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IMPROVED ROBOTICS JOINT-SPACE TRAJECTORY GENERATION WITH VIA POINT

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ABSTRACT

This paper presents novel fourth- and sixth-order polynomials to solve the problem of joint-space trajectory generation with a via point. These new polynomials use a single function rather than two polynomial functions matched at the via point as in previous methods. Another contribution of this paper is to expose the widespread, use of discontinuous acceleration functions in joint-space trajectory generation methods, which lead to **unacceptable** infinite spikes in jerk, the derivative of the acceleration.

1. INTRODUCTION

Joint-space trajectory generation is in common usage in robotics to provide smooth, continuous motion from one set of n joint angles to another, for instance for moving between two distinct Cartesian poses for which the inverse pose solution has yielded two distinct sets of n joint angles. The joint-space trajectory generation occurs at runtime for all n joints independently but simultaneously.

There is an entire body of literature devoted to trajectory generation (aka motion planning and path planning) at the joint and Cartesian (not addressed in this paper) levels. Many of these works deal with optimal motions – the current paper addresses a more simple area, those of the common methods in use today as evidenced by their appearance in the standard robotics textbooks. Our literature search turned up no papers dealing with the main contribution of this paper, a novel fourth- and sixth-order single polynomial approach to achieve joint-space trajectory generation with a via point.

Paul and Zong (1984) were among the first to suggest the use of polynomials for robot trajectory generation. A linearly-changing joint velocity, using starting and ending parabolic blends is suggested by Craig (2005), Jazar (2010), Koivo (1989), Parkin (1991), Sciavicco and Siciliano (1996), and Spong et al. (2005). A third-order polynomial approach is suggested by Craig (2005), Jazar (2010), Koivo (1989), Sciavicco and Siciliano (1996), and Spong et al. (2005), and a fifth-order polynomial approach is suggested by Craig (2005), Jazar (2010), Koivo (1989), and Sciavicco and Siciliano (1996). Fu et al. (1987) depart from these standard methods, suggesting initial, intermediate, and final polynomials of order 4-3-4, 3-5-3, or 5 third-order polynomials, for a single joint motion. These are by far the most complicated (unnecessarily) methods, and are presented without justification or comparison with simpler methods. For dealing with a via point in which the robot need not stop at the via point (such as for obstacle avoidance), Craig (2005) suggests matching two third-order polynomials.

This paper addresses two fundamental problems in these commonly-used joint-space trajectory generation methods in robotics today. The original contribution of this paper is a single sixth-order polynomial to provide smooth, continuous joint motion through a via point. Previous approaches have matched two polynomials at the via point. The second contribution of this paper is to expose the widespread use of polynomials with discontinuous accelerations functions, leading to infinite spikes in jerk (the time derivative of acceleration), which is **unacceptable** for reliable, smooth, long-life robotic systems. This has also been pointed out by Macfarlane and Croft (2003), who present jerk-bounded trajectories and Gosselin and Hadj-Messaoud (1993) and Petrinec and Kovacic (2007) who go one step further to ensure continuous, not just bounded, jerk.

This paper is organized as follows. First we review standard joint-space trajectory generation methods (linear velocity with parabolic blends, and third- and fifth-order polynomials). Then we address joint-space trajectory generation with a via point, reviewing Craig's two third-order polynomials, followed by the original single fourth- and sixth-order polynomials to accomplish the same task. Numerical examples are presented for all cases.

2. JOINT-SPACE TRAJECTORY GENERATION

Standard joint-space trajectory generation assumes two sets of n discrete joint parameters (angles for revolute joints, lengths for prismatic joints) are known and it is required to move smoothly in joint space from one set to the next. For instance, given required Cartesian poses $\mathbf{X_S}$ and $\mathbf{X_F}$, inverse pose kinematics calculates the required n joint value sets for achieving each of these; call them $\mathbf{\Theta_S}$ and $\mathbf{\Theta_F}$. Note subscript **S** stands for start (or initial) and subscript **F** stands for finish (or final).

The standard joint-space trajectory method then moves smoothly from $\mathbf{\Theta_S}$ and $\mathbf{\Theta_F}$ on all n joints independently but simultaneously. Generally it is assumed that all joint motions start and end at rest, i.e. zero joint velocities.

Polynomials are natural choices for providing smooth, continuous motion, with some level of continuous derivatives. Now we review two popular methods for polynomial-based joint-space trajectory generation.

2.1 Linear Velocity with Parabolic Blends

Many robotics textbooks (e.g. Craig 2005, Jazar 2010, Koivo 1989, Parkin 1991, Sciavicco and Siciliano 1996, and Spong et al. 2005) suggest a linearly-changing joint velocity, using starting and ending parabolic blends in order to start and end at zero velocity. The plots of joint angle and its derivatives for this standard method are shown in Figure 1.

