Proceedings of DETC '97
1997 ASME Design Technical Conferences
September 14-17, 1997, Sacramento, California

DETC97/DAC-3851

INVERSE KINEMATICS FOR PLANAR PARALLEL MANIPULATORS

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ABSTRACT

This paper presents algebraic inverse position and velocity kinematics solutions for a broad class of three degree-of-freedom planar in-parallel-actuated manipulators. Given an end-effector pose and rate, all active and passive joint values and rates are calculated independently for each serial chain connecting the ground link to the end-effector link. The solutions are independent of joint actuation. Seven serial chains consisting of revolute and prismatic joints are identified and their inverse solutions presented. To reduce computations, inverse Jacobian matrices for overall manipulators are derived to give only actuated joint rates. This matrix yields conditions for invalid actuation schemes. Simulation examples are given.

INTRODUCTION

Parallel manipulators are robots that consist of separate serial chains that connect the fixed link to the end-effector link. The following are potential advantages over serial robots: better stiffness and accuracy, lighter weight, greater load bearing, higher velocities and accelerations, and less powerful actuators. A major drawback of the parallel robot is reduced workspace.

Parallel robotic devices were proposed over 17 years ago (MacCallion and Pham, 1979). Some configurations have been built and controlled (e.g. Sumpter and Soni, 1985). Numerous works analyze kinematics, dynamics, workspace and control of parallel manipulators (see Williams, 1988 and references therein). Hunt (1983) conducted preliminary studies of various parallel robot configurations. Cox and Tesar (1981) compared the relative merits of serial and parallel robots.

These past works have focused on only a few different architectures. For example, Aradyfio and Qiao (1985) examine the inverse kinematics solutions for three different 3-dof planar parallel robots. Williams and Reinholtz (1988a and 1988b) study dynamics and workspace for a limited number of parallel manipulators. Shirkhodaie and Soni (1987) , Gosselin and Angeles (1988), and Pennock and Kassner (1990) each present a kinematic study of one planar parallel robot. Gosselin et.al. (1996) present the position, workspace, and velocity kinematics of one planar parallel robot.

Recently, more general approaches have been presented. Daniali et.al. (1995) present an in-depth study of actuation schemes, velocity relationships, and singular conditions for general planar parallel robots. Gosselin (1996) presents general parallel computation algorithms for kinematics and dynamics of planar and spatial parallel robots. Merlet (1996) solved the forward position kinematics problem for a class of planar parallel robots (see Fig. 1).

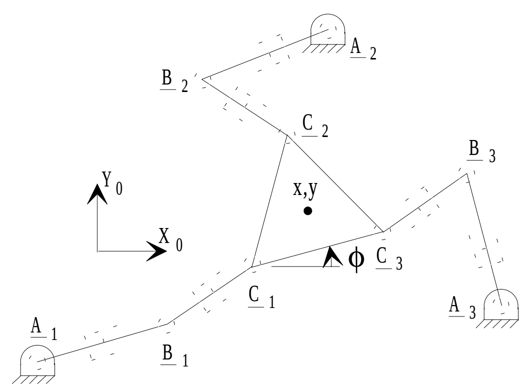

Figure 1. General Class of Manipulators

The current paper presents a purely algebraic approach to solve inverse position and velocity kinematics for Merlet's class of 3-dof planar parallel manipulators. Geometric approaches have been applied with success in the past for specific architectures (e.g. Williams and Reinholtz, 1988b and Gosselin and Angeles, 1988). The algebraic methods in this paper are suitable when one desires kinematic solutions for control of all members in a broad class of planar parallel manipulators. For instance, the current paper would enable implementation of modular reconfigurable 3-dof planar parallel manipulators.

This paper is organized as follows. First, the class of manipulators is discussed. Next the general inverse position kinematics solutions are presented, in terms of independent solutions for all possible serial chains. The general inverse velocity solutions are presented, singularity conditions are presented for each chain, and the overall inverse Jacobian is derived. Determination of invalid actuation schemes is presented. Simulation examples are given for position and velocity.

