number of hands with at least one void is 32,427,298,180. (For experts, this number is 4C(39, 13) – 6C(26, 13) + 4.)

In the same way you can find the number of hands with any particular distribution.

Probabilities, Clues to Making the Best Play

Computing probabilities simply involves comparing particular outcomes with *equally likely* possible outcomes. Here *equally likely* is an important part of the definition. For example, it would be incorrect to say that the probability of being first in a field of 48 teams in a major tournament is ½, because either you come in first or you don't. For most of us, these are not equally likely events!

The **probability** of an event occurring is the number of ways the event can occur divided by the number of *equally likely* possible outcomes.

Example 10. Find the probability of having a void in clubs.
Solution: From Example 9, the number of ways this can occur is 8,122,425,444, and from Example 7 the number of possible bridge hands is 635,013,559,600. Thus the probability of a void in clubs is 8,122,425,444 / 635,013,559,600 = .0127909, or about 1.3%.

How to open a 13-point hand with exactly 4 spades, 4 hearts, 3 diamonds, and 2 clubs is often a matter of considerable debate as not bidding a 4-card major forces bidding a 3- or 2-card suit. How often does this 4-4-3-2 distribution occur?

Example 11. Find the probability of having a hand with 4 spades, 4 hearts, 3 diamonds, and 2 clubs.
Solution: There are C(13,4) = 715 ways to choose the spades. For each of these, there are C(13,4) = 715 ways to choose the hearts, C(13, 3) = 286 ways to choose the diamonds, and finally C(13, 2) = 78 ways to choose the clubs. (The numbers 715, 286, and 78 can be found in row 13 of the first chart.) The number of such hands is 715 x 715 x 286 x 78 = 11,404,407,300, so the probability of having such a hand is 11,404,407,300 / 635,013,559,600 = .0179593, or about 1.8%.

Such large numbers can be hard to read. It may actually be more meaningful to show the solutions to Examples 10 and 11 this way:

P(void in clubs) = $\dfrac{C(13,0)\ x\ C(39,13)}{C(52,13)}$, and

P(4S-4H-3D-2C distribution) = $\dfrac{C(13,4)\ x\ C(13,4)\ x\ C(13,3)\ x\ C(13,2)}{C(52,13)}$.

Example 12. Find the probability of holding any 4-4-3-2 distribution.
Solution: There are 4 suits to pick for the doubleton. For each of these there are 3 ways to pick the 3-card suit. For each of these there is only 1 way to pick the remaining two suits, so there are 4 x 3 x 1 = 12 ways to pick the suits. Thus the probability is 12 times the answer computed in Example 11, which is 21.6%.

You are now able to compute the numbers in the opening paragraph:

Example 13. If South holds ♦AJ75 and dummy holds ♦Q10984, find the probability West will have all four diamonds.
Solution: Declarer's combined hands consist of 9 diamonds and 17 other cards, so the defenders' combined hands consist of 4 diamonds and 39 – 17 = 22 other cards. There is C(4, 4) = 1 way to choose the diamonds West holds. (Choose them all.) And there are C(22, 9) = 497,420 ways to choose West's remaining cards. This is out of C(26, 13) = 10,400,600 possible hands that West could have using the unseen cards. Thus, the probability West will have all 4 diamonds is

P = $\dfrac{C(4,4)\ x\ C(22,9)}{C(26,13)} = \dfrac{1\ x\ 497{,}420}{10{,}400{,}600}$ = 4.8%

Example 14. If South holds ♦AJ75 and dummy holds ♦Q10984, find the probability West or East will have a singleton ♦K, which will fall if declarer leads ♦A.
Solution: First find the probability that West has the singleton ♦K. This is similar to the previous example, except that this time there is *C*(1,1) = 1 way to choose the ♦K, *C*(3,0) = 1 way to choose the other three hearts (don't choose any of them), and C(22, 12) = 646,646 ways to choose West's remaining 12 cards, so

$$P = \frac{C(1,1)\ x\ C(3,0)\ x\ C(22,12)}{C(26,13)} = \frac{1\ x\ 1\ x\ 646{,}646}{10{,}400{,}600} = 6.2\%.$$

By the same reasoning, the probability East has the singleton ♦K is also 6.2%. Since these are mutually exclusive, the probabilities can be added to give 12.4%.

The examples presented here can serve as a template for computing new probabilities which may arise. Understanding the math behind bridge can allow people to enjoy the game in a new way. It even may provide an appreciation for why some of the math you learned in school and never used since is actually useful!

Phillip Alder’s web site: https://www.bridgeforeveryone.com/