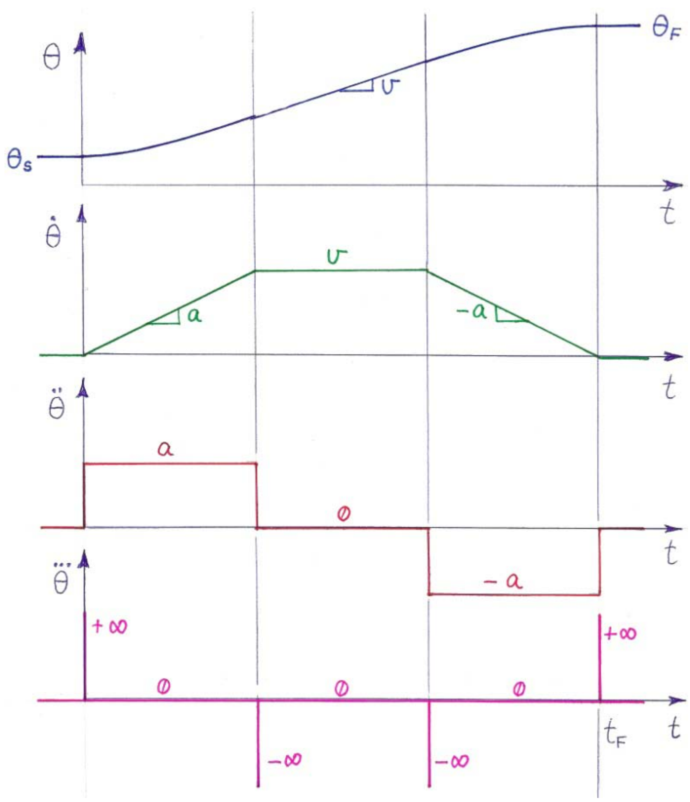

Figure 1. Linear Velocity with Parabolic Blends

All the textbooks plot this method's curves only through acceleration. As seen in Figure 1, the angular acceleration is discontinuous, leading to infinite spikes in angular jerk every time the joint angle function changes, four times in Figure 1. Instantaneous changes in acceleration are IMPOSSIBLE in the real world, leading to infinite spikes in jerk which are **unacceptable** in mechanical design and control. Therefore, this popular method should be **discontinued**. Another drawback is it requires three functions where one will do.

2.2 Third-Order Polynomial

Using a **smooth motion** criterion popular in most robotics textbooks (e.g. Craig 2005, Jazar 2010, Koivo 1989, Sciavicco and Siciliano 1996, and Spong et al. 2005), we must ensure the joint angle position and velocity functions are continuous. Further, the initial and final joint angles must match the given angles and the initial and final joint rates must be zero for starting and stopping at rest. In this paper, $t = 0$ indicates the initial time, and $t = t_f$ indicates the final time. Also, θ_S is the starting, or initial, angle and θ_F is the final angle. Note all $i = 1,2,\ldots,n$ joint values will have distinct starting and final joint values; the subscript i is dropped for clarity in notation. The methods presented below apply independently for all n joints, but occurring simultaneously in time, from 0 to t_f.

The four smooth motion joint constraints are:

$$\begin{aligned} \theta(0) &= \theta_S & \qquad \theta(t_f) &= \theta_F \\ \dot{\theta}(0) &= 0 & \qquad \dot{\theta}(t_f) &= 0 \end{aligned} \tag{1}$$

Note that the velocity values can be non-zero if necessary. With four constraints, a third-order polynomial is required. The functions for the joint i angle and joint i angular velocity are:

$$\begin{aligned} \theta(t) &= a_3t^3 + a_2t^2 + a_1t + a_0 \\ \dot{\theta}(t) &= 3a_3t^2 + 2a_2t + a_1 \end{aligned} \tag{2}$$

Four linear equations in the four unknown polynomial coefficients a_i, $i = 0,1,2,3$ result from the four smooth motion constraints. Two of the unknown polynomial coefficients are found immediately, from the initial time constraints: $a_0 = \theta_S$ and $a_1 = 0$. The 2x2 matrix/vector equation to solve for the remaining two unknowns is:

$$\begin{bmatrix} 1 & t_F \\ 2 & 3t_F \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} (\theta_F - \theta_S)/t_F^2 \\ 0 \end{Bmatrix} \tag{3}$$

The single third-order polynomial solution for each joint is:

$$\theta(t) = -\frac{2}{t_F^3}(\theta_F - \theta_S)t^3 + \frac{3}{t_F^2}(\theta_F - \theta_S)t^2 + \theta_S \tag{4}$$

Example – Third-Order Polynomial

Use a third-order polynomial fit for smooth joint space trajectory generation, demonstrated for one joint only. Given $\theta_S = 30^\circ$, $\theta_F = 120^\circ$, $t_f = 3$ *sec*, find the third-order polynomial $\theta(t)$ and plot $\theta(t), \dot{\theta}(t), \ddot{\theta}(t), \dddot{\theta}(t)$. Result:

$$
\begin{aligned}
\theta(t) &= -6.67t^3 + 30t^2 + 30 \\
\dot{\theta}(t) &= -20t^2 + 60t \\
\ddot{\theta}(t) &= -40t + 60 \\
\dddot{\theta}(t) &= -40
\end{aligned}
\qquad (5)
$$

Note: *deg* units are used throughout this example. These results are plotted in Figure 2.

