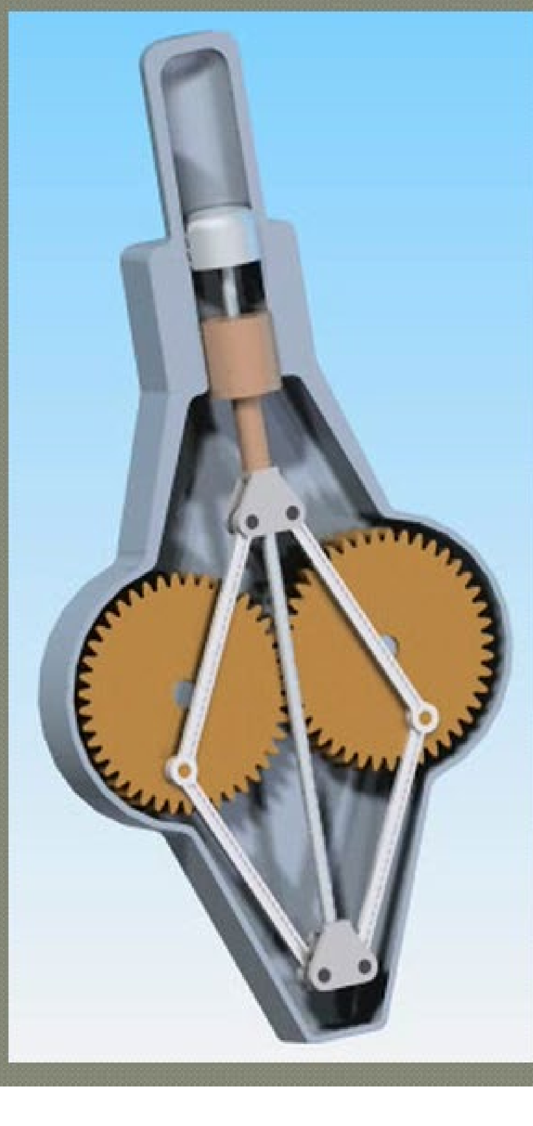
Bob Williams

Mechanism Kinematics & Dynamics



Mechanism Kinematics & Dynamics

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NotesBook self-published for
ME 3011 Kinematics & Dynamics of Machines

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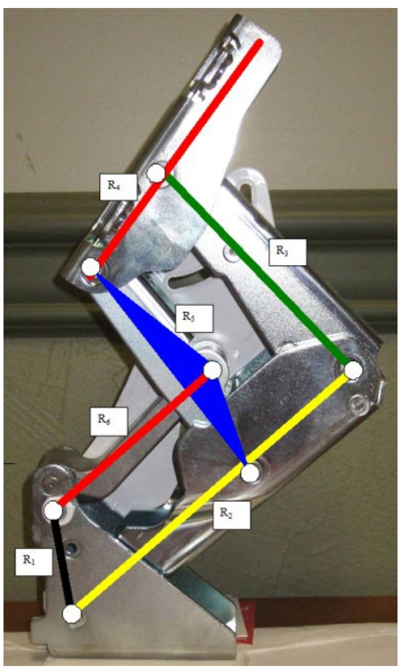
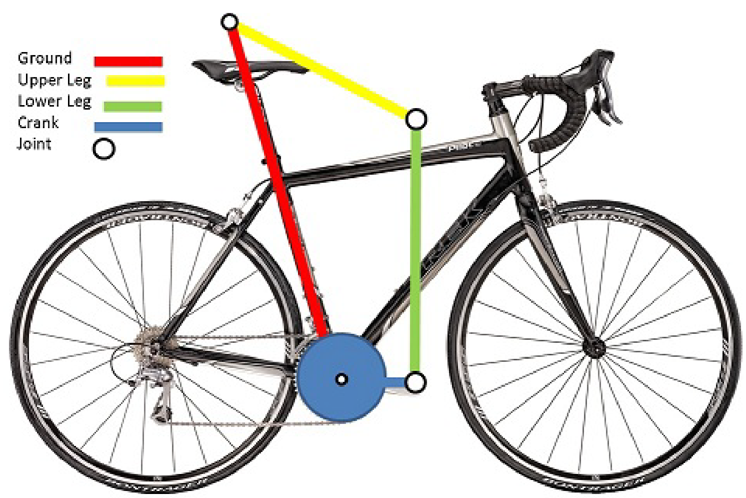


6-bar Hettich Hinge



Bicycle with Human Leg 4-bar Mechanism

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This ME 3011 NotesBook is augmented by the on-line ME 3011 NotesBook Supplement:

people.ohio.edu/williams/html/PDF/Supplement3011.pdf.

Mechanism Kinematics & Dynamics

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No bits nor bytes were harmed in the making of this electronic NotesBook.

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Cover photo: Still from a geared 9-bar Stirling engine mechanism.
https://people.ohio.edu/williams/html/MechanismAnimations.html

The body text is set in 12-pt Times New Roman, and the headings, sub-headings, and sub-sub-headings are set in 16-pt, 14-pt, and 12-pt Arial, respectively.

This NotesBook is intended for ME 3011 Kinematics and Dynamics of Machines, a required one-semester course in Mechanical Engineering at Ohio University. Covered are mobility and position, velocity, acceleration, and jerk kinematics, plus inverse dynamics for planar 1-dof linkage, gear, and cam mechanisms. MATLAB Software is used as a tool throughout for mechanisms analyses and animations. Warning: my NotesBook concept serves both as textbook and notebook – some equations, figures, and examples are blank and must be completed in class. Readers external to Ohio University are welcome with that caveat in mind.

Keywords: mechanisms, kinematics, dynamics, linkages, cams, gears, mechanical engineering, MATLAB

ME 3011 NotesBook Table of Contents

1. Introduction

1.1 Kinematics and Dynamics Concepts

Below are presented some important definitions for kinematics, dynamics, and mechanisms concepts, used throughout this ME 3011 NotesBook. You may already be familiar with some these terms – if some are unfamiliar, don't freak out, we will discuss them later.

kinematics	The study of motion without regard to forces/torques. **kinema** – Greek for *motion*
dynamics	The study of motion with regard to forces/torques. **dynamikos** – Greek for *powerful*
statics	The study of forces/torques without regard to motion. The study of force/moment equilibrium in structures.
free-body diagram (FBD)	A diagram drawn out of context for each separate link mass/inertia with all external and internal forces and moments shown to give the context.
mobility (M)	The number of degrees-of-freedom of a device.
degrees-of-freedom (dof)	The number of independent parameters required to fully specify the location of a device. The number of motors required to drive a device.
planar motion	Two-dimensional (2D) motion (projected onto a common plane) with two independent translations *XY* and one independent rotation, pitch.
spatial motion	Three-dimensional (3D) motion with three independent translations *XYZ* and three independent rotations, roll, pitch, and yaw.
robot	A device with more than one degrees-of-freedom.
mechanism	A device with one degree-of-freedom.
structure	A device with zero degrees-of-freedom (statically-determinate structure) or less than zero degrees-of-freedom (statically-indeterminate structure); i.e. having no motion.
input	The external forcing element that drives a mechanism.
actuator	The input element (motor) of a mechanism.
output	The variable of interest in motion of a mechanism.
linkage	A mechanism consisting of links connected by joints.
gear train	A mechanism consisting of toothed wheels converting angular speed and torque between shafts.

cam/follower A mechanism consisting of a lobed disk and a translating or rotating output.

link A body capable of motion.

ground link The fixed link incapable of motion. Also called the base or frame. There can only be one ground link in a mechanism.

joint A pairing element connecting (and allowing motion) between two links.

revolute joint (R) A 1-dof rotating joint.

prismatic joint (P) A 1-dof sliding joint.

gear joint (G) A 2-dof sliding and rotating joint between the teeth of two gears.

cam joint (C) A 2-dof sliding and rotating joint between a cam and its follower.

slotted-pin joint (SP) A 2-dof sliding and rotating joint between a pin on a link and a slot on another link.

analysis Determination of translational and rotational position, velocity, acceleration, and dynamic forces for a given mechanism in motion.

synthesis Design of an unknown mechanism to accomplish a specific task.

mass (*m*) Idealized mechanical element that models the inertia in a translational dynamic system, kg.

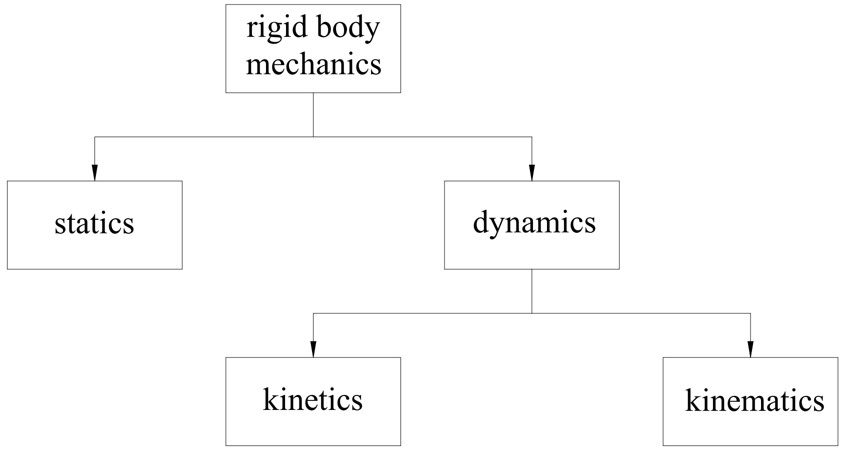
center of gravity (*CG*) The point at which a link is balanced with respect to gravity. The point at which all link mass is considered to act, m.

mass moment of inertia (*I*) Idealized mechanical element that models the rotational inertia in a rotational dynamic system, kg-m^2.

kinematic chain Any number of links connected by joints. An open kinematic chain is represented by a serial robot and a closed kinematic chain is represented by a mechanism or parallel robot.

kinematic inversion For a given mechanism, changing one of the moving links to the fixed link and freeing the fixed link. For a four-link mechanism, there are four kinematic inversions.

Rigid Body Mechanics Diagram



<u>Mechanisms</u> linkages, gear trains, cams/followers

<u>Analysis</u> vs. **<u>Synthesis</u>**

- Analysis – determination of position, velocity, acceleration, and dynamic forces for a given mechanism in motion
- Synthesis – design of an unknown mechanism to do a specific job

Mobility – number of **degrees-of-freedom (dof)** which is the number of independent parameters required to fully specify the location of a device.

- structure – static, no gross motion, device with zero or even negative dof
- mechanism – 1-dof device with rigid links connected by joints
- machine – a collection of mechanisms to transmit force (input / output)
- robot – electromechanical device having greater than 1 dof, programmable for a variety of tasks.

Motion – Translation and Rotation

planar	2D motion (projected onto a common plane); two independent translations and one independent rotation
helical	3D motion; rotation about fixed axis and translation along axis – screw
spherical	3D motion; all points in a body move about a fixed point, on the surface of a sphere
spatial	3D motion; three independent translations and three independent rotations

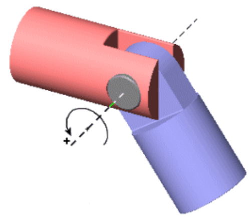
Matrices and Linear Algebra

A brief review of these important topics is given in Dr. Bob's Matrices and Linear Algebra Review:

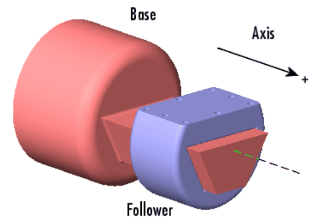
people.ohio.edu/williams/html/PDF/MatricesLinearAlgebra.pdf.

Joints – Pairing elements

Lower – surface contact

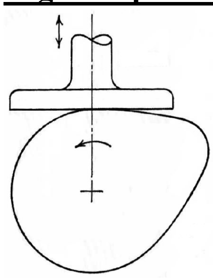


revolute joint R – pin joint, turning pair

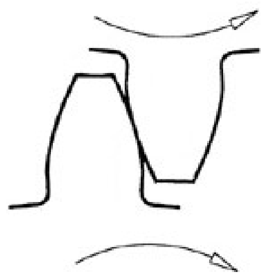


prismatic joint P – sliding pair

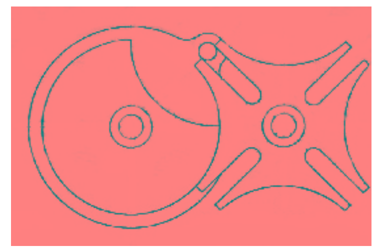
Higher – point or line contact



cam joint C

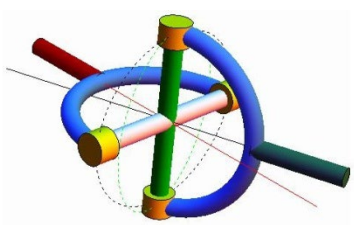


gear joint G

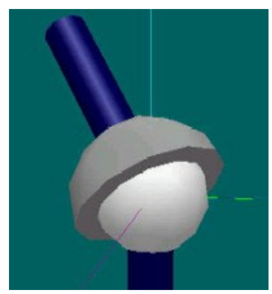


slotted pin joint SP

3D joints – beyond the scope of ME 3011



universal joint U



spherical joint S

Link – rigid body between joints

Kinematic chain – number of links connected by joints
open – serial robot
closed – mechanism, parallel robot

Kinematic Inversion – change which link is fixed. This yields the same relative motion, but different absolute motion.

Required math

The kinematics & dynamics of machines & mechanisms requires the following math topics.

- geometry and trigonometry
- calculus (differentiation and integration)
- vectors, matrices, linear algebra
- algebra

Here are some useful trigonometric identities.

$$\cos^2\phi + \sin^2\phi = 1 \qquad \begin{aligned} \sin(-a) &= -\sin(a) \\ \cos(-a) &= \ \ \cos(a) \end{aligned}$$

sum of angles formulae:

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

law of cosines $C^2 = A^2 + B^2 - 2AB\cos c$ law of sines $\dfrac{\sin a}{A} = \dfrac{\sin b}{B} = \dfrac{\sin c}{C}$

tangent half-angle substitution:

$$\text{if} \quad t = \tan\left(\frac{\theta_4}{2}\right) \qquad \text{then} \qquad \cos\theta_4 = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin\theta_4 = \frac{2t}{1+t^2}$$

derived in: people.ohio.edu/williams/html/PDF/Supplement3011.pdf

trigonometric functions are often abbreviated:

$$\text{cosine}(\theta_i) = \cos\theta_i = c_i$$
$$\text{sine}(\theta_i) = \sin\theta_i = s_i$$
$$\text{tangent}(\theta_i) = \tan\theta_i = t_i$$

Angle Units

Angles are of course super-important in mechanisms analysis and design. I prefer angle units of degrees in HW assignments and in your project. However, angular velocity ω must be in *rad / sec*, and angular acceleration α must be in *rad / sec*^{2}.

Since I am old-school, I enter angles in degree units into MATLAB and immediately convert to radians, multiplying by **`DR = pi / 180`**. This is because the built-in trigonometric functions such as **`sin(th)`** and **`cos(th)`** require angle inputs in radians. Then for reporting angle results, including plotting, I convert back to degrees for the human, by dividing by **`DR`**.

Increasingly students use MATLAB functions **`sind(th)`** and **`cosd(th)`**, which accepts angles in degree units. These functions simply did not exist when I was the first professor to bring MATLAB to Ohio University in 1994. It is paramount for you to develop the correct usage of degrees and/or radians in ME 3011.

Note that angles expressed in radians units are actually unitless, because the definition of an angle is the arc length subtended by the angle divided by the circle radius, $\theta = s / r$. The length units cancel out; however, it is convenient to assign a unit (radians) to such angles. But the appearance of radians in more complicated units does not affect said units. Example (Euler's Rotational Dynamics Law):

$$\sum \underline{M}_G = I_{GZ}\underline{\alpha} \qquad \text{units: } Nm \equiv kgm^2\frac{rad}{sec^2} \equiv \frac{kgm}{sec^2}m \cdot rad \equiv \frac{kgm}{sec^2}m \equiv Nm$$

Connection to Machine Design

In ME 3011 we focus on **kinematics** & **dynamics analysis**, not synthesis (design). However, the skills gained in this course support general mechanical design.

Before one can design a machine, the required motion must be satisfied. All design candidates must be analyzed regarding the motion each would provide (position, velocity, and acceleration, both translational and rotational). This requires **kinematics analysis**.

Before one can size the links, joints, bearings, gear box, and actuators (motors) in a machine, the worst-case force and moment loading condition(s) must be known, for statics and dynamics. This requires **dynamics analysis**.

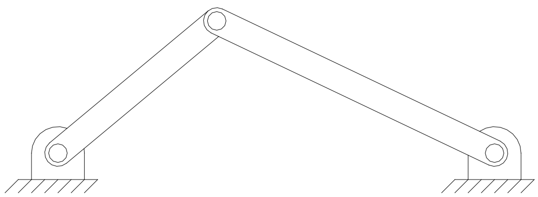
Engineering design is **iterative** by nature: each candidate design must be thoroughly analyzed to determine its performance relative to the design specifications and relative to other design candidates.