CLASS OF MANIPULATORS

Overall Manipulator

The class of manipulators in this paper are planar and actuated in parallel. Three serial chains with 3-dof each connect the fixed link to the end-effector link. One joint per chain is actuated and the remaining two are passive. The active and/or passive joints may be revolute (*R*) and/or prismatic (*P*) (see Fig. 1). If serial chain *i* is *RPR*, there is a revolute joint at $\underline{A}_i$, one of the two binary links is variable, the angle at $\underline{B}_i$ is fixed, and there is a revolute joint at $\underline{C}_i$.

Serial Chains

When solving the forward position kinematics problem for this robot class the actuated joint in each serial chain must be specified. Merlet, (1996) identifies eighteen possible serial chains, given in Table 1. Underlining indicates actuation. Any one of the three joints in a chain may be actuated unless the remaining passive joints are *PP*. Therefore, the combinations $\underline{R}PP$, $P\underline{R}P$, and $PP\underline{R}$ are omitted from Table I.

$\underline{R}RR$	$R\underline{R}R$	$RR\underline{R}$	$\underline{R}RP$	$R\underline{R}P$	$RR\underline{P}$
$\underline{R}PR$	$R\underline{P}R$	$RP\underline{R}$	$R\underline{P}P$	$RP\underline{P}$	$\underline{P}RR$
$P\underline{R}R$	$PR\underline{R}$	$\underline{P}RP$	$PR\underline{P}$	$\underline{P}PR$	$P\underline{P}R$

Table 1. Actuated Serial Chain Combinations

The *PPP* chain is not used because only two planar *P* joints in a chain are independent. For the inverse position problem of the current paper, the actuation scheme does not affect the solutions. Table 1 reduces to seven chains, given in Table 2 and Fig. 2.

RRR	*RRP*	*RPR*	*RPP*	*PRR*	*PRP*	*PPR*

Table 2. Serial Chains for Inverse Kinematics

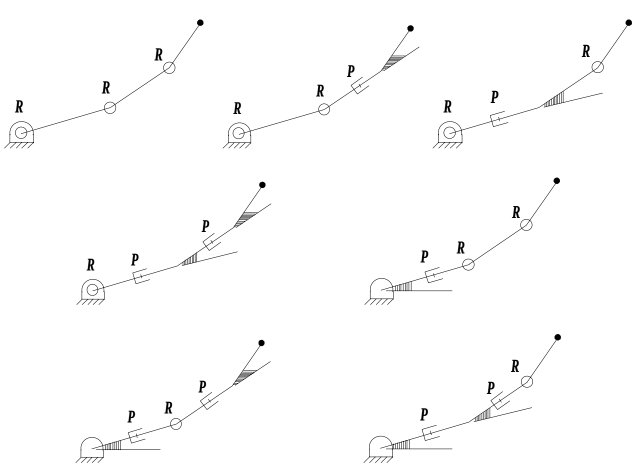

Figure 2. Seven Serial Chains

For an overall manipulator, there are three independent serial chains, each having seven possible configurations. There are 343 (7^3) robots (the number is considerably lower if the ordering of distinct chains is not considered important). A general method is required to solve inverse kinematics for all.

INVERSE POSITION KINEMATICS SOLUTIONS

The inverse position kinematics problem is stated: Given the end-effector pose $\{x \quad y \quad \phi\}^T$, calculate the three actuated joint (*R* or *P*) values. The passive joint values may also be determined for use in velocity, acceleration, dynamics. The inverse position problem may be solved for each serial chain independently. The solution is not dependent on joint actuation.

The kinematic diagram for the i^{th} chain is Fig. 3, for all seven chains in Fig. 2. Equation 1 is the position vector equation for and Eq. 2 is the angle relationship (ψ_i are defined in Fig. 4).