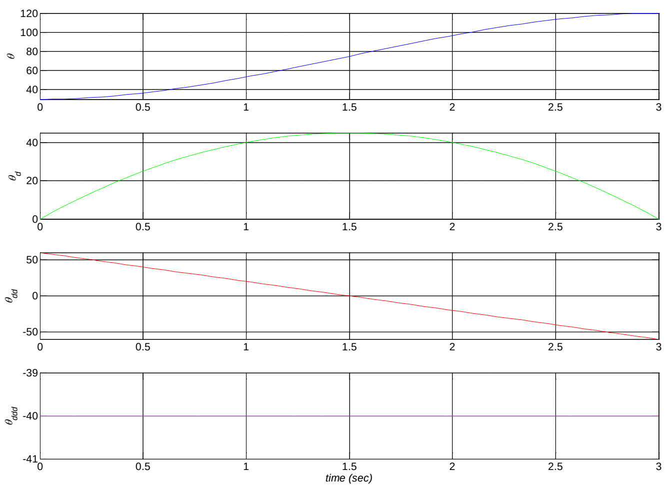

Figure 2. Third-Order Polynomial Results

We see the joint angle is cubic as specified, the joint velocity is quadratic, the joint acceleration is linear, and the joint jerk is constant. The joint jerk looks fine, but actually there is an infinite spike in jerk at the initial and final times due to the discontinuous accelerations at these points in time.

Once $\theta(t)$ is known, one must discretize the $\theta(t)$ function at the controller time step size and send these numerical values to the single robot joint controller at the proper times.

2.3 Fifth-Order Polynomial

Now, the third-order polynomial will provide smooth motion with zero velocity at the start and end. But the resulting *jerk* has infinite spikes at the start and end. To avoid this problem, use the same four smooth motion joint constraints from the third-order polynomial, plus two more constraints on acceleration to avoid the infinite *jerk* spikes:

$$
\begin{aligned}
\theta(0) &= \theta_S & \theta(t_f) &= \theta_F \\
\dot{\theta}(0) &= 0 & \dot{\theta}(t_f) &= 0 \\
\ddot{\theta}(0) &= 0 & \ddot{\theta}(t_f) &= 0
\end{aligned}
\qquad (6)
$$

Again, the velocity and acceleration values may be non-zero if necessary. With six constraints, a fifth-order polynomial is required, as presented by Craig (2005), Jazar (2010), Koivo (1989), and Sciavicco and Siciliano (1996). The functions for joint *i* angle, joint *i* angular velocity, and joint *i* acceleration are:

$$
\begin{aligned}
\theta(t) &= a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \\
\dot{\theta}(t) &= 5a_5 t^4 + 4a_4 t^3 + 3a_3 t^2 + 2a_2 t + a_1 \\
\ddot{\theta}(t) &= 20a_5 t^3 + 12a_4 t^2 + 6a_3 t + 2a_2
\end{aligned}
\qquad (7)
$$

The constraints yield six linear equations in the six unknown polynomial coefficients a_i, i = 0,1,2,3,4,5. Three of the unknown polynomial coefficients are found immediately, from the initial time constraints: $a_0 = \theta_S$, $a_1 = 0$, and $a_2 = 0$. The simplified 3x3 matrix/vector equation to solve for the remaining three unknowns is:

$$
\begin{bmatrix} 1 & t_F & t_F^2 \\ 3 & 4t_F & 5t_F^2 \\ 6 & 12t_F & 20t_F^2 \end{bmatrix}
\begin{Bmatrix} a_3 \\ a_4 \\ a_5 \end{Bmatrix}
=
\begin{Bmatrix} (\theta_F - \theta_S)/t_F^3 \\ 0 \\ 0 \end{Bmatrix}
\qquad (8)
$$

The single fifth-order polynomial solution for each joint is:

$$
\theta(t) = \frac{6}{t_F^5}(\theta_F - \theta_S)t^5 - \frac{15}{t_F^4}(\theta_F - \theta_S)t^4 + \frac{10}{t_F^3}(\theta_F - \theta_S)t^3 + \theta_S \qquad (9)
$$

Example – Fifth-Order Polynomial

Use a fifth-order polynomial fit for smooth joint space trajectory generation and finite jerk, demonstrated for one joint only. Given $\theta_S = 30^\circ$, $\theta_F = 120^\circ$, $t_f = 3$ *sec*, find the fifth-order polynomial $\theta(t)$ and plot $\theta(t), \dot{\theta}(t), \ddot{\theta}(t), \dddot{\theta}(t)$. Result:

$$
\begin{aligned}
\theta(t) &= 2.22t^5 - 16.67t^4 + 33.33t^3 + 30 \\
\dot{\theta}(t) &= 11.11t^4 - 66.67t^3 + 100t^2 \\
\ddot{\theta}(t) &= 44.44t^3 - 200t^2 + 200t \\
\dddot{\theta}(t) &= 133.33t^2 - 400t + 200
\end{aligned}
\qquad (10)
$$

Again, *deg* units are used throughout this example. These results are plotted in Figure 3.