This kinematics & dynamics analysis is facilitated using a computer. Without the computer, it is difficult to determine the worst-case loading cases, and over-designed factors of safety may be inefficiently applied.

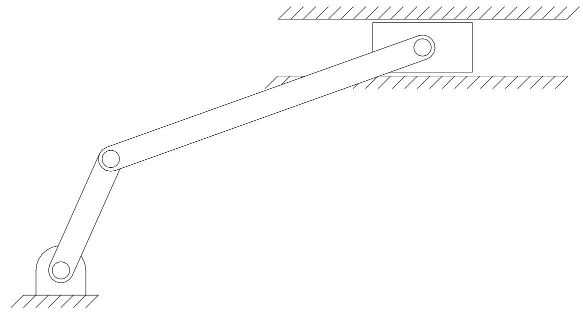
The goal of ME 3011 is to give the student general skills in general matrix/vector-based kinematics and dynamics analysis which may be applied in later classes and later careers.

1.2 An Atlas of Structures, Mechanisms, and Robots

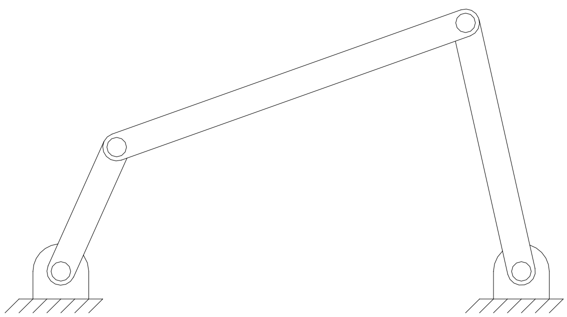
Dr. Bob's Atlas of Structures, Mechanisms, and Robots, presents a broad array of mechanisms and robots (people.ohio.edu/williams/html/PDF/MechanismAtlas.pdf), including real-world applications. A small subset of this atlas is given below.



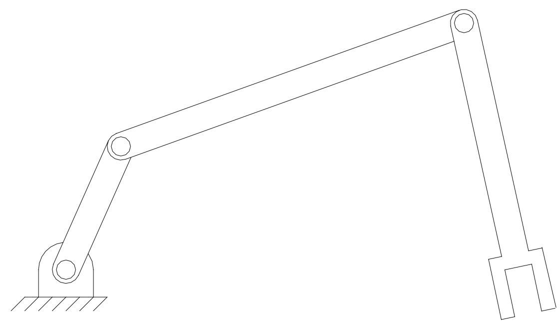
Statically Determinate Structure



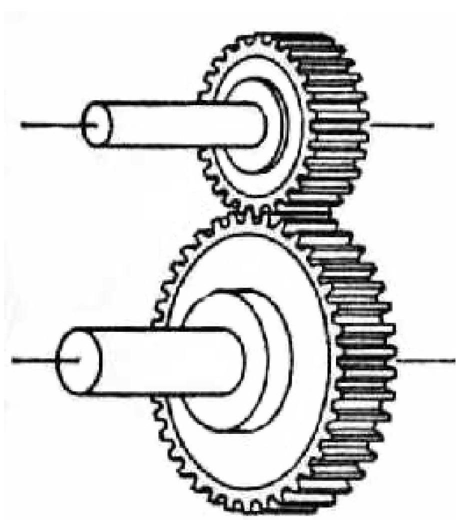
Offset Slider-Crank Mechanism



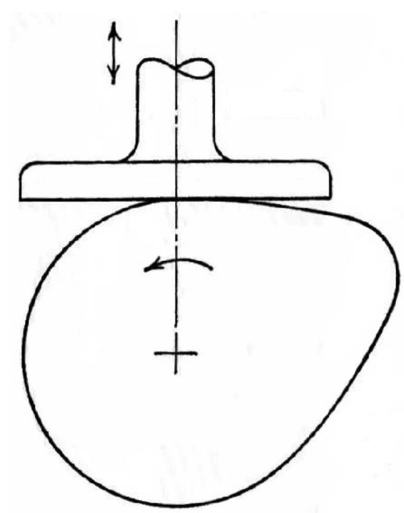
Four-bar Mechanism



Planar three-dof Robot



Spur Gear Mechanism



Cam-and-Follower Mechanism

Also presented on-line are many Mechanism Animations:

people.ohio.edu/williams/html/MechanismAnimations.html

1.3 Vectors

A planar vector is an arrow in the plane with magnitude and direction. Planar *XY* vectors are used in engineering mechanics to represent planar translational positions, velocities, accelerations, and forces. *Z* vectors normal to the *XY* plane are used to represent rotational velocities, angular accelerations, and torques (moments). In general, vectors have magnitude, direction (including sense), and point of application.

Cartesian *XY* representation

Polar representation magnitude at angle $\|\underline{P}\| \quad @ \quad \theta$

Cartesian ↔ Polar transformation

`atan2` is the quadrant-specific inverse tangent function (introduced later).

$$\underline{P}_1 = \begin{Bmatrix} p_{1x} \\ p_{1y} \\ 0 \end{Bmatrix} \qquad \underline{P}_2 = \begin{Bmatrix} p_{2x} \\ p_{2y} \\ 0 \end{Bmatrix}$$

Vector Addition

Vectors add tail-to-head (or they subtract head-to-tail). One must express all vector components in the same coordinate frame for addition. Vector addition yields a vector result.

Graphical interpretation

Vector Dot Product

The vector dot product is the projection of one vector onto another. The vector dot product yields a scalar result.

Graphical interpretation

Vector Cross Product

The vector cross product of two vectors gives a third vector mutually perpendicular to the original two vectors. The vector cross product yields a vector result.

Graphical interpretation

The resulting cross product direction is found via the right-hand-rule: Put your right hand fingers along the first vector $\underline{P}_1$ and rotate them into the second vector $\underline{P}_2$. Then your right thumb is pointing in the direction of $\underline{P}_1 \times \underline{P}_2$ (i.e., perpendicular to both vectors $\underline{P}_1$ and $\underline{P}_2$).

$\hat{k}$ Vectors

In planar mechanics, angular velocity, angular acceleration, and torque (moment) vectors are arrows along the $\hat{k}$ axis (the unit direction for the Z axis, perpendicular to the XY plane). $\hat{k}$ vectors also have magnitude and direction, but can be represented by a single Z component with $\pm$ sign. We will often represent these $\hat{k}$ vectors by curled arrows in the XY plane.

$\underline{\omega} = \pm \omega \hat{k}$

+	CCW	(curling in the direction of the right hand fingers)
–	CW	(curling in the opposite direction of the right hand fingers)

Vector Examples $\underline{P}_1 = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix}$ $\underline{P}_2 = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$

Addition $\underline{P}_1 + \underline{P}_2 =$

$\underline{P}_2 + \underline{P}_1 =$

Dot Product $\underline{P}_1 \bullet \underline{P}_2 =$

$\underline{P}_2 \bullet \underline{P}_1 =$

Cross Product $\underline{P}_1 \times \underline{P}_2 =$

$\underline{P}_2 \times \underline{P}_1 =$

Same Vector Examples using MATLAB

```
%-------------------------------
% Vectors.m - vector examples
%           Dr. Bob, ME 3011
%-------------------------------
clear; clc;
P1 = [3;1;0];             % define two 3x1 vectors
P2 = [2;3;0];
sum1 = P1+P2              % vector addition
sum2 = P2+P1
dot1 = dot(P1,P2)         % vector dot product
dot2 = dot(P2,P1)
cross1 = cross(P1,P2)     % vector cross product
cross2 = cross(P2,P1)
```

Output of Vectors.m

sum1 = 5 4 0 sum2 = 5 4 0	dot1 = 9 dot2 = 9	cross1 = 0 0 7 cross2 = 0 0 -7

For an overview of matrices, please see Dr. Bob's on-line Matrices and Linear Algebra Review:

people.ohio.edu/williams/html/PDF/MatricesLinearAlgebra.pdf

1.4 MATLAB Introduction

MATLAB is a general engineering analysis and simulation software. MATLAB stands for **MAT**rix **LAB**oratory. It was originally developed specifically for control systems simulation and design engineering, but it has grown over the years to cover many engineering and scientific fields. MATLAB is based on the C language, and its programming is vaguely C-like, but simpler. MATLAB is sold by Mathworks Inc. (www.mathworks.com) and Ohio University has a site license. For an extensive introduction to the MATLAB software, please see Dr. Bob's on-line MATLAB Primer:

people.ohio.edu/williams/html/PDF/MATLABPrimer.pdf

1.5 Mobility

Mobility

The number of **degrees-of-freedom** of a device.

Degrees-of-freedom (dof)

The number of independent parameters required to fully specify the location of a device. The number of motors required to drive a device.

How many degrees-of-freedom does an unconstrained planar link have?

What is the effect of constraining that link with a revolute joint?

Kutzbach's Mobility Equation for Planar Jointed Devices

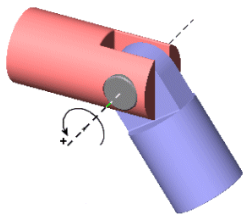
where M – mobility

N – total number of links, including ground

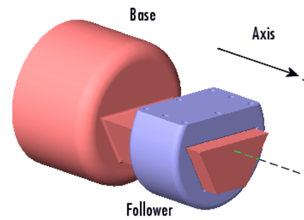
J_1 – number of one-degree-of-freedom joints

J_2 – number of two-degree-of-freedom joints

One-degree-of-freedom joints (J_1)

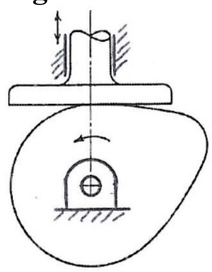


Revolute



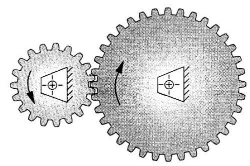
Prismatic

Two-degree-of-freedom joints (J_2, all have rolling and sliding)

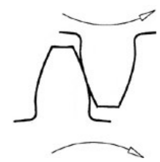


1-dof Cam mechanism

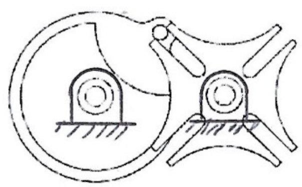
2-dof Cam joint



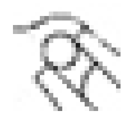
1-dof Gear mechanism



2-dof Gear joint



1-dof Geneva Wheel mechanism



2-dof Slotted-pin joint

Each of these three mechanisms have 1-dof overall. Each named joint has 2-dof, thus being a J_2.

If there are p links joining at one revolute location, you must count p–1 revolute joints.

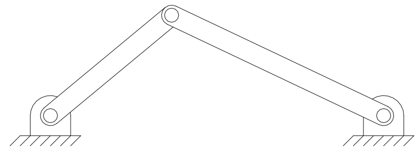
You must count the ground link (its freedom is subtracted in the mobility formula with N–1).

Planar mechanical device classification

$M > 1$	EE/ME 4290/5290 robotics
$M = 1$	ME 3011 kinematics & dynamics
$M = 0$	ET 2200 statics
$M < 0$	ET 2200 statics

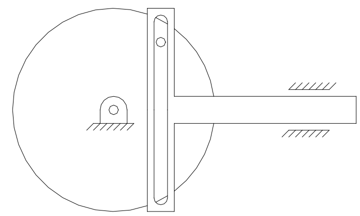
Solved Planar Mobility Examples

1) Statically-determinate structure



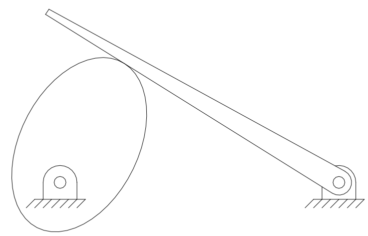
$N = 3$ $J_1 = 3$ $J_2 = 0$ $M = 3(3-1) - 2(3) - 1(0) = 0$ dof

2) Scotch yoke mechanism



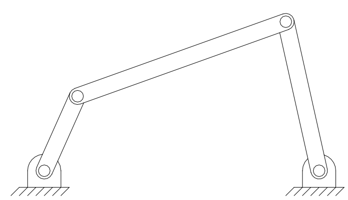
$N = 3$ $J_1 = 2$ $J_2 = 1$ $M = 3(3-1) - 2(2) - 1(1) = 1$ dof

3) Cam and follower mechanism



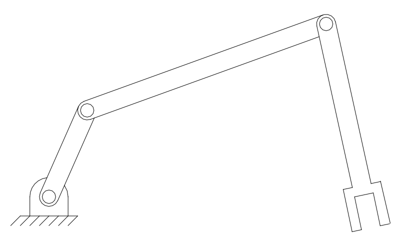
$N = 3$ $J_1 = 2$ $J_2 = 1$ $M = 3(3-1) - 2(2) - 1(1) = 1$ dof

4) Four-bar mechanism



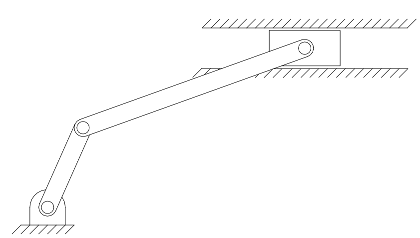
$N = 4$ $J_1 = 4$ $J_2 = 0$ $M = 3(4-1) - 2(4) - 1(0) = 1$ dof

5) Three-link serial robot



$N = 4$ $J_1 = 3$ $J_2 = 0$ $M = 3(4-1) - 2(3) - 1(0) = 3$ dof

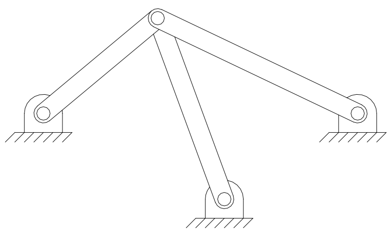
6) Offset slider-crank mechanism



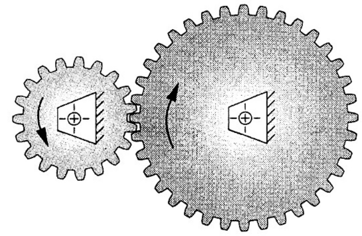
$N = 4$ $J_1 = 4$ $J_2 = 0$ $M = 3(4-1) - 2(4) - 1(0) = 1$ dof

Other Planar Mobility Examples (not solved)

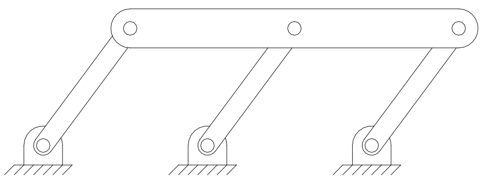
7) Statically-indeterminate structure



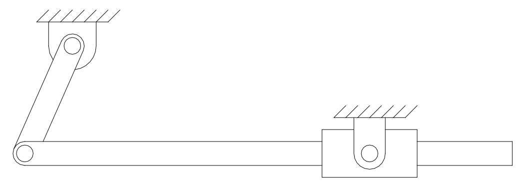
8) External spur gear mechanism



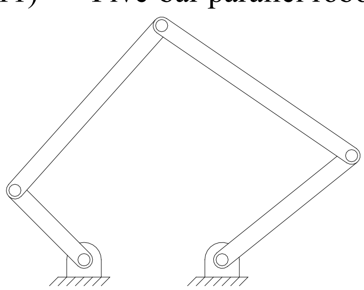
9) Four-bar mechanism with parallel link