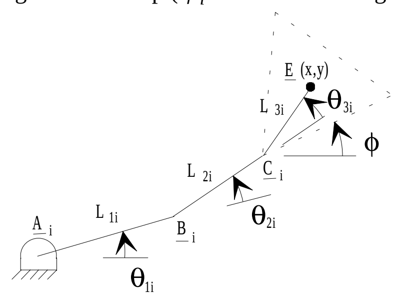

Figure 3. Kinematic Diagram for all Seven Serial Chains

$$\underline{E} = \begin{Bmatrix} x \\ y \end{Bmatrix} = \underline{A}_i + L_{1i} e^{j\theta_{1i}} + L_{2i} e^{j(\theta_{1i}+\theta_{2i})} + L_{3i} e^{j(\theta_{i1}+\theta_{2i}+\theta_{3i})} \quad (1)$$

$$\theta_{1i} + \theta_{2i} + \theta_{3i} + \pi = \phi + \psi_i \quad (2)$$

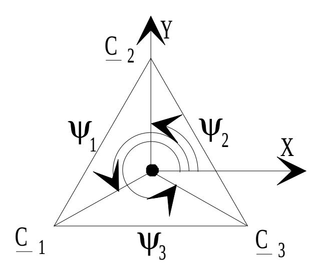

Figure 4. End-Effector Triangle Geometry

The inverse kinematics problem requires three independent inverse solutions, one for each serial chain. The coupled transcendental Eqs. 1 and 2 must be solved three times, *i*=1,2,3. If joint *k* on chain *i* is revolute, θ_{ki} is variable; if joint *k* on chain *i* is prismatic, L_{ki} is variable. The remaining terms are fixed.

The given manipulator pose fixes the position and orientation of the end-effector triangle in the plane. Each $\underline{C}_i$ is:

$$\begin{aligned} C_{ix} &= x + L_{3i}\cos(\phi + \psi_i) \\ C_{iy} &= y + L_{3i}\sin(\phi + \psi_i) \end{aligned} \quad (3)$$

Making use of Eq. 3 and expanding Eq. 1 to x and y components:

$$\begin{aligned} A_{ix} + L_{1i}c_1 + L_{2i}c_{12} &= C_{ix} \\ A_{iy} + L_{1i}s_1 + L_{2i}s_{12} &= C_{iy} \end{aligned} \tag{4}$$

where $c_{12} = \cos(\theta_{1i} + \theta_{2i})$ and $s_{12} = \sin(\theta_{1i} + \theta_{2i})$. Equations 4 and 2 are three equations in three unknowns to solve for each of the three serial chains in a given manipulator. Inverse position solutions are given below for the seven serial chains.

***RRR* Chain** Given $\{x \quad y \quad \phi\}^T$, determine $\theta_{1i}, \theta_{2i}, \theta_{3i}$. Isolating $\theta_{1i} + \theta_{2i}$ terms in Eqs. 4, squaring and adding yields Eq. 5 in θ_{1i}. Equation 6 is solved for θ_{1i} using the tangent half-angle substitution (Mabie and Reinholtz, 1987); the result is Eq. 6. There are two solutions (*elbow-up* and *elbow-down*). For each θ_{1i}, a unique θ_{2i} is found by a y to x ratio of Eqs. 4, given in Eq. 7. For each (θ_{1i}, θ_{2i}), a unique θ_{3i} is found from Eq. 2.

$$Ec_1 + Fs_1 + G = 0 \qquad \begin{aligned} E &= 2(C_{ix} - A_{ix})L_{1i} \\ F &= 2(C_{iy} - A_{iy})L_{1i} \\ G &= L_{2i}{}^2 - L_{1i}{}^2 - (C_{ix} - A_{ix})^2 - (C_{iy} - A_{iy})^2 \end{aligned} \tag{5}$$

$$\theta_{1i_{1,2}} = 2\tan^{-1}\left(\frac{-F \pm \sqrt{E^2 + F^2 - G^2}}{G - E}\right) \tag{6}$$

$$\theta_{2i_{1,2}} = a\tan 2\left(C_{iy} - A_{iy} - L_{1i}s_{1_{1,2}}, C_{ix} - A_{ix} - L_{1i}c_{1_{1,2}}\right) - \theta_{1i_{1,2}} \tag{7}$$

***RRP* Chain** Given $\{x \quad y \quad \phi\}^T$, determine $\theta_{1i}, \theta_{2i}, L_{2i}$. Equation 2 is rewritten $\theta_{2i} = \zeta_i - \theta_{1i}$ where ζ_i is the constant $\phi + \psi_i - \theta_{3i} - \pi$. Eqs. 4 are simplified to eliminate θ_{1i}. L_{2i} is isolated in the x equation (Eq. 8) and substituted into the y equation to give the Eq. 5 form, where E, F, and G are Eq. 9. Two values for θ_{1i} are obtained using Eq. 6. θ_{2i} and L_{2i} may now be evaluated from $\theta_{2i} = \zeta_i - \theta_{1i}$ and Eq. 8. The two solutions correspond to a *shorter* and *longer* L_{2i} joint length.