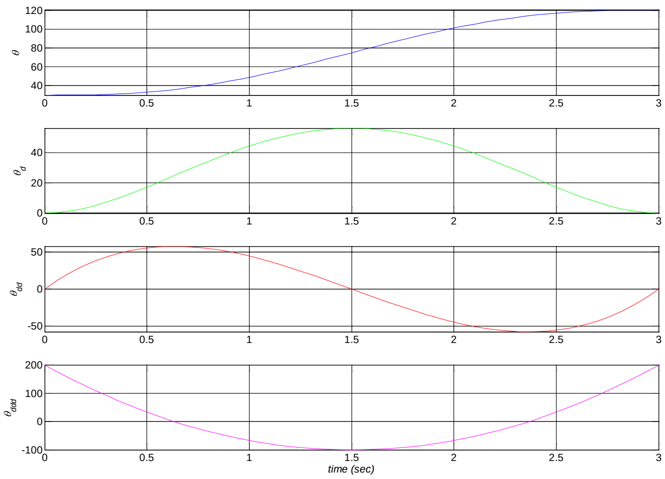

Figure 3. Fifth-Order Polynomial Results

Now the *jerk* has a discontinuous jump at the start and end, but it is a finite discontinuity. This fifth-order polynomial is thus superior to the previous third-order polynomial.

3. JOINT-SPACE TRAJECTORY GENERATION WITH VIA POINT

In many practical robotics joint-space motion planning situations, the robot must pass through intermediate point(s) between the start and finish poses, such as for obstacle avoidance. The joint rates and accelerations do not need to go to zero at these so-called via points, but they must be matched between functions meeting at the via point(s).

3.1 Two Third-Order Polynomials

Craig (2005) suggests the use of two third-order polynomials meeting at the via point. Two third-order polynomials will provide smooth motion with continuous position and velocity and zero velocity at the start and end. The same four smooth motion joint constraints from the single third-order polynomial given earlier apply, but now for two different third-order polynomials:

$$\begin{aligned} &\theta_1(0) = \theta_S && \theta_2(t_2) = \theta_F \\ &\dot{\theta}_1(0) = 0 && \dot{\theta}_2(t_2) = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} &\theta_1(t) = a_{13}t^3 + a_{12}t^2 + a_{11}t + a_{10} && \theta_2(t) = a_{23}t^3 + a_{22}t^2 + a_{21}t + a_{20} \\ &\dot{\theta}_1(t) = 3a_{13}t^2 + 2a_{12}t + a_{11} && \dot{\theta}_2(t) = 3a_{23}t^2 + 2a_{22}t + a_{21} \\ &\ddot{\theta}_1(t) = 6a_{13}t + 2a_{12} && \ddot{\theta}_2(t) = 6a_{23}t + 2a_{22} \end{aligned} \tag{12}$$

It is convenient to shift the time axis so that the $\theta_2(t)$ polynomial has zero time starting at the via time. t_1 is the end time of the first range ($t_1 = t_V$) and t_2 is the relative end time of the second range ($t_2 = t_F - t_V$).

We need four more constraints since we now have two third-order polynomials. We force the first polynomial to end at the via angle θ_V and the second polynomial to start at the via angle θ_V (two constraints). We also ensure the velocities match at the via point (they need not go to zero, they must only match – this is one constraint). We also ensure the accelerations match at the via point (again, they need not go to zero, only match – this is also one constraint). This will ensure that the *jerk* stays finite during this via point transition between the two polynomials. These four additional constraints are:

$$\begin{aligned} &\theta_1(t_1) = \theta_V && \dot{\theta}_1(t_1) = \dot{\theta}_2(0) \\ &\theta_2(0) = \theta_V && \ddot{\theta}_1(t_1) = \ddot{\theta}_2(0) \end{aligned} \tag{13}$$

The constraints yield eight linear equations in the eight unknowns (two third-order polynomials, four unknown coefficients each). Three of the unknown polynomial coefficients are found immediately, from the initial and via time constraints: $a_{10} = \theta_S$, $a_{11} = 0$, and $a_{20} = \theta_V$. The simplified 5x5 matrix/vector equation to solve for the remaining three unknowns is:

$$\begin{bmatrix} 1 & t_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & t_2 & t_2^2 \\ 0 & 0 & 1 & 2t_2 & 3t_2^2 \\ 2t_1 & 3t_1^2 & -1 & 0 & 0 \\ 2 & 6t_1 & 0 & -2 & 0 \end{bmatrix} \begin{Bmatrix} a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{Bmatrix} = \begin{Bmatrix} (\theta_V - \theta_S)/t_1^2 \\ (\theta_F - \theta_V)/t_2 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{14}$$

Solve (14) for the remaining five unknowns, for use in (12).