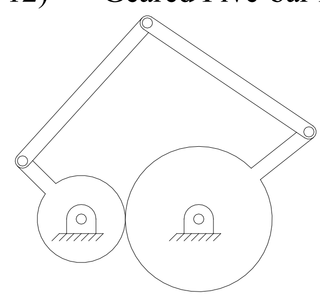
10) Slider-crank mechanism, inversion 3



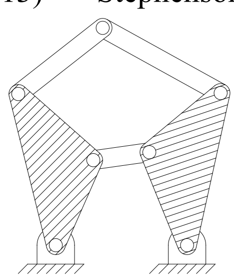
11) Five-bar parallel robot



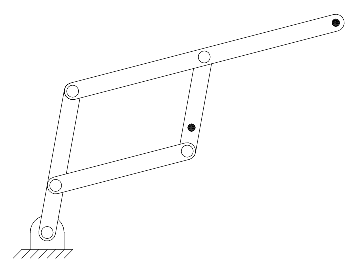
12) Geared Five-bar mechanism



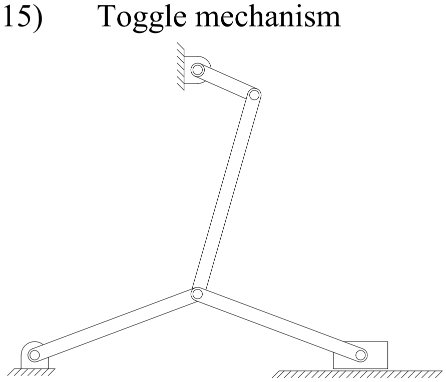
13) Stephenson I Six-bar mechanism



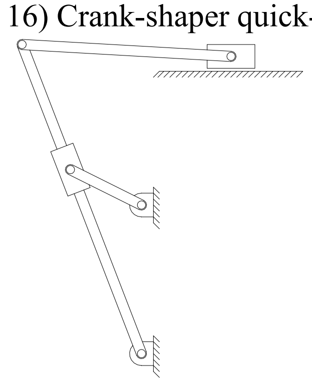
14) Pantograph robot



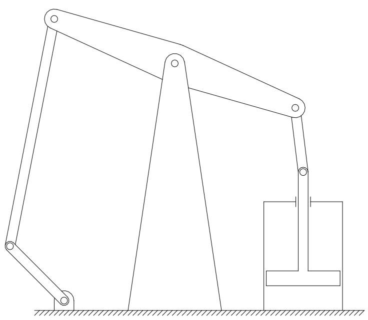
15) Toggle mechanism



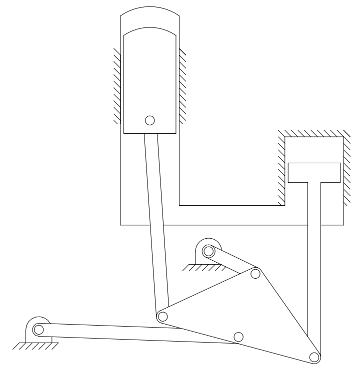
16) Crank-shaper quick-return mechanism



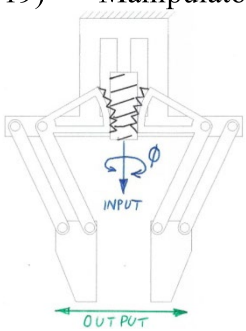
17) Watt steam engine mechanism



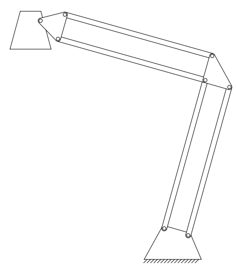
18) Ross Stirling engine mechanism



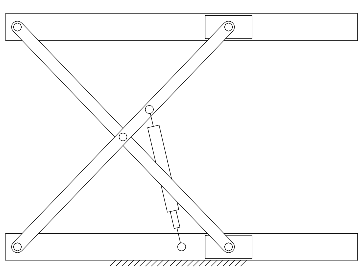
19) Manipulator gripper mechanism



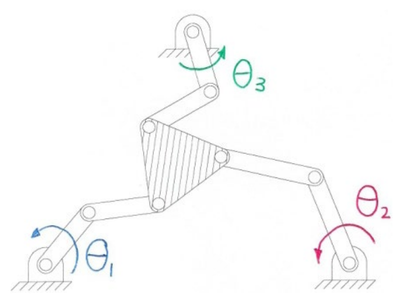
20) Desk lamp robot



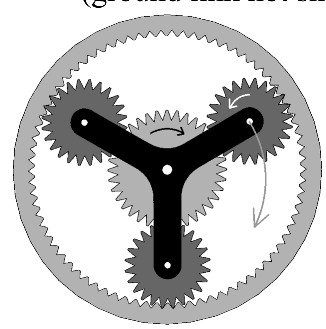
21) Scissor-lift mechanism



22) 3-RRR parallel robot



23) Planetary gear train (ground link not shown)



For more unsolved mobility examples, see Dr. Bob's Atlas of Structures, Mechanisms, and Robots:

people.ohio.edu/williams/html/PDF/MechanismAtlas.pdf

MATLAB function to calculate mobility

```
% Function for calculating planar mobility
%   Dr. Bob, ME 3011
function M = dof(N,J1,J2)
M = 3*(N-1) - 2*J1 - 1*J2;
```

Usage

```
mob = dof(4,4,0);   % for four-bar and slider-crank mechanisms
```

Result

```
mob = 1
```

2. Position Kinematics Analysis

Kinematics analysis is concerned with relating the **position**, **velocity**, and **acceleration** parameters (given and unknown) in the motion of planar mechanisms. We also consider **jerk** (the time derivative of the **acceleration**). **Kinematics** analysis is performed first, followed by inverse dynamics analysis.

Kinematics is the study of motion without regard to forces.

Position analysis is the first step in general **kinematics** analysis. It relates the translational positions and angles of the links for a mechanism in motion.

Position (Displacement) Analysis:

Position analysis is determination of the position/orientation of all links in a mechanism. It is required for testing the motion of a synthesized mechanism. It is also required for further analysis: velocity, acceleration, dynamics, and forces/moments.

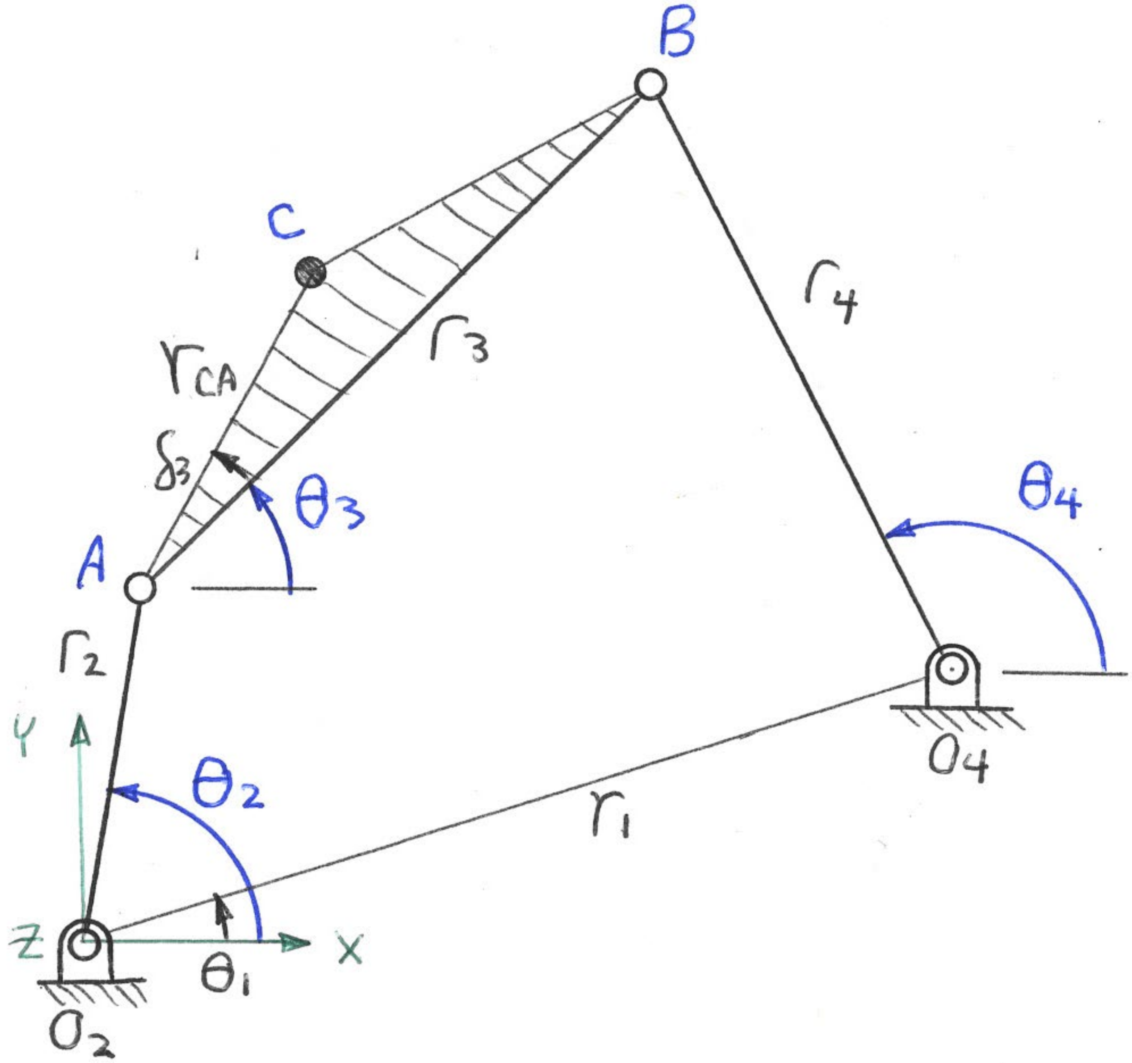
Generic Mechanism Position Analysis Problem Statement

Given the mechanism and one-dof of position input, calculate the position unknowns.

2.1 Four-Bar Mechanism Position Analysis

2.1.1 Four-Bar Mechanism Position Analysis Steps and Solution

Step 1. Draw the **Kinematic Diagram**



r_1 – constant ground link length	θ_1 – constant ground link angle
r_2 – constant input link length	θ_2 – variable input angle
r_3 – constant coupler link length	θ_3 – variable coupler angle
r_4 – constant output link length	θ_4 – variable output angle
standard names for important points	O_2 O_4 A B C
r_{CA} – constant length from A to C	δ_3 – constant angle in coupler triangle

All angles must be measured in a right-handed sense from the right horizontal to the link.

Step 2. **State the Problem**

Step 3. Draw the **Vector Diagram**. Define all angles in positive sense, measured from the right horizontal to the link vector (tail-to-head). Don't try to force acute angles; the relationships we can see so easily in the first quadrant hold for all four quadrants:

$$\underline{r_i} = \begin{Bmatrix} r_i \cos\theta_i \\ r_i \sin\theta_i \end{Bmatrix} \qquad \text{holds good for all } \theta_i .$$

Step 4. Derive the **Vector-Loop-Closure Equation**. Starting at one point, add vectors tail-to-head until you reach a second point. Write the VLCE by starting and ending at the same points, but choosing a different path.

Step 5. Write the ***XY* Components** for the Vector-Loop-Closure Equation. Separate the one vector equation into its two scalar components (*X* and *Y*).

<u>Step 6.</u> <u>Solve for the Unknowns</u> from the *XY* equations. There are two coupled nonlinear equations in the two unknowns θ_3 and θ_4. Isolate and eliminate θ_3 and solve for θ_4. Then return to the same equations to find θ_3. Square and add the *XY* equations to accomplish this.

This equation has the form

$$E = 2r_4(r_1c_1 - r_2c_2)$$
$$F = 2r_4(r_1s_1 - r_2s_2)$$
$$G = r_1^2 + r_2^2 - r_3^2 + r_4^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$$

For G we used the trigonometric identity $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$. Solve this equation using the tangent half-angle substitution (this is derived in the on-line ME 3011 Supplement, people.ohio.edu/williams/html/PDF/Supplement3011.pdf, along with an alternate solution).

Let $t = \tan\left(\frac{\theta_4}{2}\right)$ then $\cos\theta_4 = \frac{1-t^2}{1+t^2}$ and $\sin\theta_4 = \frac{2t}{1+t^2}$

The mathematical form $(G - E)\,(G + E) = G^2 - E^2$ is called the **difference of two squares**.

We converted a complicated coupled transcendental set of equations into a quadratic polynomial. This is much easier to solve (but we doubled the order of the equation). There are two solutions for θ_4.

With the multiplier 2, there is no need to use the **`atan2`** function.

Why are there two solutions? Demonstrate the two branches.

Complete the four-bar position solution by finding θ_3

θ_4 has now been solved with two results corresponding to the two (open and crossed) branches. Now we must go back to find θ_3, one for each solution branch. Return to the original two *XY* scalar equations.

$$r_3 c_3 = r_1 c_1 + r_4 c_4 - r_2 c_2$$
$$r_3 s_3 = r_1 s_1 + r_4 s_4 - r_2 s_2$$

Use a ratio of the *Y* to *X* equations

Show the graphical interpretation of this result $\theta_3 = \tan^{-1}\left(\dfrac{B_Y - A_Y}{B_X - A_X}\right)$

In this case we must use the **`atan2`** function to automatically choose the correct quadrant for the answer θ_3.

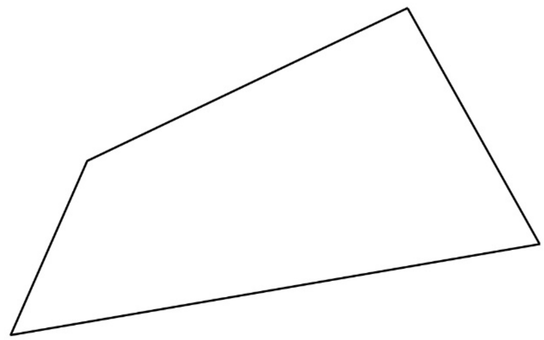
Four-Bar Mechanism Position Analysis Graphical Solution

The Four-Bar mechanism position analysis problem may be solved ***graphically***, by drawing the mechanism, determining the mechanism closure, and measuring the answers θ_3 and θ_4. This is an excellent method to validate your computer results at a given snapshot.

- Draw the known ground link (points O_2 and O_4 separated by r_1 at the fixed angle θ_1).
- Draw the given input link length r_2 at the given angle θ_2 (the endpoint of r_2 is point A).
- Draw a circle of radius r_3 centered at point A.
- Draw a circle of radius r_4 centered at point O_4.
- These circles intersect in general in two places to yield two possible points B. Connect the two branches and measure the unknown angles θ_3 and θ_4 for each branch.
- What if there is only one solution for point B? What if there are no solutions for point B (i.e. the two circles do not intersect)?

Four-bar mechanism transmission angle

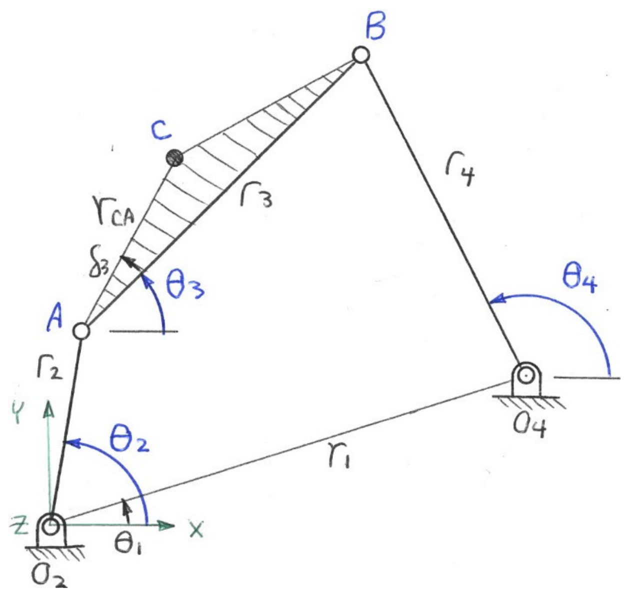
The transmission angle μ is the relative angle between the coupler and output links (3 and 4). The transmission angle is a measure of mechanical advantage of the four-bar mechanism. $\mu = 90^\circ$ is ideal for maximum transmission and $\mu = 0,180^\circ$ yields zero transmission. As a rule of thumb, the absolute value of μ should remain in the range $40^\circ < \mu < 140^\circ$ for good transmission in a mechanism. The transmission angle can be found by geometry (using the Vertical Angles Theorem).



For an alternate geometric method to derive the four-bar mechanism transmission angle μ, please see the on-line ME 3011 NotesBook supplement.

Position of a point on the four-bar mechanism

The basic four-bar mechanism position analysis problem is solved now that we have found θ_3 and θ_4. Now that we know these angular unknowns, we can find the translational **position of any point** on the mechanism, e.g. coupler point *C*.