$$L_{2i} = \frac{C_{ix} - A_{ix} - L_{1i}c_1}{c\zeta_i} \tag{8}$$

$$\begin{aligned} E &= L_{1i}s\zeta_i \\ F &= -L_{1i}c\zeta_i \\ G &= (C_{iy} - A_{iy})c\zeta_i + A_{ix} - C_{ix} \end{aligned} \tag{9}$$

***RPR* Chain** Given $\{x \quad y \quad \phi\}^T$, determine $\theta_{1i}, L_{1i}, \theta_{3i}$. The $\underline{A}_i$ terms of Eqs. 4 are brought to the *RHS*, sum of angles formulas are used, and L_{1i} is isolated in the x equation (Eq. 10). L_{1i} is substituted into the y equation and greatly simplified to the Eq. 5 form, where E, F, and G are Eq. 11. Two θ_{1i} are found from Eq. 6. L_{1i} may now be evaluated from Eq. 10, and θ_{3i} from Eq. 2. There are two solutions but one has negative L_{1i}.

$$L_{1i} = \frac{C_{ix} - A_{ix} - L_{2i}(c_1c_2 - s_1s_2)}{c_1} \tag{10}$$

$$\begin{aligned} E &= C_{iy} - A_{iy} \\ F &= A_{ix} - C_{ix} \\ G &= -L_{2i}s_2 \end{aligned} \tag{11}$$

***RPP* Chain** Given $\{x \quad y \quad \phi\}^T$, determine $\theta_{1i}, L_{1i}, L_{2i}$. A unique θ_{1i} is found from Eq. 2. Eqs. 4 are then two linear equations in the unknowns L_{1i}, L_{2i} with a unique solution.

$$L_{1i} = \frac{(C_{ix} - A_{ix})s_{12} - (C_{iy} - A_{iy})c_{12}}{s_2} \tag{12}$$

$$L_{2i} = \frac{-(C_{ix} - A_{ix})s_1 + (C_{iy} - A_{iy})c_1}{s_2} \tag{13}$$

***PRR* Chain** Given $\{x \quad y \quad \phi\}^T$, determine $L_{1i}, \theta_{2i}, \theta_{3i}$. The solution starts with Eq. 11 (and Eq. 6) of the *RPR*. In this case the variable to be solved is θ_{2i}, not the constant θ_{1i}. Two solutions exist, a *near* (Eq. 21) and *far* ($\pi - \theta_{2i}$) solution. L_{1i} and θ_{3i} are solved using Eqs. 10 and 2.

$$(C_{ix} - A_{ix})s_1 + L_{2i}s_2 = (C_{iy} - A_{iy})c_1 \tag{14}$$

$$\theta_{2i} = \sin^{-1}\left(\frac{(C_{iy} - A_{iy})c_1 - (C_{ix} - A_{ix})s_1}{L_{2i}}\right) \tag{15}$$

***PRP* Chain** Given $\{x \quad y \quad \phi\}^T$, determine $L_{1i}, \theta_{2i}, L_{2i}$. A unique θ_{2i} is found from Eq. 2. The remaining unknowns L_{1i} and L_{2i} are found from Eqs. 12 and 13.

***PPR* Chain** Given $\{x \quad y \quad \phi\}^T$, determine $L_{1i}, L_{2i}, \theta_{3i}$. θ_{1i} and θ_{2i} are constant so the unknowns L_{1i} and L_{2i} are found from Eqs. 12 and 13. A unique θ_{3i} is solved with Eq. 2.

Overall Manipulator

For the inverse position kinematics problem for all 343 robots, the three serial chains are solved independently. Given the commanded pose $\{x \quad y \quad \phi\}^T$, Eq. 3 is used. Then the unknown joint parameters for each serial chain are solved by choosing the proper algorithm from above. The number of inverse position solutions is the product of the number of solutions (1 or 2) in each serial chain.