For the special case of $t_1 = t_2 = t_F/2 = t_V = T$, the analytical solution is $a_{10} = \theta_S$, $a_{11} = 0$, and $a_{20} = \theta_V$, and:

$$\begin{aligned} a_{12} &= \frac{-3(\theta_F - \theta_V) + 9(\theta_V - \theta_S)}{4T^2} \\ a_{13} &= \frac{3(\theta_F - \theta_V) - 5(\theta_V - \theta_S)}{4T^3} \\ a_{21} &= \frac{3(\theta_F - \theta_S)}{4T} \\ a_{22} &= \frac{3(\theta_F - \theta_V) - 3(\theta_V - \theta_S)}{2T^2} \\ a_{23} &= \frac{-5(\theta_F - \theta_V) + 3(\theta_V - \theta_S)}{4T^3} \end{aligned} \tag{15}$$

<u>Example – Two Third-Order Polynomials with Via Point</u>

Use two third-order polynomials for smooth joint space trajectory generation, plus motion through a via point (with no need to stop at the via point, but ensure smooth motion), for one joint. Given $\theta_S = 30^\circ$, $\theta_V = 180^\circ$, $\theta_F = 120^\circ$, $t_V = 1.5, t_F = 3$ ($t_1 = t_2 = 1.5$) *sec*, find $\theta_1(t)$ and $\theta_2(t)$ and plot the joint angle vs. time, with three derivatives. Result:

$$
\begin{aligned}
&\theta_1(t) = -68.89t^3 + 170t^2 + 30 && \theta_2(t) = 55.56t^3 - 140t^2 + 45t + 180 \\
&\dot{\theta}_1(t) = -206.67t^2 + 340t && \dot{\theta}_2(t) = 166.67t^2 - 280t + 45 \\
&\ddot{\theta}_1(t) = -413.33t + 340 && \ddot{\theta}_2(t) = 333.33t - 280 \\
&\dddot{\theta}_1(t) = -413.33 && \dddot{\theta}_2(t) = 333.33
\end{aligned}
\tag{16}
$$

Again, *deg* units are used throughout this example. These results are plotted in Figure 4.

Note that the joint angle passes through the via point $\theta_V = 180^\circ$ and keeps going for a brief time – this is because the velocity is still positive beyond the first 1.5 *sec*. The maximum angle is $\theta_{MAX} = 183.9^\circ$, occurring at t = 1.68 sec. At θ_{MAX} the angular velocity goes to zero (the slope of the angle is zero at that point since the angle is changing direction). Also note that the velocity term in $\theta_2(t)$, the t coefficient, is non-zero since the velocity does not need to go to zero at this via point. Further note that the velocity and acceleration have been successfully matched at the 1.5 *sec* via point transition as required. Note that the *jerk* is not matched at the transition (the slope of the two polynomial accelerations are the same magnitude but different signs), but it remains finite since the acceleration is matched.

Also, the *jerk* still has an infinite spike at the start and end as we saw in the single third-order polynomial example, which is **unacceptable**.

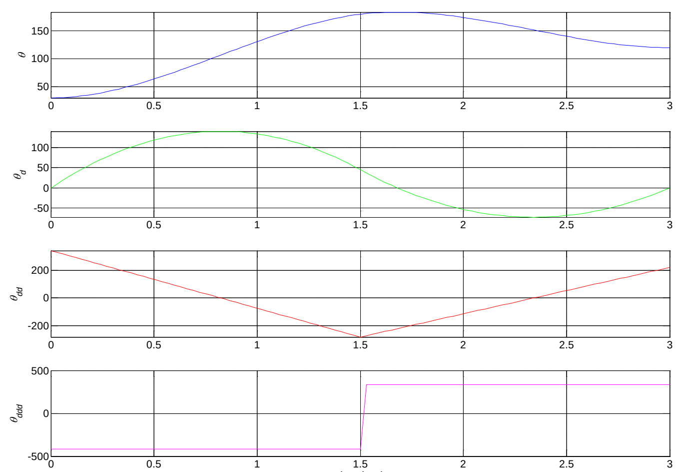

Figure 4. Two Third-Order Polynomials with Via Point

3.2 Two Fifth-Order Polynomials

The two third-order polynomials presented above, matched at a via point, appear in many robotics textbooks (e.g. Craig, 2005). However, this approach suffers from the same problem identified earlier: since the acceleration functions are discontinuous at the initial and final time, the jerk functions have infinite spikes at these points in time. We know this is unacceptable for the repeated motions of mechanical systems, due to unacceptable wear, noise, and dynamic excitation. To fix this, we could match two fifth-order polynomials at the via point, specifying zero acceleration at the initial and final times (not at the via time – instead matching velocities and accelerations between the two fifth-order polynomial functions at the via time). However, this will not presented because we came up with a better method, a single polynomial vs. two matched polynomials – this original work is presented in the next two subsections.