Four-Bar Mechanism Snapshot MATLAB m-file

This program solves the four-bar position analysis problem for both branches given a single input angle θ_2. The resulting mechanism branches are drawn to the screen.

```
%---------------------------------------------------------------------------------
%   FBarKinSnap.m - four-bar linkage snapshot position analysis with both branches
%      Dr. Bob, ME 3011, with graphical output
%---------------------------------------------------------------------------------

clc; clear;    % clear cursor and clear previously defined variables

%  Inputs
DR  = pi/180;
R   = input('Enter [r1  r2  r3  r4  rca] (length units):  ');
r1  = R(1); r2 = R(2); r3 = R(3); r4 = R(4); rca = R(5);
Ang = input('Enter [th1  th2  del3] (deg):  ');
th1 = Ang(1)*DR; th2 = Ang(2)*DR; del3 = Ang(3)*DR;     % change deg to rad
r1x = r1*cos(th1); r1y = r1*sin(th1);
ax  = r2*cos(th2);                                      % Pt A branch independent
ay  = r2*sin(th2);

%  Position analysis:  theta4
E      =  2*r4*(r1*cos(th1) - r2*cos(th2));
F      =  2*r4*(r1*sin(th1) - r2*sin(th2));
G      =  r1^2 + r2^2 - r3^2 + r4^2 - 2*r1*r2*cos(th1-th2);
t(1)   = (-F + sqrt(E^2 + F^2 - G^2)) / (G-E);          % crossed branch
t(2)   = (-F - sqrt(E^2 + F^2 - G^2)) / (G-E);          %  open   branch
th4(1) = 2*atan(t(1));
th4(2) = 2*atan(t(2));

%  th3, coupler point, transmission angle; calculate for both branches
for i = 1:2,
   bx     = r4*cos(th4(i)) + r1x;                       % Pt B changes w/ branch
   by     = r4*sin(th4(i)) + r1y;
   th3(i) = atan2(by-ay,bx-ax);                         % theta3
   mu(i)  = abs(th4(i)-th3(i));                         % transmission angle
   bet    = th3(i) + del3;                              % coupler point
   pcx(i) = r2*cos(th2) + rca*cos(bet);
   pcy(i) = r2*sin(th2) + rca*sin(bet);

%  Draw four-bar to the screen, each branch
   x2 = [0                r2*cos(th2)];                 % link 2 coordinates
   y2 = [0                r2*sin(th2)];
   x3 = [r2*cos(th2)  r1x+r4*cos(th4(i))  pcx(i)];      % link 3 coordinates
   y3 = [r2*sin(th2)  r1y+r4*sin(th4(i))  pcy(i)];
   x4 = [r1x          r1x+r4*cos(th4(i))];              % link 4 coordinates
   y4 = [r1y          r1y+r4*sin(th4(i))];
   figure;
   plot(x2,y2,'r',x4,y4,'b');
   patch(x3,y3,'g');
   axis('square'); axis([-r2 r1+r4 -(r1+r2+r4)/2 (r1+r2+r4)/2]); grid;
   set(gca,'FontSize',18);
   xlabel('\itX (\itm)'); ylabel('\itY (\itm)');
end
```

Four-Bar Mechanism Full-Range-Of-Motion (F.R.O.M.) MATLAB m-file

It is straightforward to extend the previous four-bar position analysis snapshot program to perform F.R.O.M. analysis, with graphical animation.

- Refer to **`MATEx2.m`** given in Dr. Bob's on-line MATLAB Primer for F.R.O.M. analysis for a single link animation. Use that program structure and drop the four-bar position program into the **`for`** loop. people.ohio.edu/williams/html/PDF/MATLABPrimer.pdf

- Eliminate the two-branch solution – use only the assigned branch and use the **`i`** index for different input angle values instead of the two branches. To choose between branches, choose either **`+`** or **`–`** in the **`t`** intermediate variable for the θ_4 calculation – be sure to use MATLAB graphics (done for you in the previous and following code examples) to ensure you have the correct branch.

- Use an **`(i)`** in left-hand-side assignments only when you want to save that variable for later plotting. If a variable has the **`(i)`** upon its creation, you must use the **`(i)`** in subsequent calculations with that variable, or else you will get the entire array for that term.

- Inside the loop, after the mechanism is drawn to the screen, you must use **`pause(dt)`** where **`dt`** is a small time in seconds. If this is omitted, you will not see the animation since it zips by too fast. You can experiment with setting **`dt`** (**`dt`** $= \Delta\theta_2 / \omega_2$) in attempt to obtain real-time animations, but MATLAB is not a real-time software and the Windows operating system is certainly not real-time.

- I use a permanent **`pause`** for the first time through the loop only, so you can see what the mechanism looks like, re-size the window, and orient yourself for the coming animation. Hit **`<enter>`** to continue when ready.

- Plotting of variables is done outside the loop with the entire arrays and thus no need for the **`(i)`** notation.

- You can put more than one curve on each plot window, using color and/or different linetypes to distinguish amongst the different curves. You can use plot titles, axis labels (with names and units), equal axes where appropriate (Y vs. X mechanism plots), larger font size, and a legend to distinguish the curves.

- The following page gives the resulting m-file for your use.

```
%---------------------------------------------------------------------------
%   FBarAnim.m - four-bar linkage F.R.O.M. position analysis, open branch only
%      Dr. Bob, ME 3011, with graphical animation
%---------------------------------------------------------------------------

clc; clear;                % clear cursor and clear previously defined variables

%  Inputs
DR = pi/180;
R   = input('Enter [r1  r2  r3  r4  rca] (length units):  ');
r1  = R(1); r2 = R(2); r3 = R(3); r4 = R(4); rca = R(5);
Ang = input('Enter [th1  del3] (deg):  ');
th1 = Ang(1)*DR; del3 = Ang(2)*DR;                            % change deg to rad
r1x = r1*cos(th1); r1y = r1*sin(th1);

th20 = 0;  dth2 = 5;  th2f = 360;  th2  = [th20:dth2:th2f]*DR;  % th2 array
N = (th2f-th20)/dth2 + 1;  % number of times to repeat loop for F.R.O.M.

figure;
for i=1:N,                 % F.R.O.M. loop over all input th2
   % Position analysis:  theta4
   E =  2*r4*(r1*cos(th1) - r2*cos(th2(i)));
   F =  2*r4*(r1*sin(th1) - r2*sin(th2(i)));
   G =  r1^2 + r2^2 - r3^2 + r4^2 - 2*r1*r2*cos(th1-th2(i));
   t = (-F - sqrt(E^2 + F^2 - G^2)) / (G-E);              % open branch only
   th4(i) = 2*atan(t);

   % th3, coupler point, transmission angle; open branch only
   ax = r2*cos(th2(i));                                   % Point A
   ay = r2*sin(th2(i));
   bx = r4*cos(th4(i)) + r1x;                             % Point B
   by = r4*sin(th4(i)) + r1y;
   th3(i) = atan2(by-ay,bx-ax);                           % theta3
   mu(i)  = abs(th4(i)-th3(i));                           % transmission angle
   bet    = th3(i) + del3;                                % coupler point
   pcx(i) = r2*cos(th2(i)) + rca*cos(bet);
   pcy(i) = r2*sin(th2(i)) + rca*sin(bet);

   % Draw four-bar to the screen, open branch only
   x2 = [0                  r2*cos(th2(i))];              % link 2 coordinates
   y2 = [0                  r2*sin(th2(i))];
   x3 = [r2*cos(th2(i))  r1x+r4*cos(th4(i)) pcx(i)];      % link 3 coordinates
   y3 = [r2*sin(th2(i))  r1y+r4*sin(th4(i)) pcy(i)];
   x4 = [r1x             r1x+r4*cos(th4(i))];             % link 4 coordinates
   y4 = [r1y             r1y+r4*sin(th4(i))];
   plot(x2,y2,'r',x4,y4,'b');
   patch(x3,y3,'g');
   set(gca,'FontSize',18); xlabel('\itX (\itm)'); ylabel('\itY (\itm)');
   axis('square'); axis([-r2 r1+r4 -(r1+r2+r4)/2 (r1+r2+r4)/2]); grid;
   pause(1/4);             % If this is left out, animation will ZIP right by
   if i==1
      pause;               % Hit <Enter> to proceed, first time only
   end
end

%  Plots outside loop
figure;
plot(th2/DR,th3/DR,'r',th2/DR,th4/DR,'g',th2/DR,mu/DR,'b'); grid;
set(gca,'FontSize',18); legend('{\it\theta_3}', '{\it\theta_4}', '{\it\mu}');
xlabel('{\it\theta_2} ({\itdeg})'); ylabel('{\itAngles} ({\itdeg})');

figure;
plot(pcx,pcy); grid; axis('equal');
set(gca,'FontSize',18); title('Four-Bar Mechanism Coupler Curve');
xlabel('{\itX} ({\itm})'); ylabel('{\itY} ({\itm})');
```

Term Example 1 presents snapshot and F.R.O.M. examples for an example four-bar mechanism position analysis (that will be continued for velocity, acceleration, and inverse dynamics analysis). This is example is now initiated.

Four-Bar Mechanism Position Analysis – Term Example 1

Given: $r_1 = 11.18$, $r_2 = 3$, $r_3 = 8$, $r_4 = 7$ in; $r_1 = 0.284$, $r_2 = 0.076$, $r_3 = 0.203$, $r_4 = 0.178$ m

and $\theta_1 = 10.3^\circ$ (Ground link is 11" over and 2" up). Also given $r_{CA} = 5$ in (0.127 m) and $\delta_3 = 36.9^\circ$ for the coupler link point of interest.

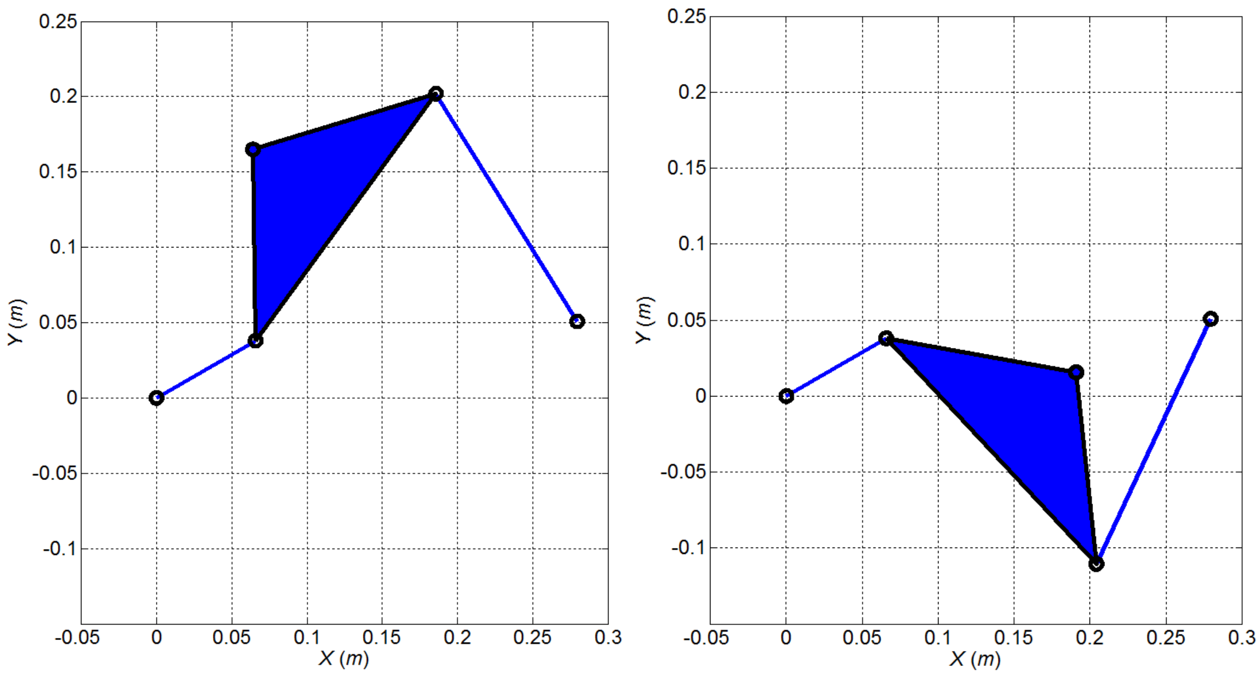
Snapshot Analysis (for one given input angle θ_2)

Given this mechanism and $\theta_2 = 30^\circ$, calculate θ_3, θ_4, μ, and $\underline{P}_C$ for both branches. Results:

$$E = 0.076$$
$$F = 0.005$$
$$G = 0.036$$

branch	***t***	θ_3	θ_4	μ	$\underline{P}_C$
open	1.799	53.8°	121.7°	67.9°	(0.064, 0.165)
crossed	–1.570	313.0°	245.0°	67.9°	(0.191, 0.016)

These two branch solutions are demonstrated in the figures below. The length units are *m*. Note μ is identical for both branches due to the conventions presented earlier.

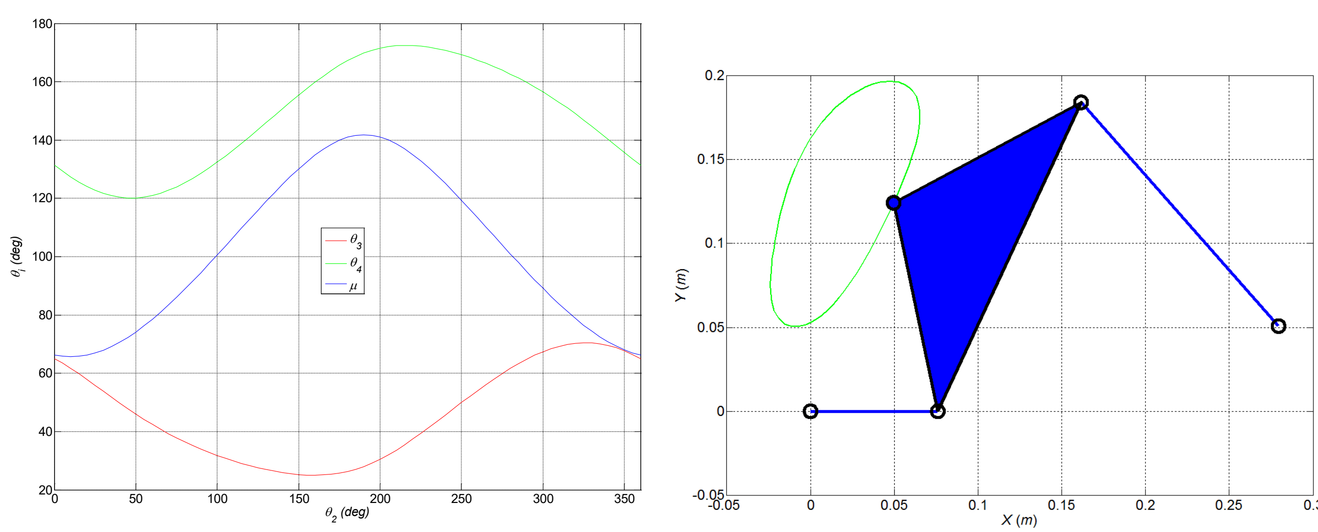


Open Branch ($t(2)$) **Crossed Branch ($t(1)$)**

Term Example 1 Snapshot

<u>Full-Range-Of-Motion (F.R.O.M.) Analysis – Term Example 1</u>

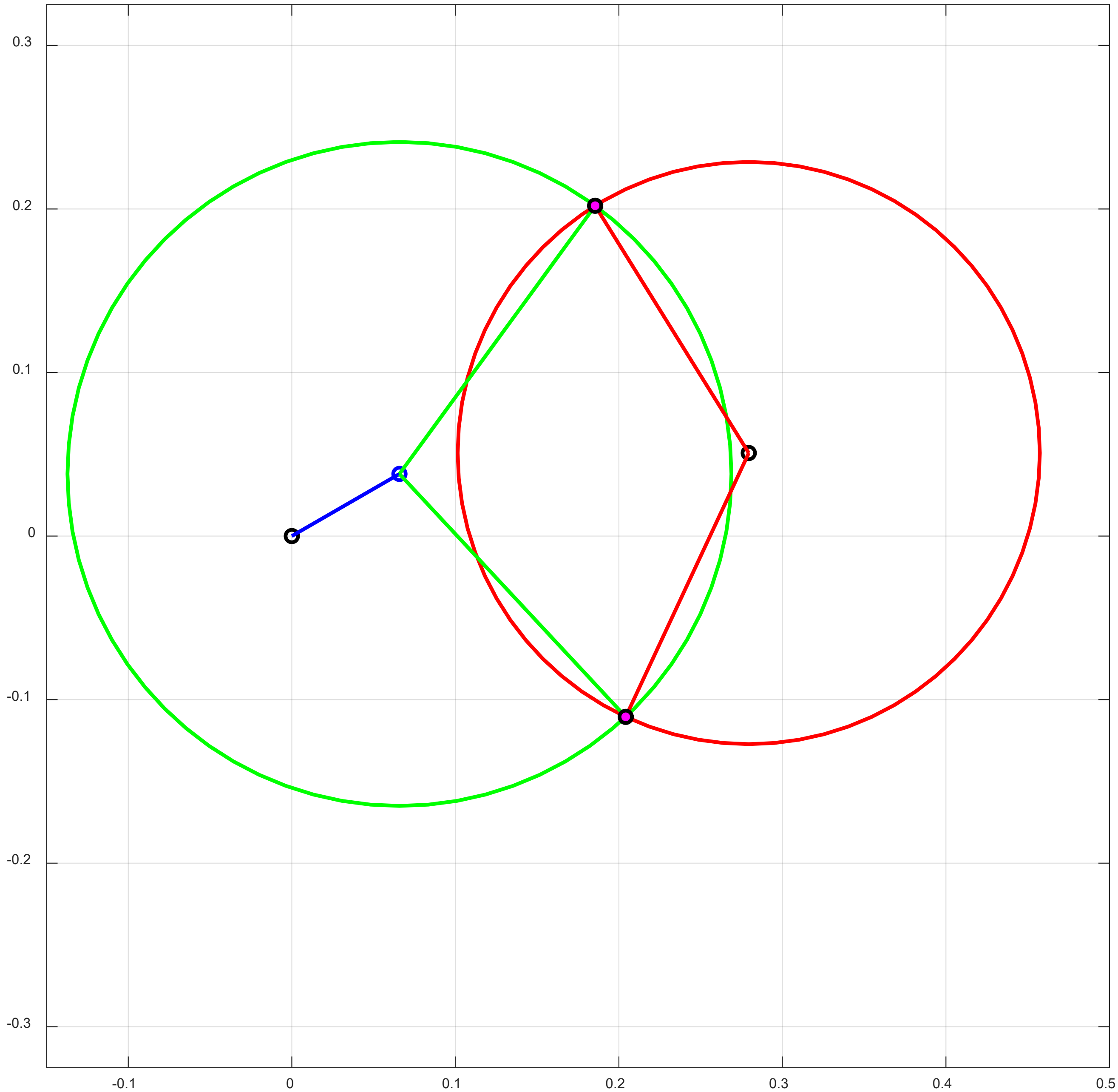
A more meaningful result from position analysis is to solve and plot the position analysis unknowns for the entire range of mechanism motion. The left plot below gives θ_3 (red), θ_4 (green), and μ (blue), all *deg*, for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, the open branch only. The right plot below gives the initial (and final) animation position, for $\theta_2 = 0, 360^\circ$. It also shows the coupler curve to scale for the open branch, plotting P_{CY} vs. P_{CX} in green.