INVERSE VELOCITY KINEMATICS SOLUTIONS

The inverse velocity kinematics problem is stated: Given the end-effector rate command $\dot{X} = \{\dot{x} \quad \dot{y} \quad \omega_z\}^T$ and all position variables, calculate the three actuated joint (R or P) rates. Additionally, the passive joint rates may be determined for use in acceleration and dynamics. The inverse velocity problem may be solved for each serial chain independently. The solution is not dependent on actuation scheme.

Velocity relationships for each independent serial chain i are obtained from the first time derivatives of Eqs. 1 and 2 (where the index i is dropped for notational convenience):

$$\begin{Bmatrix}\dot{x}\\ \dot{y}\end{Bmatrix} = L_1 je^{j\theta_1}\dot{\theta}_1 + \dot{L}_1 e^{j\theta_1} + L_2 je^{j(\theta_1+\theta_2)}\left(\dot{\theta}_1+\dot{\theta}_2\right) + \dot{L}_2 e^{j(\theta_1+\theta_2)} + L_3 je^{j(\theta_1+\theta_2+\theta_3)}\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right) + \dot{L}_3 e^{j(\theta_1+\theta_2+\theta_3)} \quad (16)$$

$$\omega_z = \dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \quad (17)$$

Separating Eq. 16 into real and imaginary parts yields the x and y velocities. Note that $\dot{L}_{3i} = 0$ because L_{3i} is part of the rigid triangle link. The general translational and rotational velocities for all seven serial chains are:

$$\begin{aligned}
\dot{x} &= -L_1 s_1\dot{\theta}_1 + \dot{L}_1 c_1 - L_2 s_{12}\left(\dot{\theta}_1+\dot{\theta}_2\right) + \dot{L}_2 c_{12} - L_3 s_{123}\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right)\\
\dot{y} &= L_1 c_1\dot{\theta}_1 + \dot{L}_1 s_1 + L_2 c_{12}\left(\dot{\theta}_1+\dot{\theta}_2\right) + \dot{L}_2 s_{12} + L_3 c_{123}\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right) \quad (18)\\
\omega_z &= \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3
\end{aligned}$$

The velocity relationship for each serial chain can be expressed:

$$\dot{X} = J_i \dot{\rho}_i \quad (19)$$

where $\dot{X} = \{\dot{x} \quad \dot{y} \quad \omega_z\}^T$ is the Cartesian end-effector velocity, J_i is the i^{th} serial chain 3x3 Jacobian matrix, and $\dot{\rho}_i$ is the vector of joint rates for the i^{th} serial chain (one active and two passive, R and/or P). Since Eqs. 18 are written with respect to $\{0\}$ coordinates, that is the frame of expression for $\dot{X}$ and J_i. Jacobian matrices for each of the seven chains are determined from Eqs. 18 by zeroing terms that do not apply to a specific chain. Jacobian matrices for each of the seven serial chains are:

$$\textbf{RRR}\begin{bmatrix} j_{11} & j_{12} & j_{13}\\ j_{21} & j_{22} & j_{23}\\ 1 & 1 & 1\end{bmatrix} \textbf{RRP}\begin{bmatrix} j_{11} & j_{12} & c_{12}\\ j_{21} & j_{22} & s_{12}\\ 1 & 1 & 0\end{bmatrix} \textbf{RPR}\begin{bmatrix} j_{11} & c_1 & j_{13}\\ j_{21} & s_1 & j_{23}\\ 1 & 0 & 1\end{bmatrix}$$

$$\textbf{RPP}\begin{bmatrix} j_{11} & c_1 & c_{12}\\ j_{21} & s_1 & s_{12}\\ 1 & 0 & 0\end{bmatrix} \textbf{PRR}\begin{bmatrix} c_1 & j_{12} & j_{13}\\ s_1 & j_{22} & j_{23}\\ 0 & 1 & 1\end{bmatrix}$$

$$\textbf{PRP}\begin{bmatrix} c_1 & j_{12} & c_{12}\\ s_1 & j_{22} & s_{12}\\ 0 & 1 & 0\end{bmatrix} \textbf{PPR}\begin{bmatrix} c_1 & c_{12} & j_{13}\\ s_1 & s_{12} & j_{23}\\ 0 & 0 & 1\end{bmatrix} \quad \text{(Eqs. 20-26)}$$