3.3 Single Fourth-Order Polynomial

Finally we present the first original contribution of this paper. We can achieve the same goals as the two third-order polynomials meeting at a via point much more simply: let us use only one polynomial, forced to go through the via point. Here are the constraints for meeting the required angles with smooth motion (since we use a single continuous polynomial, the velocity and accelerations are guaranteed to match and be continuous at the via point):

$$
\begin{matrix}
\theta(0) = \theta_S & & \theta(t_f) = \theta_F \\
 & \theta(t_V) = \theta_V & \\
\dot{\theta}(0) = 0 & & \dot{\theta}(t_f) = 0
\end{matrix}
\tag{17}
$$

Note in this case since there is only one time range, it is convenient to treat all times as absolute, rather than relative as when we did matching of two third-order polynomials. With five constraints we can use a single fourth-order polynomial:

$$
\begin{aligned}
\theta(t) &= a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \\
\dot{\theta}(t) &= 4a_4 t^3 + 3a_3 t^2 + 2a_2 t + a_1
\end{aligned}
\tag{18}
$$

Five linear equations in the five unknown polynomial coefficients a_i, i = 0,1,2,3,4 result from the five constraints (17). Two of the unknown polynomial coefficients are found immediately, from the initial time constraints: $a_0 = \theta_S$ and a_1 = 0. The simplified 3x3 matrix/vector equation to solve for the remaining three unknowns is:

$$
\begin{bmatrix} 1 & t_V & t_V^2 \\ 1 & t_F & t_F^2 \\ 2 & 3t_F & 4t_F^2 \end{bmatrix}
\begin{Bmatrix} a_2 \\ a_3 \\ a_4 \end{Bmatrix}
=
\begin{Bmatrix} (\theta_V - \theta_S)/t_V^2 \\ (\theta_F - \theta_S)/t_F^2 \\ 0 \end{Bmatrix}
\tag{19}
$$

The single fourth-order polynomial solution for each joint is:

$$
\begin{aligned}
a_2 &= \frac{1}{(t_F - t_V)^2}\left[\frac{(\theta_V - \theta_S)t_F^2}{t_V^2} - \frac{(\theta_F - \theta_S)(4t_F - 3t_V)t_V}{t_F^2}\right] \\
a_3 &= \frac{2}{(t_F - t_V)^2}\left[-\frac{(\theta_V - \theta_S)t_F}{t_V^2} + \frac{(\theta_F - \theta_S)(2t_F^2 - t_V^2)}{t_F^3}\right] \\
a_4 &= \frac{1}{(t_F - t_V)^2}\left[\frac{(\theta_V - \theta_S)}{t_V^2} - \frac{(\theta_F - \theta_S)(3t_F - 2t_V)}{t_F^3}\right]
\end{aligned}
\tag{20}
$$

For the special case of $t_V = t_F/2$, the solution is:

$$\begin{aligned} a_2 &= \frac{1}{t_F^2}\left[16(\theta_V - \theta_S) - 5(\theta_F - \theta_S)\right] \\ a_3 &= -\frac{2}{t_F^3}\left[16(\theta_V - \theta_S) - 7(\theta_F - \theta_S)\right] \\ a_4 &= \frac{8}{t_F^4}\left[\,2(\theta_V - \theta_S) - (\theta_F - \theta_S)\right] \end{aligned} \tag{21}$$

<u>Example – Single Fourth-Order Polynomial with Via Point</u>

Use a single fourth-order polynomial for smooth joint space trajectory generation, plus motion through a via point (with no need to stop at the via point, but ensure smooth motion), for one joint. Given $\theta_S = 30^\circ, \theta_V = 180^\circ, \theta_F = 120^\circ$, $t_V = 1.5, t_F = 3$ *sec*, find $\theta(t)$ and plot $\theta(t), \dot{\theta}(t), \ddot{\theta}(t), \dddot{\theta}(t)$. Result:

$$\begin{aligned} \theta(t) &= 20.74t^4 - 131.11t^3 + 216.67t^2 + 30 \\ \dot{\theta}(t) &= 82.96t^3 - 393.33t^2 + 433.33t \\ \ddot{\theta}(t) &= 248.89t^2 - 786.67t + 433.33 \\ \dddot{\theta}(t) &= 497.78t - 786.67 \end{aligned} \tag{22}$$

Again, *deg* units are used throughout this example. These results are plotted in Figure 5.