θ_3, θ_4, and μ **Coupler Curve**

Term Example 1 F.R.O.M. Position Results

Four-Bar Mechanism Graphical Position Analysis, Term Example 1



FBarGraphical.m

2.1.2 Trigonometric Uncertainty

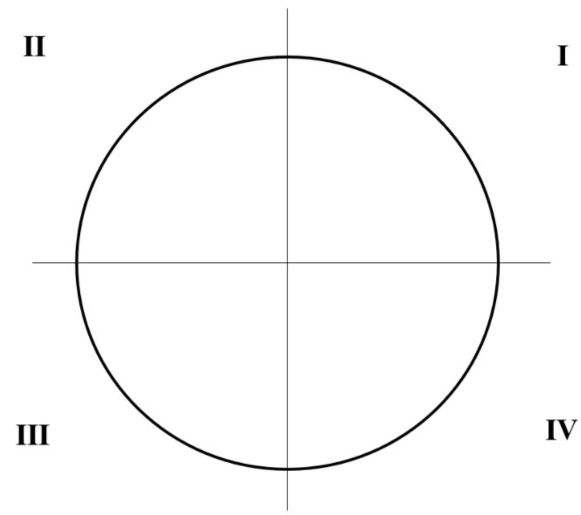
Return to the four-bar mechanism θ_3 solution; again, the *XY* scalar equations are:

$$r_3c_3 = r_1c_1 + r_4c_4 - r_2c_2$$

$$r_3s_3 = r_1s_1 + r_4s_4 - r_2s_2$$

Since θ_4 has been solved, why not find θ_3 using the *Y* equation only?

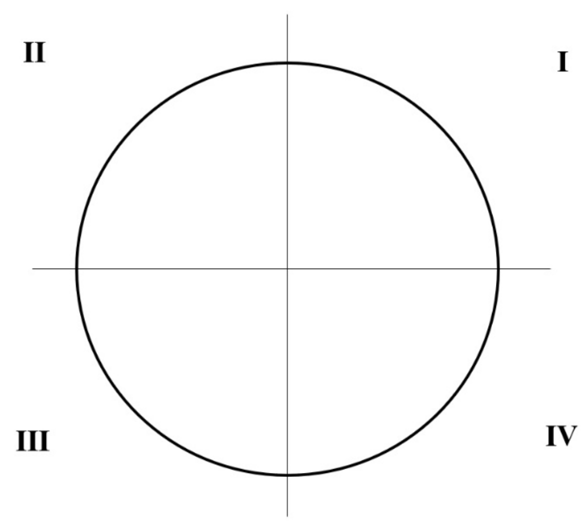
example $\theta_3 = \sin^{-1}(0.5)$ trig unit circle



Problem: the **`asin`** *inverse sine function* is **double-valued**. For each θ_4 there are two possible θ_3 solutions, only one of which is correct.

Why not find θ_3 using the *X* equation only? The inverse cosine function has a similar problem:

example $\theta_3 = \cos^{-1}(0.866)$ trig unit circle

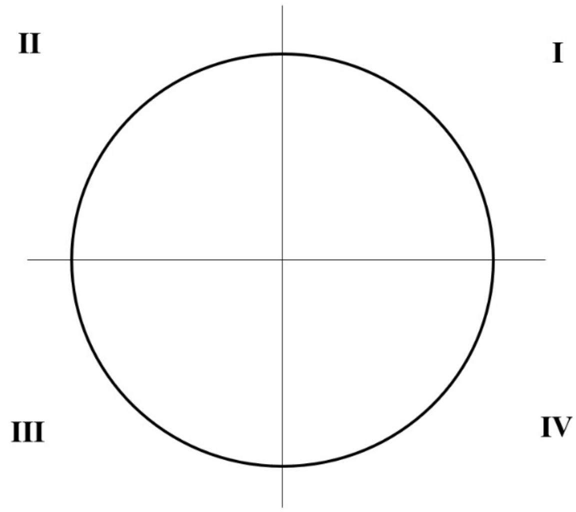


Problem: the **`acos`** *inverse cosine function* is **double-valued**. For each θ_4 there are two possible θ_3 solutions, only one of which is correct.

So we must use information from both sine and cosine (i.e. both *Y* and *X* equations) – this suggests using the tangent (as we did earlier in the θ_3 solution).

$$\theta_3 = \tan^{-1}\left[\frac{r_1 s_1 + r_4 s_4 - r_2 s_2}{r_1 c_1 + r_4 c_4 - r_2 c_2}\right]$$

example $\theta_3 = \tan^{-1}(0.5774)$ trig unit circle



Problem the plain **`atan`** *inverse tangent function* is still **double-valued.** For each θ_4 there are two possible θ_3 solutions, only one of which is correct.

Solution use the **quadrant-specific** inverse tangent function **`atan2`**. The input to this function is both the numerator and denominator. The function has built-in logic to determine the correct quadrant for the angle answer, given the $\pm$ signs of the numerator and denominator. The plain **`atan`** function takes a single quotient input; hence this sign information is lost and the true quadrant for the answer is unknown. There is **no trigonometric uncertainty** with the **quadrant-specific** inverse tangent function **`atan2`**.

example:

$$\theta_3 = \text{atan2}(+0.5, +0.866) = 30^\circ$$

$$\theta_3 = \text{atan2}(-0.5, -0.866) = 210^\circ$$

usage:

$$\tan\theta_3 = \frac{num}{den}$$

$$\theta_3 = \text{atan2}(num, den)$$

e.g. $\theta_3 = \text{atan2}(r_1 s_1 + r_4 s_4 - r_2 s_2, r_1 c_1 + r_4 c_4 - r_2 c_2)$

`` % % TrigUn.m - trigonometric uncertainty examples % Dr. Bob, ME 3011 % clear; clc; DR = pi/180; % sine phi1s = asin(0.5000); asinang1 = phi1s/DR phi2s = pi - phi1s; asinang2 = phi2s/DR sine1 = sin(phi1s) sine2 = sin(phi2s) % cosine phi1c = acos(0.8660); acosang1 = phi1c/DR phi2c = -phi1c; acosang2 = phi2c/DR cos1 = cos(phi1c) cos2 = cos(phi2c) % tangent phi1t = atan(0.5774); atanang1 = phi1t/DR phi2t = phi1t+pi; atanang2 = phi2t/DR tan1 = tan(phi1t) tan2 = tan(phi2t) % quadrant-specific tangent phi1t = atan2(+0.5,+0.8660); atan2ang1 = phi1t/DR ratio1 = +0.5/+0.8660 ttan1 = tan(phi1t) phi2t = atan2(-0.5,-0.8660); atan2ang2 = phi2t/DR ratio2 = -0.5/-0.8660 ttan2 = tan(phi2t) ```	``` Output of TrigUn.m % sine asinang1 = 30.0 asinang2 = 150.0 sine1 = 0.5 sine2 = 0.5 % cosine acosang1 = 30.0 acosang2 = -30.0 cos1 = 0.866 cos2 = 0.866 % tangent atanang1 = 30.0 atanang2 = 210.0 tan1 = 0.5774 tan2 = 0.5774 % quadrant-specific tangent atan2ang1 = 30.0 ratio1 = 0.5774 ttan1 = 0.5774 atan2ang2 = -150.0 ratio2 = 0.5774 ttan2 = 0.5774 ```

Now, having just cleared up this **Trigonometric Uncertainty**, we already have an exception in the θ_4 tangent half-angle solution. Recall the tangent half-angle definition was

$$t = \tan\left(\frac{\theta_4}{2}\right)$$

and so the solution for θ_4 was

$$\theta_4 = 2\tan^{-1}(t)$$

There are two branches, one for each t value; only one is shown above. Recall we have a numerator and denominator for each t, but we can ignore them and form a single ratio for the **`atan`** function.

With the 2 multiplying the inverse tangent result, it doesn't matter whether we use **`atan`** or **`atan2`** since the final answer will come to the same angle. See the example directly below.

Example

For $\frac{\theta_4}{2} = \tan^{-1}(0.5774)$, we don't know if the solution is:

$$\frac{\theta_4}{2} = 30^\circ \qquad \text{or} \qquad \frac{\theta_4}{2} = 210^\circ$$

However, the multiplier 2 takes care of this uncertainty, since angles are identical every 360°:

$$\theta_4 = 60^\circ \qquad \text{or} \qquad \theta_4 = 420^\circ = 60^\circ$$

2.1.3 Four-Bar Mechanism Solution Irregularities

Recall the equation for t, the tangent half-angle substitution variable to calculate angle θ_4:

$$t_{1,2} = \frac{-F \pm \sqrt{E^2 + F^2 - G^2}}{G - E}$$

$$\theta_{4_{1,2}} = 2\tan^{-1}(t_{1,2})$$

Do the solutions for θ_4, both branches, always exist?

- What if $E = G$? This is a divide-by-zero singularity where t_1 and t_2 are infinite and the associated θ_4 solution is unreliable (see the on-line ME 3011 Supplement, people.ohio.edu/williams/html/PDF/Supplement3011.pdf, Four Bar Mechanism Solution Irregularities).

- What if $E^2 + F^2 - G^2 < 0$? This yields an imaginary solution for θ_4, which physically means the mechanism cannot assemble for that input angle. The two circles from the four-bar graphical position analysis fail to intersect. See the following section on Grashof's Law.

- What if $E^2 + F^2 - G^2 = 0$? This represents the boundary between real and imaginary solutions for θ_4. These cases physically represent joint limits for the input link, i.e. angular limits on θ_2. The two solution branches become one in these cases, i.e. the two circles from the four-bar graphical position analysis intersect only in one point, not two (see the on-line ME 3011 Supplement, Four-Bar Mechanism Joint Limits). In this situation, links 3 and 4 have become collinear, preventing link 2 from rotating any further.

These conditions are the same for both solution branches. Next we present Grashof's Law to determine the rotatability of the input and output links in a four-bar mechanism. This is related to the condition $E^2 + F^2 - G^2 < 0$ from above.

2.1.4 Grashof's Law

Grashof was a German Engineer in the late 1800s. **Grashof's Law** is used to determine the rotatability of the input and output links in a four-bar mechanism.

crank full rotation, no limits for the link, $E^2+F^2-G^2>0$ always if input is a crank
rocker not full rotation, link rotates back-and-forth between limits

For an input link **rocker**, $E^2+F^2-G^2<0$ for some θ_2 range (for some input link range, there is an imaginary θ_4 solution for which the four-bar mechanism fails to assemble).

<u>**Mechanism types (input / output links)**</u>

crank-rocker and rocker-crank
crank-crank (double crank, a.k.a. drag-link mechanism)
rocker-rocker (double rocker)

Identify the longest L, shortest S, and intermediate 2 link lengths P,Q L, S, P, Q

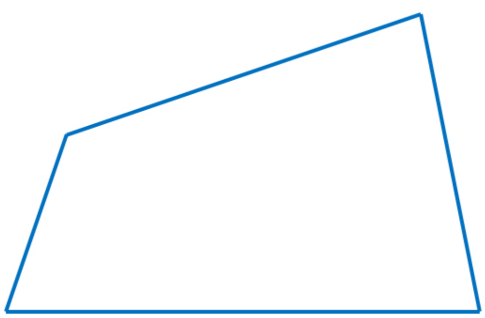
1) If $L+S<P+Q$, this is called a ***Grashof Mechanism*** and there are four different mechanisms and rotation conditions.

The 4 Four-Bar Mechanism Kinematic Inversions

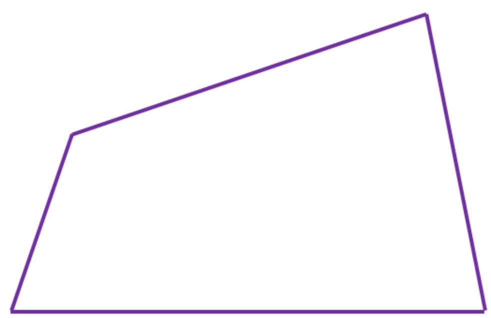
a.

b.

c.



d.



2) If $L+S>P+Q$, this is called a ***Non-Grashof Mechanism*** and the four different mechanism inversions yield only one rotation condition.

3) If $L+S=P+Q$, this is called a ***Special Grashof Mechanism*** and the four different mechanism inversions yield the identical rotation conditions from 1) ***Grashof Mechanism***. However, there is the additional interesting and troublesome feature that the mechanism may jump branches. The links become collinear at $\theta_2 = 0,180^\circ$.

Grashof's Law Examples

1) $L=10,\ S=4,\ P=8,\ Q=7$ **Grashof Mechanism**; demonstrate the 4 possibilities

2) $L=10,\ S=6,\ P=8,\ Q=7$ **Non-Grashof Mechanism**; all double rockers

3) $L=10,\ S=5,\ P=8,\ Q=7$ **Special Grashof Mechanism**; demonstrate branch jumping

4) $L=P=10,\ S=Q=4$ **Special-case Special Grashof Mechanism**

Example 4 is the parallel four-bar locomotive linkage – it is subject to branch jumping unless constrained. Also, the kinematics analysis is very easy, assuming it remains parallel.

$$\theta_2 = \theta_4 = \mu \qquad \theta_3 = 0$$

$$\omega_2 = \omega_4 \qquad \omega_3 = 0$$

$$\alpha_2 = \alpha_4 \qquad \alpha_3 = 0$$

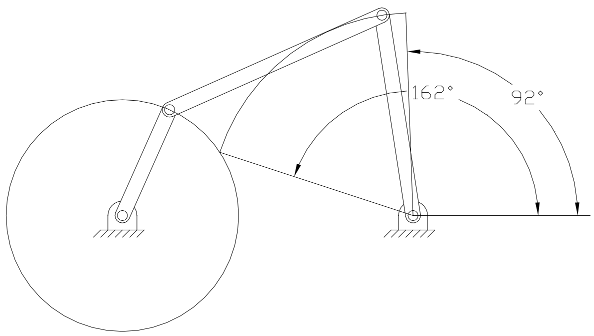
for all motion.

For more on Grashof's Law (including coupler link rotatability), see:

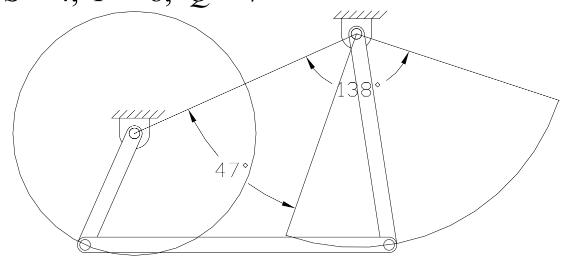
R.L. Williams II and C.F. Reinholtz, 1987, "Mechanism Link Rotatability and Limit Position Analysis Using Polynomial Discriminants", *Journal of Mechanisms, Transmissions, and Automation in Design*, Transactions of the ASME, 109(2): 178-182.

R.L. Williams II and C.F. Reinholtz, 1986, "Proof of Grashof's Law Using Polynomial Discriminants", *Journal of Mechanisms, Transmissions, and Automation in Design*, Transactions of the ASME, 108(4): 562-564.