where: $j_{11} = -L_1 s_1 - L_2 s_{12} - L_3 s_{123}$, $j_{12} = -L_2 s_{12} - L_3 s_{123}$, $j_{13} = -L_3 s_{123}$, $j_{21} = L_1 c_1 + L_2 c_{12} + L_3 c_{123}$, $j_{22} = L_2 c_{12} + L_3 c_{123}$, $j_{23} = L_3 c_{123}$, $c_{123} = \cos\left(\theta_1+\theta_2+\theta_3\right)$, $s_{123} = \sin\left(\theta_1+\theta_2+\theta_3\right)$.

The inverse velocity solution for the i^{th} serial chain is:

$$\dot{\rho}_i = J_i^{-1}\dot{X} \quad (27)$$

providing J_i has full rank. These inverse Jacobians are presented symbolically below for the seven serial chains.

$$\textbf{RRR} \quad \frac{1}{L_1 L_2 s_2}\begin{bmatrix} L_2 c_{12} & L_2 s_{12} & L_2 L_3 s_3\\ -L_1 c_1 - L_2 c_{12} & -L_1 s_1 - L_2 s_{12} & -L_3\left(L_1 s_{23} + L_2 s_3\right)\\ L_1 c_1 & L_1 s_1 & L_1\left(L_2 s_2 + L_3 s_{23}\right)\end{bmatrix}$$

$$\textbf{RRP} \quad \frac{1}{L_1 c_2}\begin{bmatrix} -s_{12} & c_{12} & -L_2 - L_3 c_3\\ s_{12} & -c_{12} & L_1 c_2 + L_2 + L_3 c_3\\ L_1 c_1 & L_1 s_1 & L_1\left(L_2 s_2 + L_3 s_{23}\right)\end{bmatrix}$$

RPR

$$\frac{-1}{L_1 + L_2 c_2}\begin{bmatrix} s_1 & -c_1 & L_3 c_{23}\\ -L_1 c_1 - L_2 c_{12} & -L_1 s_1 - L_2 s_{12} & -L_3\left(L_1 s_{23} + L_2 s_3\right)\\ -s_1 & c_1 & -L_1 - L_2 c_2 - L_3 c_{23}\end{bmatrix}$$

$$\textbf{RPP} \quad \frac{1}{s_2}\begin{bmatrix} 0 & 0 & s_2\\ s_{12} & -c_{12} & L_1 c_2 + L_2 + L_3 c_3\\ -s_1 & c_1 & -L_1 - L_2 c_2 - L_3 c_{23}\end{bmatrix}$$

$$\textbf{PRR} \quad \frac{1}{L_2 c_2}\begin{bmatrix} L_2 c_{12} & L_2 s_{12} & L_2 L_3 s_3\\ -s_1 & c_1 & -L_3 c_{23}\\ s_1 & -c_1 & L_2 c_2 + L_3 c_{23}\end{bmatrix}$$

$$\textbf{PRP} \quad \frac{1}{s_2}\begin{bmatrix} s_{12} & -c_{12} & L_2 + L_3 c_3\\ 0 & 0 & s_2\\ -s_1 & c_1 & -L_2 c_2 - L_3 c_{23}\end{bmatrix}$$

$$\textbf{PPR} \quad \frac{1}{s_2}\begin{bmatrix} s_{12} & -c_{12} & L_3 c_3\\ -s_1 & c_1 & -L_3 c_{23}\\ 0 & 0 & s_2\end{bmatrix} \quad \text{(Eqs. 28-34)}$$

The singularity conditions (cases where J_i has less than full rank) for each chain are determined as follows. First it is assumed that $L_1 \neq 0$ and $L_2 \neq 0$; these correspond to degenerate robot chains. The *RRR*, *RPP*, *PRP*, and *PPR* serial chains are all singular when $s_2 = 0$

($\theta_{2i} = 0, \pi$). The *RRP* and *PRR* serial chains are singular when $c_2 = 0$ ($\theta_{2i} = \pm\pi/2$). Note that in some cases the relative angle θ_{2i} is fixed so it can be designed to avoid the singularity for all motion, where in other cases θ_{2i} is variable. The *RPR* serial chain is singular when $L_1 + L_2 c_2 = 0$.