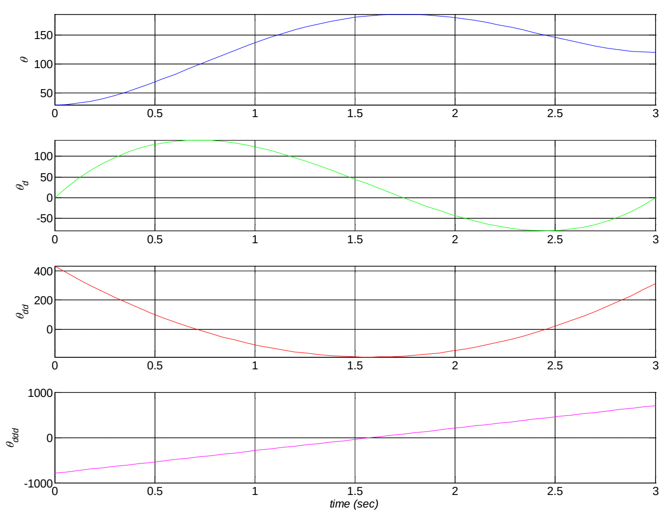

Figure 5. Single 4th Order Polynomial with Via Point

The $\theta(t)$ shape for the single fourth-order polynomial is very similar to that for the two third-order polynomials. Again the joint angle passes through the via point $\theta_V = 180^\circ$ and keeps going briefly, due to the fact that the velocity is still positive beyond the first 1.5 *sec*. The peak for this fourth-order case is slightly greater and occurs in time slightly after the peak for the two third-order polynomials case. The maximum angle is $\theta_{MAX} = 185.4^\circ$, occurring at t = 1.74 sec. For the fourth-order polynomial, a side benefit has arisen: the *jerk* is now continuous at the via time, where it was discontinuous for the two matched third-order polynomials.

The *jerk* still has an infinite spike at the start and end as we saw with both previous third-order polynomial examples, which is **unacceptable**. We now improve upon this with a single sixth-order polynomial in the next subsection – again, this work is original.

3.4 Single Sixth-Order Polynomial

We can achieve the same goals as the single fourth-order polynomial with via point and eliminate the infinite spikes in the *jerk* at the start and end as follows. To the previous 5 constraints, add two more, for zero acceleration at the start and end of the single motion range. Here are the seven constraints:

$$\begin{array}{lcl} \theta(0) = \theta_S & & \theta(t_f) = \theta_F \\ \dot{\theta}(0) = 0 & \theta(t_V) = \theta_V & \dot{\theta}(t_f) = 0 \\ \ddot{\theta}(0) = 0 & & \ddot{\theta}(t_f) = 0 \end{array} \tag{23}$$

With seven constraints we can use a single sixth-order polynomial:

$$\begin{aligned} \theta(t) &= a_6t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \\ \dot{\theta}(t) &= 6a_6t^5 + 5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1 \\ \ddot{\theta}(t) &= 30a_6t^4 + 20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2 \end{aligned} \tag{24}$$

Seven linear equations in the seven unknown polynomial coefficients a_i, i = 0,1,2,3,4,5,6 result from the seven constraints (23). Three of the unknown polynomial coefficients are found immediately, from the initial time constraints: $a_0 = \theta_S$, $a_1 = 0$, and $a_2 = 0$. The simplified 4x4 matrix/vector equation to solve for the remaining four unknowns is:

$$\begin{bmatrix} 1 & t_V & t_V^2 & t_V^3 \\ 1 & t_F & t_F^2 & t_F^3 \\ 3 & 4t_F & 5t_F^2 & 6t_F^3 \\ 6 & 12t_F & 20t_F^2 & 30t_F^3 \end{bmatrix} \begin{Bmatrix} a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \begin{Bmatrix} (\theta_V - \theta_S)/t_V^3 \\ (\theta_F - \theta_S)/t_F^3 \\ 0 \\ 0 \end{Bmatrix} \tag{25}$$

The single sixth-order polynomial solution for each joint is:

$$a_3 = \frac{1}{(t_F - t_V)^3}\left[\frac{(\theta_V - \theta_S)t_F^3}{t_V^3} - \frac{(\theta_F - \theta_S)(15t_F^2 - 24t_F t_V + 10t_V^2)t_V}{t_F^3}\right]$$

$$a_4 = \frac{3}{(t_F - t_V)^3}\left[-\frac{(\theta_V - \theta_S)t_F^2}{t_V^3} + \frac{(\theta_F - \theta_S)(5t_F^3 - 9t_F t_V^2 + 5t_V^3)}{t_F^4}\right]$$

$$a_5 = \frac{3}{(t_F - t_V)^3}\left[\frac{(\theta_V - \theta_S)t_F}{t_V^3} - \frac{(\theta_F - \theta_S)(8t_F^3 - 9t_F^2 t_V + 2t_V^3)}{t_F^5}\right]$$

$$a_6 = \frac{1}{(t_F - t_V)^3}\left[\frac{(\theta_V - \theta_S)}{t_V^3} - \frac{(\theta_F - \theta_S)(10t_F^2 - 15t_F t_V + 6t_V^2)}{t_F^5}\right] \tag{26}$$

For the special case of $t_V = t_F/2$, the analytical solution is:

$$\begin{aligned}
a_3 &= \frac{2}{t_F^3}\left[32(\theta_V - \theta_S) - 11(\theta_F - \theta_S)\right] \\
a_4 &= -\frac{3}{t_F^4}\left[64(\theta_V - \theta_S) - 27(\theta_F - \theta_S)\right] \\
a_5 &= \frac{3}{t_F^5}\left[64(\theta_V - \theta_S) - 30(\theta_F - \theta_S)\right] \\
a_6 &= -\frac{32}{t_F^6}\left[2(\theta_V - \theta_S) - (\theta_F - \theta_S)\right]
\end{aligned} \tag{27}$$