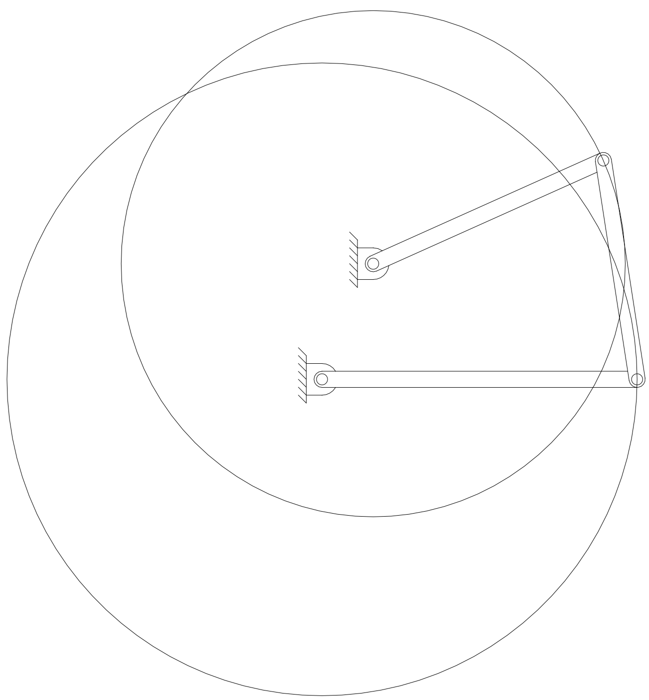
Grashof's Law Examples Figures 1) $L = 10,\ S = 4,\ P = 8,\ Q = 7$



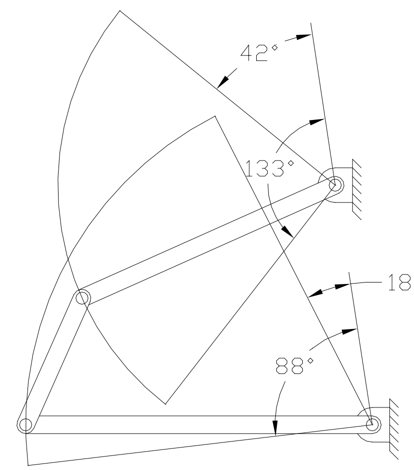
1a. Grashof Crank Rocker



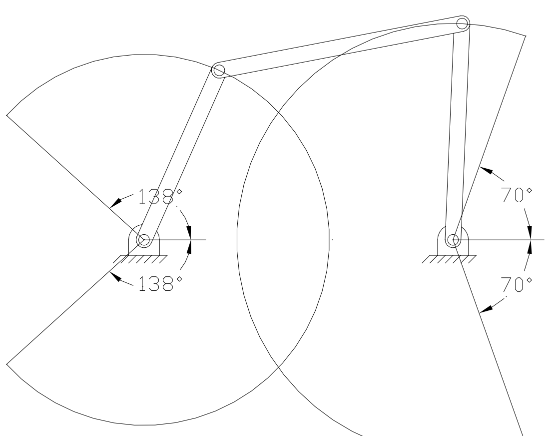
1b. Grashof Crank Rocker



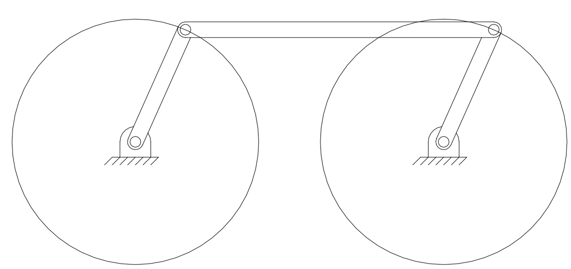
1c. Grashof Double Crank



1d. Grashof Double Rocker



2. Non-Grashof Double Rocker (first inversion)
$L = 10,\ S = 6,\ P = 8,\ Q = 7$



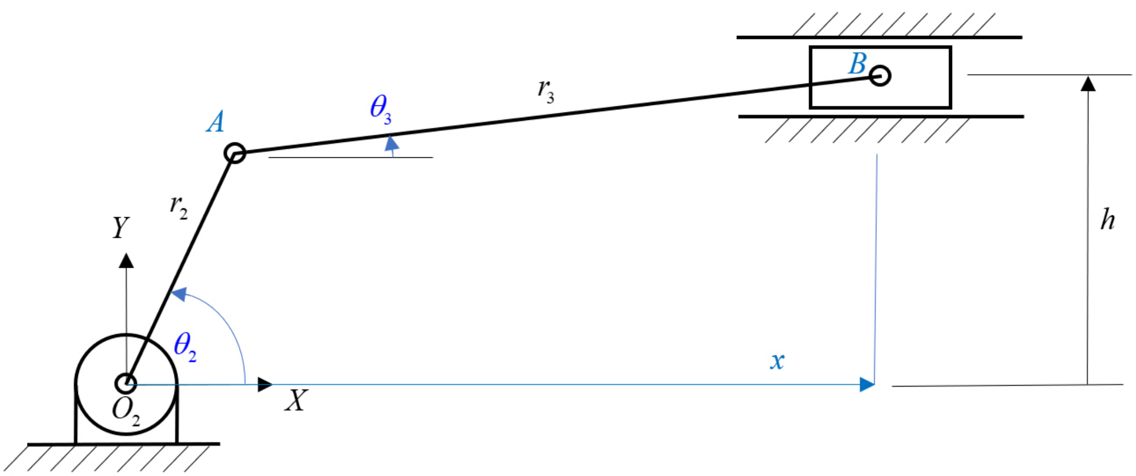
3. Special Grashof Mechanism
$L = 10,\ S = 4,\ P = 10,\ Q = 4$

For four-bar mechanism joint limits please see the on-line ME 3011 Supplement.

2.2 Slider-Crank Mechanism Position Analysis

The slider-crank mechanism converts linear motion to rotary motion or vice versa via a connecting rod link. With the **internal combustion engine** an explosion drives the input piston and the output is the drive shaft rotation. With the **air compressor** an electric motor drives the input crank and the output piston compresses air. Two dead points occur when the piston is at its translational limits. A flywheel is generally used on the crank to avoid locking at these dead points. There are four kinematic inversions of the slider-crank mechanism that yield different types of motion (see Dr. Bob's on-line Atlas of Structures, Mechanisms, and Robots; people.ohio.edu/williams/html/PDF/MechanismAtlas.pdf). We will solve the air compressor case, for the slider-crank mechanism inversion 1, where the crank is the input and the slider is the output.

Step 1. Draw the **Kinematic Diagram**



r_2 – constant input link length	θ_2 – variable input angle
r_3 – constant coupler link length	θ_3 – variable coupler angle
h – constant slider offset	x – variable output displacement
standard names for important points	O_2 A B

Link 1 is the fixed ground link. All angles are measured in a right-hand sense from horizontal to the link. The output variable x is measured horizontally from the origin to the slider/coupler revolute joint location.

Step 2. State the Problem

Step 3. Draw the **Vector Diagram**. Define all angles in a positive sense, measured with the right hand from the right horizontal to the link vector (tail-to-head).

Step 4. Derive the **Vector-Loop-Closure Equation**. Start at one point and add vectors tail-to-head until you reach a second point. Write the VLCE by starting and ending at the same points, but choosing a different vector path.

Step 5. Write the ***XY* Components** for the Vector-Loop-Closure Equation. Separate the one vector equation into two X and Y scalar component equations.

Step 6. **Solve for the Unknowns** from the XY scalar equations. Step 5 yields two partially-coupled, partially-nonlinear equations in the two unknowns x, θ_3. We could isolate θ_3, square and add the two equations, and solve for x like the four-bar mechanism approach (see the alternate solution presented on the next page). However, notice that the two XY equations are coupled and nonlinear only in θ_3 but not in x. There is a much simpler method – solve θ_3 using the Y equation only and then solve x from the X equation:

What about **trigonometric uncertainty**? The inverse sine function is **double-valued**, but in this case both solutions are correct, yielding two valid solution branches. Graphically demonstrate the right and left branches.

Step 6. Solve for the unknowns – alternate solution

Here are the same slider-crank mechanism position analysis *XY* component equations, rearranged to isolate the θ_3 terms.

$$r_3c_3 = x - r_2c_2$$
$$r_3s_3 = h - r_2s_2$$

We can square and add to eliminate θ_3, similar to the four-bar mechanism solution approach.

$$r_3^2c_3^2 = x^2 - 2xr_2c_2 + r_2^2c_2^2$$
$$r_3^2s_3^2 = h^2 - 2hr_2s_2 + r_2^2s_2^2$$

$$r_3^2 = x^2 + h^2 + r_2^2 - 2xr_2c_2 - 2hr_2s_2$$

This quadratic equation in *x* has the following form:

$$ax^2 + bx + c = 0 \qquad \begin{aligned} a &= 1 \\ b &= -2r_2c_2 \\ c &= r_2^2 - r_3^2 + h^2 - 2hr_2s_2 \end{aligned}$$

There are two solutions for *x*, corresponding to the right and left branches.

$$x_{1,2} = r_2c_2 \pm \sqrt{r_3^2 - h^2 - r_2^2s_2^2 + 2hr_2s_2}$$

Then θ_3 is found from a ratio of the *Y* to *X* equations.

$$\theta_{3_{1,2}} = \text{atan2}(h - r_2s_2, x_{1,2} - r_2c_2)$$

This alternate solution yields identical results as the earlier solution approach in the ME 3011 NotesBook for the right (θ_{3_1}, x_1) and left (θ_{3_2}, x_2) branches.

Slider-Crank Mechanism Position Analysis Graphical Solution

The Slider-Crank Mechanism position analysis problem may be solved *graphically*, by drawing the mechanism, determining the mechanism closure, and measuring the unknown variables. This is an excellent method to validate your computer results at a given snapshot.

- Place the grounded revolute joint point O_2 for the crank at the origin. Establish a suitable scale.
- Draw the line of the slider, offset vertically from the *X* axis by *h*.
- Draw the given input link length r_2 at the given angle θ_2. The endpoint of r_2 is point *A*.
- Draw a circle of radius r_3 centered at point *A*.
- This circle intersects the *h* slider line in general in two places to give two locations for point *B*.
- Connect the two branches and measure the unknowns *x* and θ_3.
- What if there is only one solution for point *B*? What if there are no solutions for point *B* (i.e. the r_3 circle and the *h* line do not intersect)?

Slider-crank mechanism transmission angle

The slider-crank transmission angle μ is defined the same as that for the four-bar: the relative angle between the coupler and output links 3 and 4. The transmission angle can be found by geometry, $\mu = \theta_4 - \theta_3 = 90^\circ - \theta_3$. μ is optimal at 90° and zero transmission results when $\mu = 0,180^\circ$; therefore, $\theta_3 = 0,180^\circ$ is optimal and $\theta_3 = 90^\circ, 270^\circ$ is the worst transmission. The latter (worst) cases are avoided if the input link is a crank (see next subsection).

Full-rotation condition

For the slider-crank mechanism position solution to exist for entire motion range (for r_2 to be a crank), the absolute value of the inverse sine argument must always be less than or equal to 1:

$$\left| \frac{h - r_2 s_2}{r_3} \right| \le 1 \qquad\qquad r_3 \ge h - r_2 s_2$$

which must hold for all motion. The worst case is $\theta_2 = -90^\circ$, which yields $r_3 \ge h + r_2$. This is much simpler than Grashof's law for the Four-Bar Mechanism.

This condition was derived assuming positive *h*; allowing negative *h*, the condition is:

$$r_3 \ge |h| + r_2 .$$

Term Example 2 presents snapshot and F.R.O.M. examples for an example slider-crank mechanism position analysis (that will be continued for velocity, acceleration, and inverse dynamics analysis). This is example is now initiated.

Slider-Crank Mechanism Position Analysis – Term Example 2

Given:

$$r_2 = 4 \qquad r_3 = 8 \ in \qquad h = 3$$

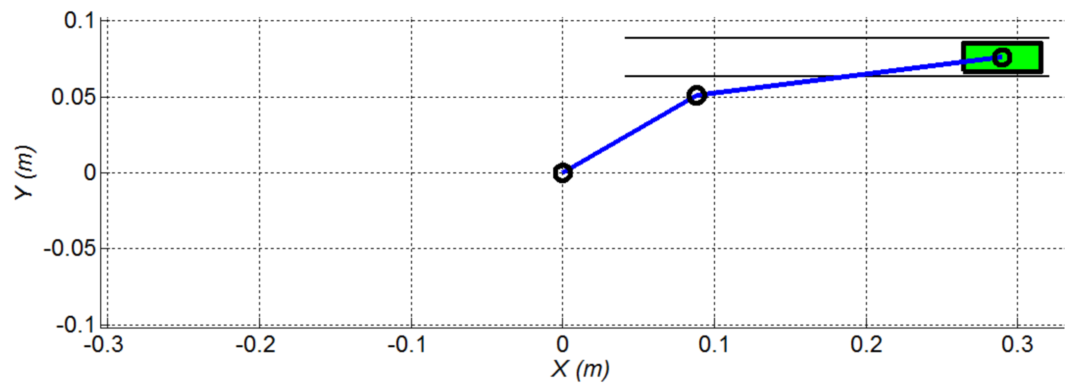
$$r_2 = 0.102 \qquad r_3 = 0.203 \ m \qquad h = 0.076$$

Snapshot Analysis (one input angle)

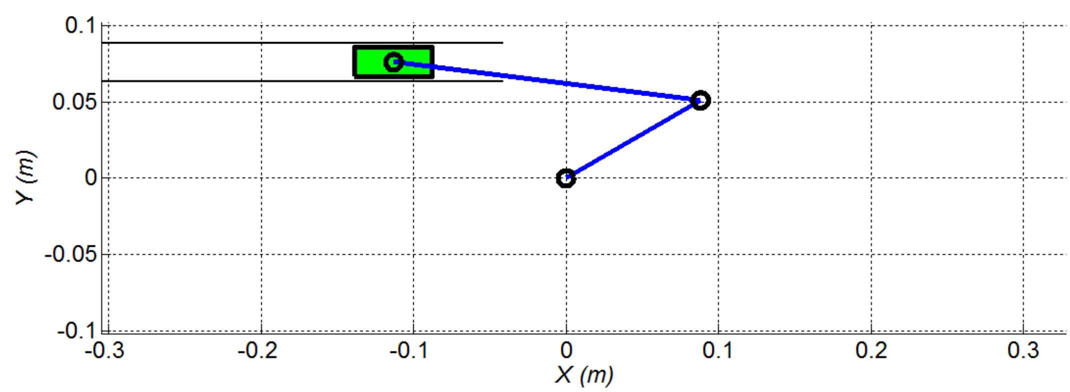
Given this mechanism and $\theta_2 = 30^\circ$, calculate x and θ_3 for both branches. Results:

branch	θ_3 **(deg)**	x **(m)**
right	7.1	0.290
left	172.9	–0.113

These two branch solutions are demonstrated in the figures below (length units are *m*).



Right Branch



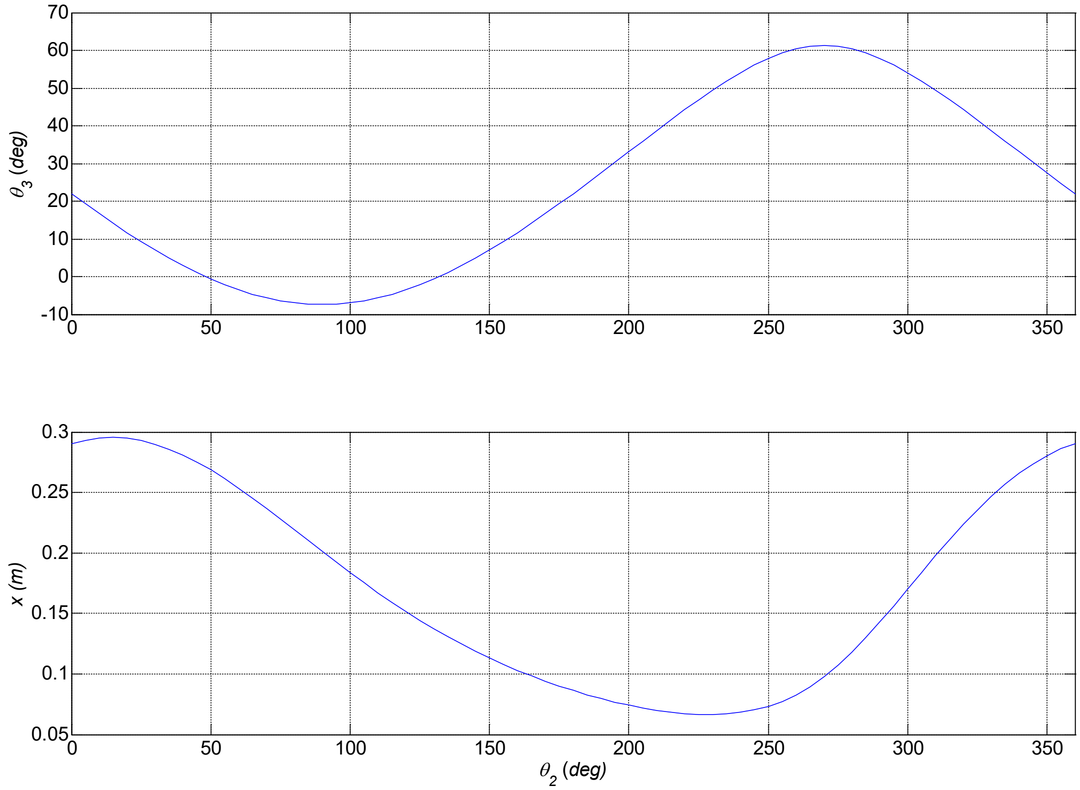
Left Branch

Term Example 2 Snapshot

Full-Range-Of-Motion (F.R.O.M.) Analysis – Term Example 2

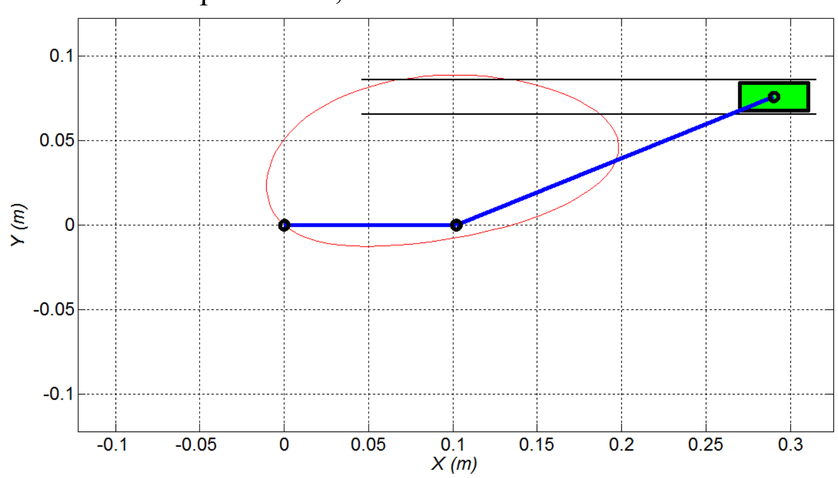
A more meaningful result from position analysis is to solve and plot the position analysis unknowns for the entire range of mechanism motion.