Overall Manipulator

The preceding section is sufficient to solve the inverse velocity problem. Three appropriate inverse mappings (Eqs. 28-34) must be chosen, one for each independent serial chain. The Cartesian rates $\dot{X} = \{\dot{x} \quad \dot{y} \quad \omega_z\}^T$ are commanded to each chain; all joint rates, active and passive, are solved.

However, the above procedure yields six passive results which are not required for resolved-rate control. Therefore, for more efficient real-time computation, a method is developed in this section to map $\dot{X}$ into active joint rates only. This mapping is called the overall manipulator inverse Jacobian matrix.

In this case, the actuated joints must be considered. The j^{th} row of the seven serial chain inverse Jacobian matrices (Eqs. 28-34) represents the mapping of the end-effector's velocity to the j^{th} joint. Therefore, for any robot configuration, its overall inverse Jacobian *M* may be constructed from the rows corresponding to the actuated joints in each independent chain. The 3*x*3 matrix is built from actuated row *j* for each serial chain *i* (*j* can be 1, 2, or 3 for *i*=1,2,3). Equation 35 gives the general overall inverse Jacobian *M*:

$$\begin{Bmatrix} \dot{\rho}_{1j_1} \\ \dot{\rho}_{2j_2} \\ \dot{\rho}_{3j_3} \end{Bmatrix} = \begin{bmatrix} Row(j_1)_{1^{st}\,chain} \\ Row(j_2)_{2^{nd}\,chain} \\ Row(j_3)_{3^{rd}\,chain} \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \omega_z \end{Bmatrix} \qquad (\dot{\mathrm{P}} = M\dot{X}) \tag{35}$$

where $\dot{\rho}_{ij_i}$ is the actuated joint rate (joint j_i) for the i^{th} serial chain and $Row(j_i)_{i^{th}chain}$ is the j_i^{th} row of the i^{th} serial chain inverse Jacobian matrix J_i^{-1}. (Note this method is equivalent to the rate kinematics in Gosselin et.al. (1996) where $A\dot{\mathrm{P}} + B\dot{X} = 0$; the relationship is thus $M = -A^{-1}B$.)

For example, consider an $RR\underline{P}$-$\underline{P}RP$-$RR\underline{R}$ manipulator. In this robot, the third, first, and third joints, respectively, are actuated for serial chains 1, 2, and 3 (j_1=3, j_2=1, j_3=3). The manipulator overall inverse Jacobian matrix is given in Eq. 36:

$$\begin{Bmatrix} \dot{L}_{2_1} \\ \dot{L}_{1_2} \\ \dot{\theta}_{3_3} \end{Bmatrix} = \begin{bmatrix} c_{1_1}/c_{2_1} & s_{1_1}/c_{2_1} & \left(L_{2_1}s_{2_1} + L_{3_1}s_{23_1}\right)/c_{2_1} \\ s_{12_2}/s_{2_2} & -c_{12_2}/s_{2_2} & \left(L_{2_2} + L_{3_2}c_{3_2}\right)/s_{2_2} \\ c_{1_3}/L_{2_3}s_{2_3} & s_{1_3}/L_{2_3}s_{2_3} & \left(L_2 s_2 + L_3 s_{23}\right)/L_{2_3}s_{2_3} \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \omega_z \end{Bmatrix} \tag{36}$$

Invalid Actuation Schemes

The matrix *M* maps Cartesian end-effector velocity into actuated joint velocities. If *M* is invertible, the forward velocity kinematics problem for the overall robot can be solved.

$$\dot{X} = M^{-1}\dot{\mathrm{P}} \tag{37}$$

The overall inverse Jacobian matrix *M* may not be invertible based on a manipulator's actuation scheme. Certain manipulators constructed of three identical chains will produce *M* matrices with columns of zero. For example, for a 3-$\underline{R}PP$ robot (where j_1=j_2= j_3=1):

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{38}$$

Clearly, *M* has rank one and is singular. The physical interpretation of this mathematical singularity is that when the three actuated joints are locked, the overall manipulator has an additional uncontrollable freedom. Due to this problem, certain parallel robots have invalid actuation schemes. This singularity condition is not configuration dependent; the 3-$\underline{R}PP$ robot has an uncontrollable fourth degree-of-freedom for all motion. Table 3 shows the invalid actuation schemes. The actuated joint in each chain is underlined.