__Example – Single Sixth-Order Polynomial with Via Point__

Use a single sixth-order polynomial for smooth joint space trajectory generation, plus motion through a via point (with no need to stop at the via point, but ensure smooth motion), for one joint. Given $\theta_S = 30^\circ$, $\theta_V = 180^\circ$, $\theta_F = 120^\circ$, $t_V = 1.5, t_F = 3$ *sec*, find $\theta(t)$ and plot $\theta(t), \dot{\theta}(t), \ddot{\theta}(t), \dddot{\theta}(t)$. Result:

$$\begin{aligned}
\theta(t) &= -9.22t^6 + 85.19t^5 - 265.56t^4 + 282.22t^3 + 30 \\
\dot{\theta}(t) &= -55.3t^5 + 425.9t^4 - 1062.6t^3 + 846.7t^2 \\
\ddot{\theta}(t) &= -276.5t^4 + 1703.7t^3 - 3186.7t^2 + 1693.3t \\
\dddot{\theta}(t) &= -1106.2t^3 + 5111.1t^2 - 6373.3t + 1693.3
\end{aligned} \tag{28}$$

Again, *deg* units are used throughout this example. These results are plotted in Figure 6.

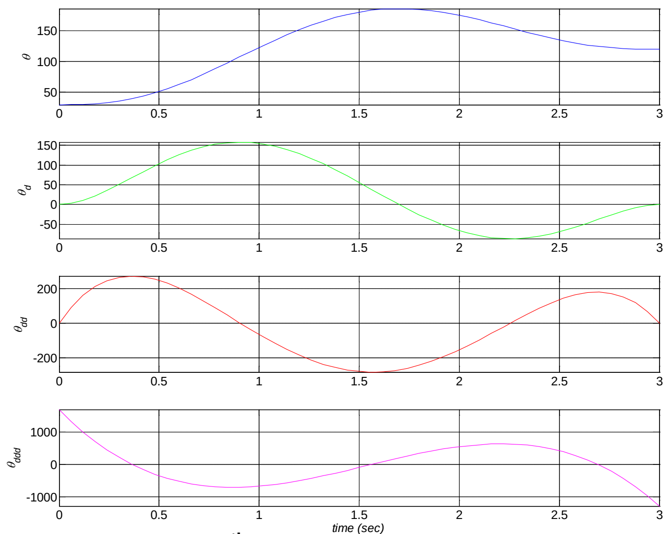

Figure 6. Single 6th-Order Polynomial with Via Point

Again the $\theta(t)$ shape is similar to the previous two cases (two third-order polynomials and single fourth-order polynomial) with via point. The maximum angle is $\theta_{MAX} = 185.6^\circ$, occurring at t = 1.70 sec. Note the magnitude of the angular velocity, acceleration, and jerk are not greatly different for the three cases we have presented with via point. Generally they are the lowest for the two third-order polynomials. This single sixth-order polynomial approach has eliminated the problem of infinite spikes in *jerk* at the start and end of each motion. Here the *jerk* is discontinuous but it remains finite, which obeys the rule of thumb for mechanical design/motion.

4. CONCLUSION

This main original contribution of this paper is to present a new joint space trajectory generation approach using a single polynomial for motion through a given via point. The robot need not stop at the via point, but the motion must be smooth and continuous through the via point. A single fourth-order polynomial was developed to achieve this smooth motion through a via point, to replace the two three-order polynomials matched at the via point in common usage today. With the new approach this matching comes automatically and there is no need for two polynomial functions.

An important secondary contribution of this paper is to expose a bad practice in common usage in joint-space trajectory generation in robotics today. We are not the first to notice this (Macfarlane and Croft 2003, Gosselin and Hadj-Messaoud 1993, and Petrinec and Kovacic 2007), but judging from the four robotics textbooks in the reference list published within the past five years, the message has not been widely understood. For any functions in which the acceleration is discontinuous, the associate jerk (time derivative of acceleration) function will have infinite spikes at those acceleration discontinuities. Cam design teaches that these infinite spikes in jerk are

unacceptable and must be avoided by the designed/motion controller. By extension, in robot joint space trajectory generation, any method which allows discontinuous acceleration functions are **unacceptable** and must not be used. Unfortunately, the linear velocity with parabolic blends, the third-order polynomial, and the two third-order polynomials matched to meet at a via point all suffer from discontinuity in acceleration and thus infinite spikes in jerk. These methods are still very much in use in the standard robotics textbooks today – these methods must be discarded. Otherwise, unacceptable noise, wear, stress, reduced life, and the introduction of bad dynamics may result.

The single fourth-order polynomial introduced in this paper to go through a via point also suffers from this discontinuous acceleration. Therefore, to remedy this we introduced a novel sixth-order polynomial to move smoothly and continuously through a via point, with finite jerk throughout the entire motion, which is **acceptable**.

For joint-space trajectory generation in robotics, the recommendations of this paper are simple – for all joint motions without a via point, use the standard fifth-order polynomial from Craig (2005), Koivo (1989), and Sciavicco and Siciliano (1996). For all joint motions with a via point, use the new single sixth-order polynomial introduced in this paper. Not only does this new sixth-order polynomial keep the jerk finite, it also achieves the via point with only one function.

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