The top plot gives θ_3 (*deg*) and the bottom plot gives x (*m*), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 2, the right branch only.

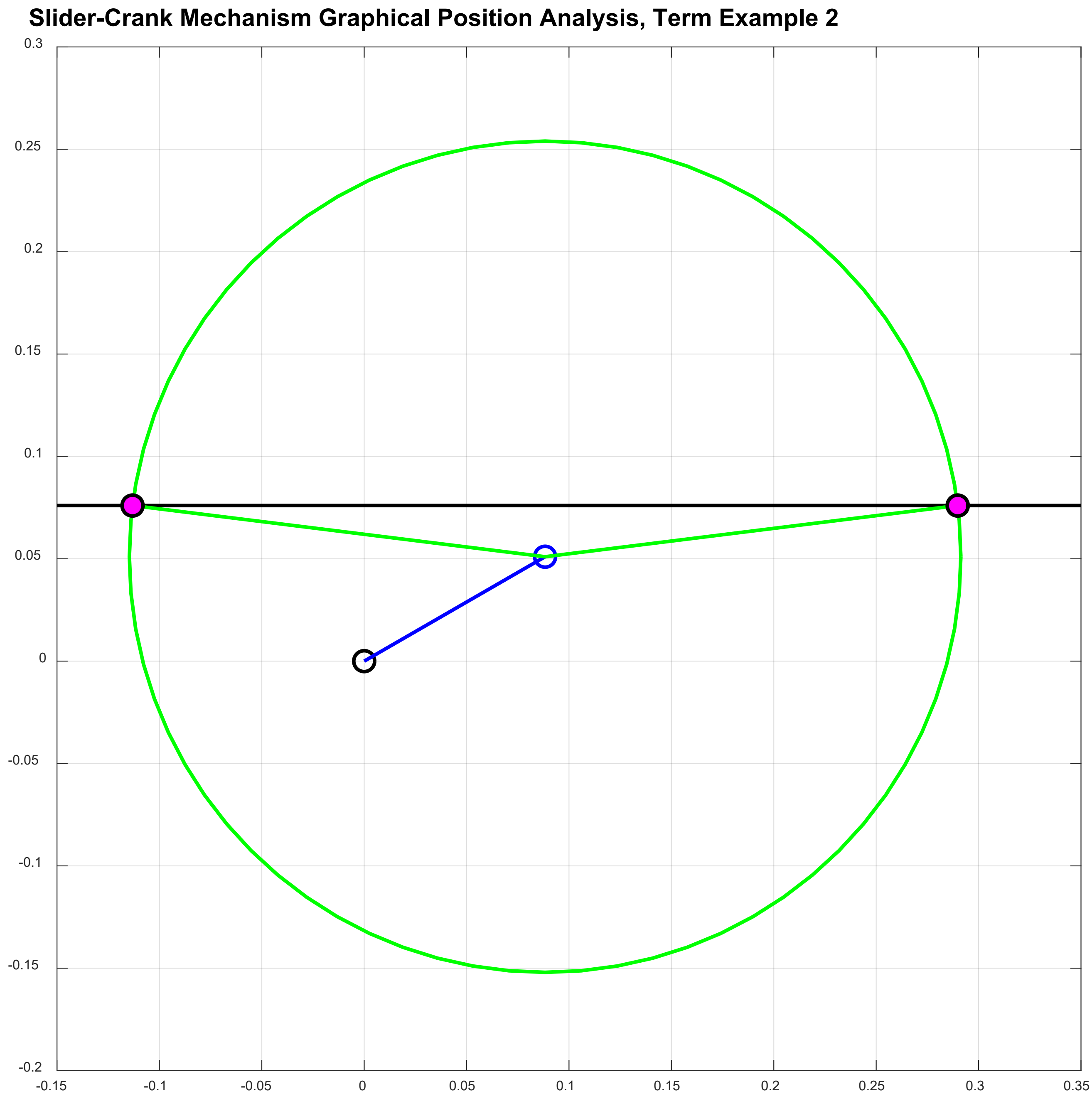


θ_3 and x

The plot below gives the initial (and final) animation position, for $\theta_2 = 0, 360^\circ$. It also gives the coupler curve to scale for the right branch, plotting P_{CY} vs. P_{CX} in red. In this case the coupler point is taken to be the midpoint of coupler link 3. For Term Example 2, the slider translation limits are $0.067 \le x \le 0.295$, as seen in the x plot above, calculated from the x translational limit equations.



Term Example 2 F.R.O.M. Position Results



SCGraphical.m

Slider-Crank Mechanism Snapshot and F.R.O.M. MATLAB m-files

No sample m-files are given for the slider-crank mechanism since you can readily adapt the snapshot and F.R.O.M. m-files given for the four-bar mechanism previously.

However, below we include a partial m-file to show how to draw the slider and fixed piston walls for the slider-crank mechanism graphics, since this was not required for the four-bar mechanism.

Outside the loop:

```
Lp     = put a number here;     % length of piston (slider link)
Hp     = put a number here;     % height of piston
Xp     = [-1 -1   1   1]*Lp/2;
Yp     = [-1  1   1  -1]*Hp/2;
```

This establishes the rectangular corner coordinates for the slider link, centered at the origin of your coordinate frame. It can be done once, outside the loop. Instead of typing numbers for **Lp** and **Hp**, I scale them to a fraction of r_2, for generality in different-sized slider-crank mechanisms. Note I only included the four corner points – MATLAB **patch** (below) closes the rectangular figure, i.e. back to the starting point.

Inside the loop (right after the **plot** command where links 2 and 3 are drawn to the screen)

```
patch(Xp+x(i),Yp+h,'g');        % draw piston to screen
```

where **x(i)** is the variable horizontal slider displacement and **h** is the constant vertical offset. These position parameters shift the piston coordinates from the origin to the correct location in each loop. You can use any piston color you like (I show **green** here, **'g'**).

Further, to draw the horizontal lines representing the piston walls:

Outside the loop

```
Xpt = [xMin-Lp    xMax+Lp];     % fixed piston walls
Ypt = [h+wall/2  h+wall/2];
Xpb = [xMin-Lp    xMax+Lp];
Ypb = [h-wall/2  h-wall/2];
```

Inside the loop (right after the **plot** command where links 2 and 3 are drawn to the screen)

```
line(Xpt,Ypt,'LineWidth',2); line(Xpb,Ypb,'LineWidth',2);
```

Set the piston wall width **wall** to allow a small clearance between the piston and the walls. Again, it can be scaled to a small fraction of r_2 for generality. The **xMin** and **xMax** coordinates used above are calculated from the piston x translation limits presented a couple of pages later, for the right branch.

MATLAB `subplot` feature

In a slider-crank mechanism full-range-of-motion (F.R.O.M.) simulation you will need to plot both θ_3 and x vs. the independent variable θ_2. Since the units of θ_3 (deg) and x (m) are dissimilar, they may not fit clearly on the same plot. In this situation you should use a sub-plot arrangement.

Outside the F.R.O.M. loop you can do the subplot in this way:

```
subplot(211);             % 2x1 arrangement of plots, first plot
plot(th2/DR,th3/DR);
subplot(212);             % 2x1 arrangement of plots, second plot
plot(th2/DR,x);
```

Now, you can use the standard axis labels, linetypes, titles, axis limits, grid, etc., for each plot within a subplot (repeat these formatting commands after each **`plot`** statement above to use similar formatting for each). These options are not shown, for clarity.

The generalized usage of **`subplot`** is shown below.

```
subplot(mni);             % m x n arrangement of plots, ith plot
plot( . . . );
```

As seen in the example syntax above, the integers need not be separated by spaces or commas. However, I believe they may be so separated if you desire.

Slider-Crank Mechanism Slider Limits

As mentioned earlier, the crank of the slider-crank will rotate fully through 360° without limit by design, as long as the following inequality is satisfied:

$$r_3 \geq |h| + r_2$$

However, the slider displacement variable x has limits, presented in this subsection. The derivation varies by right and left branch, and also + and – offset h. In addition to slider limits, the θ_2 angles at which these min/max values occur are given.

The slider reaches its maximum displacement when links 2 and 3 are aligned and its minimum displacement occurs when link 2 is folded onto link 3. We can draw two right triangles representing these conditions and calculate the x translational limits to be (for the right branch):

$$\sqrt{(r_3 - r_2)^2 - h^2} \leq x \leq \sqrt{(r_3 + r_2)^2 - h^2}$$

Right Branch, +*h*

$$x_{\min} = \sqrt{(r_3 - r_2)^2 - h^2} \qquad \theta_{2\min} = \tan^{-1}\left[\frac{h}{x_{\min}}\right] + \pi$$

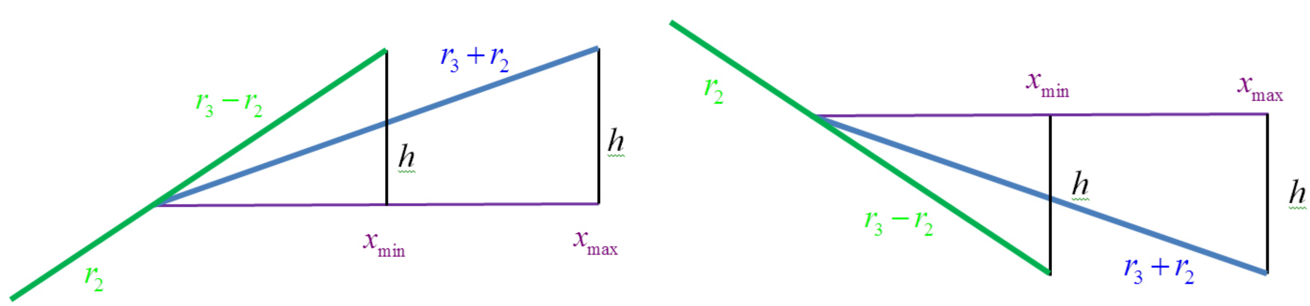
$$x_{\max} = \sqrt{(r_3 + r_2)^2 - h^2} \qquad \theta_{2\max} = \tan^{-1}\left[\frac{h}{x_{\max}}\right]$$

Right Branch, –*h*

$$x_{\min} = \sqrt{(r_3 - r_2)^2 - h^2} \qquad \theta_{2\min} = \pi - \tan^{-1}\left[\frac{|h|}{x_{\min}}\right]$$

(same)

$$x_{\max} = \sqrt{(r_3 + r_2)^2 - h^2} \qquad \theta_{2\max} = 2\pi - \tan^{-1}\left[\frac{|h|}{x_{\max}}\right]$$



Right Branch, +*h* **Right Branch, –*h***

Note both of the above Right-Branch cases yield the following simplified results in the case of *h* = 0:

Right Branch, *h* = 0

$$x_{\min} = r_3 - r_2 \qquad \theta_{2\min} = \pi$$

$$x_{\max} = r_3 + r_2 \qquad \theta_{2\max} = 0$$

Left Branch, +*h*

$$x_{\min} = -\sqrt{(r_3 + r_2)^2 - h^2} \qquad \theta_{2\min} = \pi - \tan^{-1}\left[\frac{h}{|x_{\min}|}\right]$$

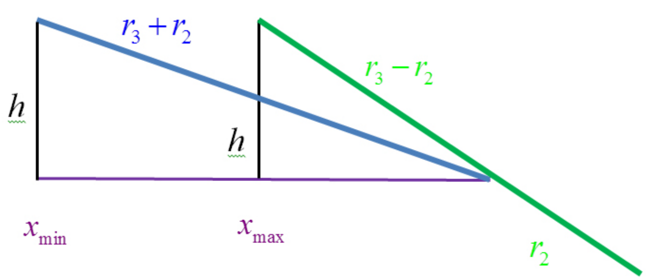
$$x_{\max} = -\sqrt{(r_3 - r_2)^2 - h^2} \qquad \theta_{2\max} = 2\pi - \tan^{-1}\left[\frac{h}{|x_{\max}|}\right]$$

Left Branch, –*h*

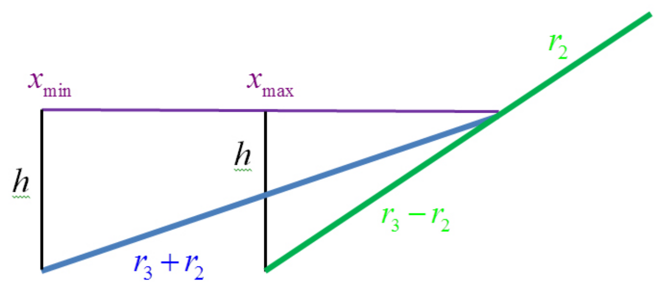
$$x_{\min} = -\sqrt{(r_3 + r_2)^2 - h^2} \qquad \theta_{2\min} = \pi + \tan^{-1}\left[\frac{|h|}{|x_{\min}|}\right]$$

(same)

$$x_{\max} = -\sqrt{(r_3 - r_2)^2 - h^2} \qquad \theta_{2\max} = \tan^{-1}\left[\frac{|h|}{|x_{\max}|}\right]$$



Left Branch, +*h*



Left Branch, –*h*

Note both of the above Left-Branch cases yield the following simplified results in the case of *h* = 0:

Left Branch, *h* = 0

$$x_{\min} = -(r_3 + r_2) \qquad \theta_{2\min} = \pi$$

$$x_{\max} = -(r_3 - r_2) \qquad \theta_{2\max} = 0$$

All of these Right- and Left-Branch slider limit results have been validated in MATLAB simulation.

3. Velocity Kinematics Analysis

3.1 Velocity Analysis Introduction

Velocity analysis is the second step in general **kinematics** analysis. It relates the translational and angular velocities of the links for a mechanism in motion. Position analysis must be completed first.

Velocity analysis is important for kinematic motion analysis because some practical tasks have timing and rates of motion. Velocity analysis is also required for dynamics and machine design: position, ***velocity***, acceleration, dynamics, forces, machine design. Velocity analysis requires the solution of coupled *linear* equations. Translational and rotational velocity is the first time derivative of the position and orientation and it is a vector quantity. The magnitude of velocity is speed; the velocity direction is also crucial in analysis. Analytical velocity analysis involves taking the first time derivatives of the *XY* component equations from position analysis and solving for the unknowns. Here are the general translational and rotational velocity expressions and units.

Mechanism Velocity Analysis

Mechanism velocity analysis involves determination of the translational and rotational velocities of the moving links in a mechanism. It is required for complete mechanism motion analysis. It is also required for further analysis: acceleration, dynamics, forces, and machine design. Linear equations result from the first time differentiation of the position equations. There is a unique velocity solution for each mechanism branch. Position analysis must be complete prior to performing velocity analysis. Since we deal with one-dof mechanisms, again one-dof of velocity input must be given.