$\underline{R}PP$	$P\underline{R}P$	$PP\underline{R}$

Table 3. Invalid Serial-Chain Actuation Schemes

Merlet (1996) states that "*each joint of a chain may be actuated* [provided] *the chain obtained when locking the actuated joint is not of the PP type*." This rule, derived in the position domain, is verified in the velocity domain as shown in Table 3. Any parallel manipulator constructed with permutations of the serial chains in Table 3 has an invalid actuation scheme (not just for identical chains).

If a manipulator is not always singular, it is still subject to the serial chain singularity conditions identified previously. For a more complete treatment of planar parallel manipulator singularities, see Daniali et. al. (1995), Sefriou and Gosselin (1995), and Merlet (1989).

SIMULATION EXAMPLES

An *RPP-RRR-PRR* robot is arbitrarily chosen to demonstrate the inverse position solutions. This robot has 4 solutions (1*x*2*x*2), as shown in Fig. 5. The *RPP* chain has a unique solution, the *RRR* chain has *elbow-up* and *elbow-down* solutions, and the *PRR* chain *near* and *far*.

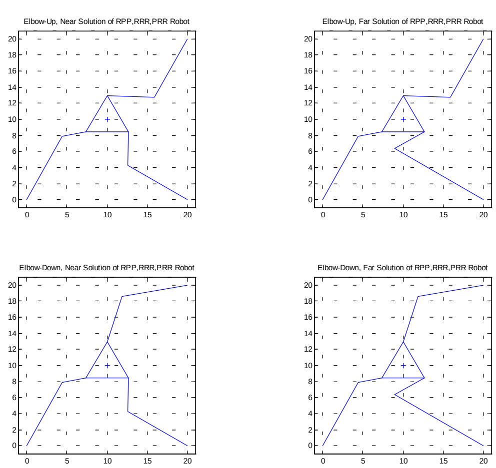

Figure 5. Four Solutions for RPP-RRR-PRR Manipulator

An *P̲RP-PR̲P-PRP̲* robot is arbitrarily chosen to demonstrate the resolved rate simulation in Fig. 6.

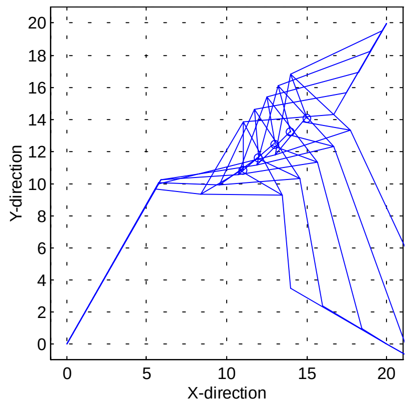

Figure 6. Resolved-Rate Simulation of P̲RP-PR̲P-PRP̲

CONCLUSION

This paper has presented the general inverse position and velocity kinematics solutions for a broad class of planar parallel robots. The class has 3-dof, three serial chains of 3-dof each connecting to the end-effector triangle, one actuator per chain, and *R* and *P* joints. Each serial chain is solved independently given the end-effector pose and rate. Since the actuation scheme does not affect the inverse solution, there are seven possible serial chains, for a total of 343 distinct robots (less if the order of chains is unimportant).

The primary contributions of this paper are: 1) general algebraic approach, where one computer program solves all possible inverse position and velocity problems; 2) all possible independent 3-dof planar serial chains with *R* and *P* joints are classified and their inverse position and velocity problems solved in a unified and succinct manner; 3) overall inverse mapping for actuated joint rates only; and 4) identification of invalid actuation schemes, which agrees with previously published results. The results of this paper can be used for the design, simulation, and control of any 3-dof planar parallel robot.

ACKNOWLEDGMENTS

The first author would like to acknowledge the reviewers for their helpful comments.

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