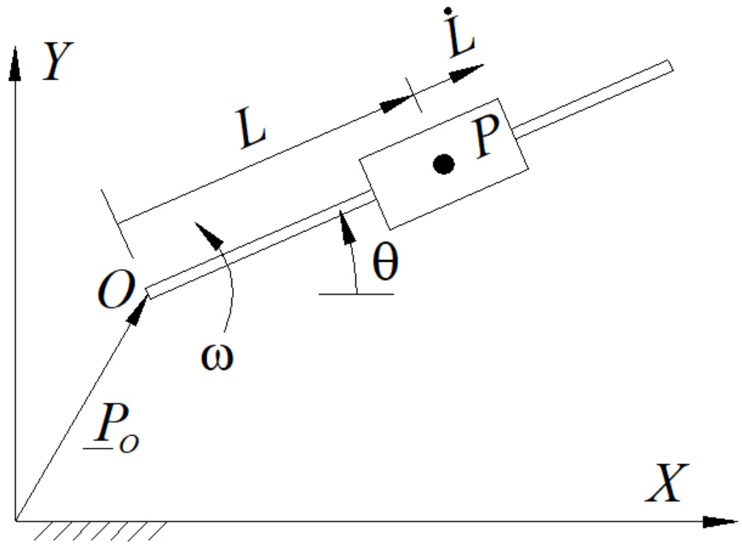
Generic Mechanism Velocity Analysis Problem Statement

Given the mechanism, complete position analysis, and one-dof of velocity input, calculate the velocity unknowns.

3.2 Three-Part Velocity Formula

In this section we will derive the three-part velocity formula, showing the most general velocity motions possible for planar devices.

Three-Part Velocity Derivation Figure



This figure presents the most general planar velocity case, a translating and rotating rigid rod with a slider on it. Find the total velocity of point *P* on the slider. Express the position vector in Cartesian coordinates.

$$\underline{P}_P = \underline{P}_O + \underline{L} =$$

The angle is changing with time (as shown below). Only the planar case is this simple; the spatial rotation case is more complicated. The length of the rod is also changing with time.

Product and Chain Rules of Differentiation

We'll need to use the product and chain rules over and over in velocity and acceleration analysis derivations.

Product rule

$$\frac{d}{dt}(x(t)y(t)) = \frac{dx(t)}{dt}y(t) + x(t)\frac{dy(t)}{dt}$$

$x(t)$, $y(t)$ are both functions of time t.

Chain rule

$$\frac{d}{dt}(f(x(t))) = \frac{df(x(t))}{dx(t)}\frac{dx(t)}{dt}$$

f is a function of $x(t)$, and thus f is an implicit function of time t.

Example

$$\frac{d}{dt}(L(t)\cos\theta(t)) = \dot{L}(t)\cos\theta(t) + L(t)\frac{d}{dt}(\cos\theta(t))$$

$$= \dot{L}(t)\cos\theta(t) + L(t)\frac{d\cos\theta(t)}{d\theta(t)}\frac{d\theta(t)}{dt}$$

$$= \dot{L}(t)\cos\theta(t) + L(t)(-\sin\theta(t))\dot{\theta}(t)$$

$$= V(t)\cos\theta(t) - L(t)\omega(t)\sin\theta(t)$$

where:

$$V(t) = \dot{L}(t) = \frac{dL(t)}{dt}$$

$$\omega(t) = \dot{\theta}(t) = \frac{d\theta(t)}{dt}$$

and all terms $L(t)$, $V(t)$, $\theta(t)$, and $\omega(t)$ are functions of time.

Now we can return to the three-part velocity derivation.

First time derivative of the position vector

$$\underline{V}_P = \frac{d\underline{P}_P}{dt} =$$

We have just derived the **Three-Part Velocity Equation**.

$$\underline{V}_P = \underline{V}_O + \underline{V} + \underline{\omega} \times \underline{L}$$

The terms for the **Three-Part Velocity Equation** can be expressed in various ways as summarized in the table below.

vector	$\underline{V}_O$	$\underline{V}$	$\underline{\omega} \times \underline{L}$
name	point *O* velocity vector	sliding velocity vector	tangential velocity vector
***XY* components**			
magnitude / direction			

Three-Part Velocity Snapshot Example

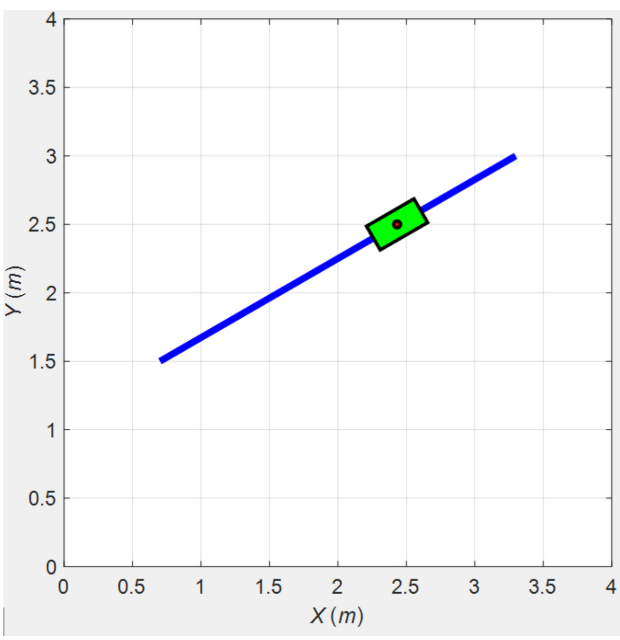
Given $L = 2\ m,\ \theta = 30^\circ,\ \omega = 1\ rad/s,\ |\underline{V}| = \dot{L} = 3\ m/s$ (outward), $\underline{V}_O = \{3 \quad 2\}^T\ m/s$, calculate $\underline{V}_P$.

$$\underline{V}_P = \begin{Bmatrix} V_{OX} + V\cos\theta - L\omega\sin\theta \\ V_{OY} + V\sin\theta + L\omega\cos\theta \end{Bmatrix} = \begin{Bmatrix} 3 + 3\cos 30^\circ - 2(1)\sin 30^\circ \\ 2 + 3\sin 30^\circ + 2(1)\cos 30^\circ \end{Bmatrix}$$

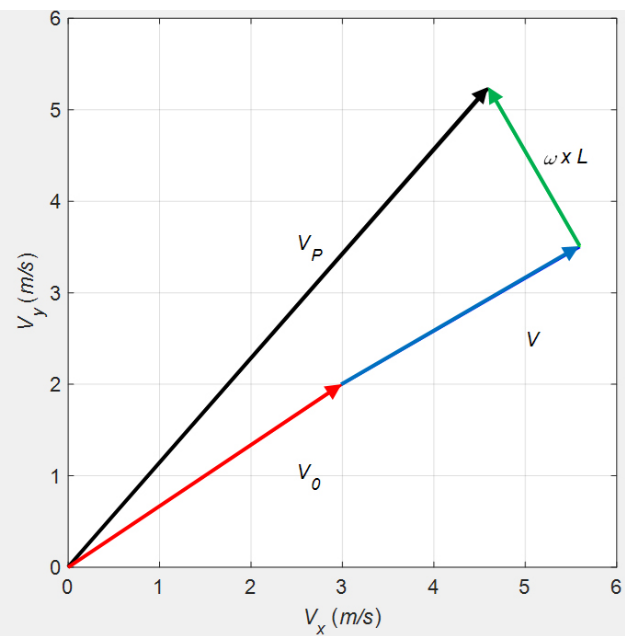
$$\underline{V}_P = \begin{Bmatrix} 3 + 2.6 - 1 \\ 2 + 1.5 + 1.7 \end{Bmatrix} = \begin{Bmatrix} 4.6 \\ 5.2 \end{Bmatrix} \frac{m}{s}$$

$$\underline{V}_P = 7.0 @ 48.7^\circ$$

vector	$\underline{V}_0$	$\underline{V}$	$\underline{\omega} \times \underline{L}$	$\underline{V}_P$
name	point *O* velocity vector	sliding velocity vector	tangential velocity vector	total velocity vector
***XY* components**	$\begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$	$\begin{Bmatrix} 2.6 \\ 1.5 \end{Bmatrix}$	$\begin{Bmatrix} -1 \\ 1.7 \end{Bmatrix}$	$\begin{Bmatrix} 4.6 \\ 5.2 \end{Bmatrix}$
magnitude / direction	3.6 @33.7°	3 @30°	2 @120°	7.0 @48.7°



Kinematic Diagram with $\underline{P}_0 = \{0.7 \quad 1.5\}^T$

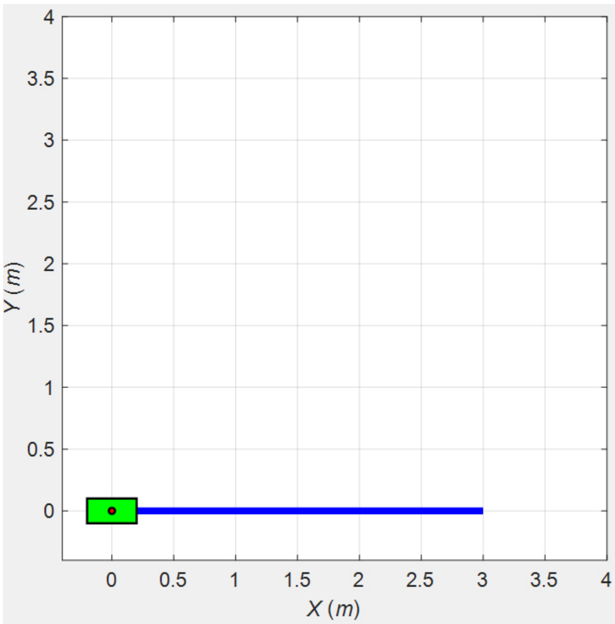


Three-Part Velocity Diagram

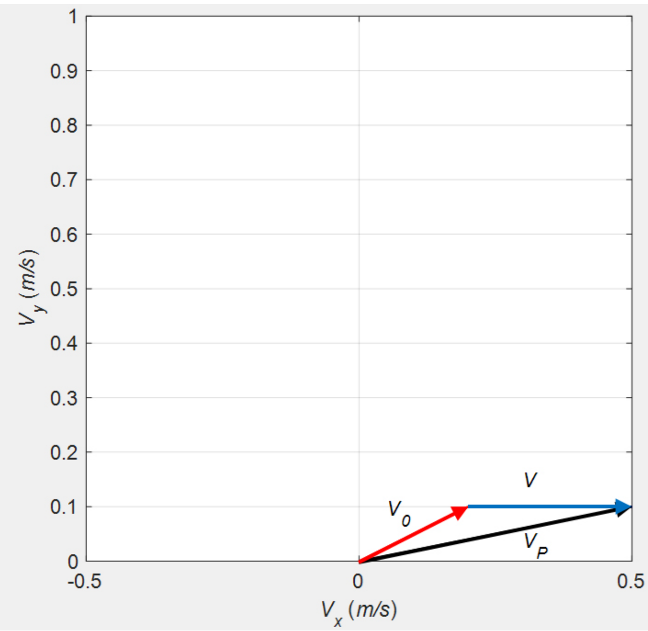
A moving example is presented for the three-part velocity formula next.

Three-Part Velocity Moving Example

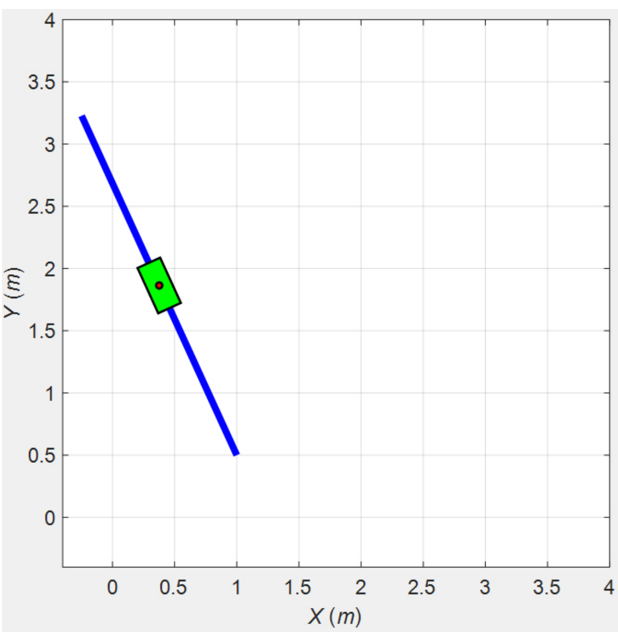
Given initial positions $\{P_{0x} \quad P_{0y} \quad L \quad \theta\} = \{0 \quad 0 \quad 0 \quad 0\}$ (m, rad) and constant velocities $\{V_{0x} \quad V_{0y} \quad V \quad \omega\} = \{0.2 \quad 0.1 \quad 0.3 \quad 0.4\}$ (m/s, rad/s), simulate this motion and determine $\underline{V}_P$ at each instant. $t_f = 5$ and $\Delta t = 0.1$ sec was used. The initial and final snapshots, with their three-part velocity diagrams, are shown below.



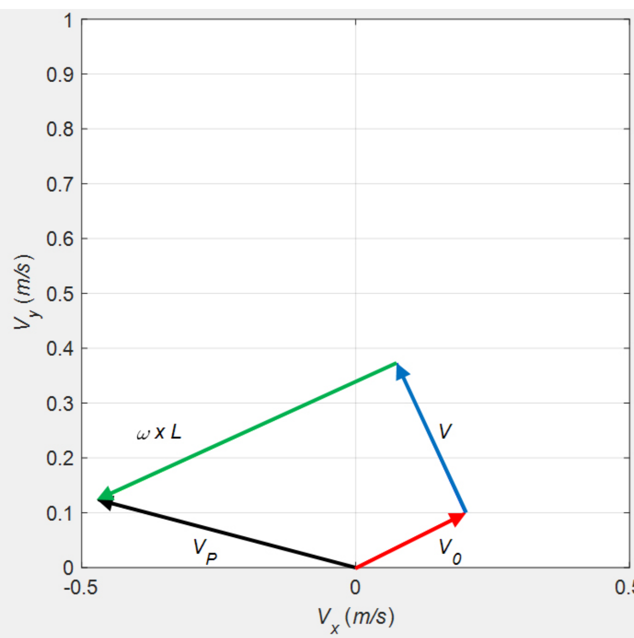
Initial Kinematic Diagram



Initial Three-Part Velocity Diagram

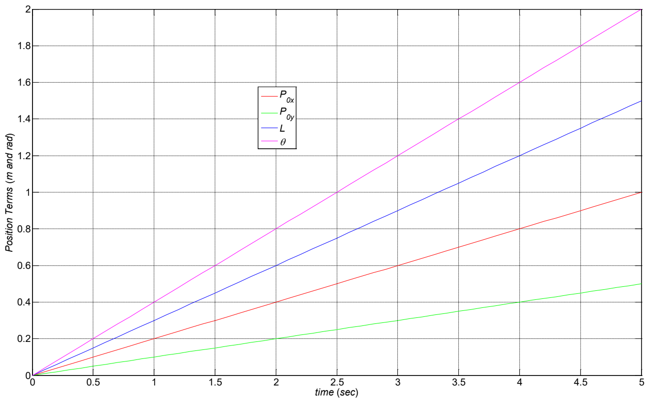


Final Kinematic Diagram

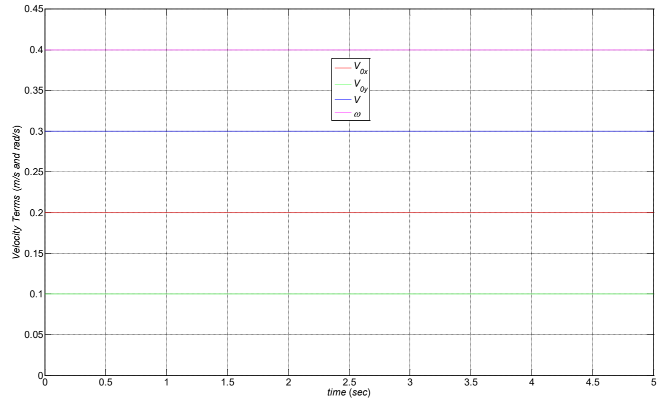


Final Three-Part Velocity Diagram

Three-Part Velocity Moving Example Plots

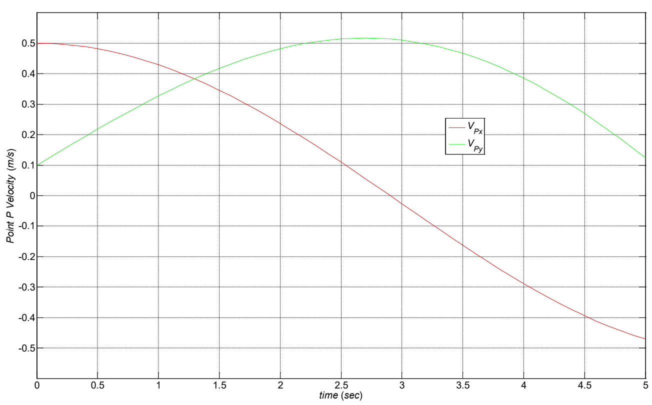


Position Terms

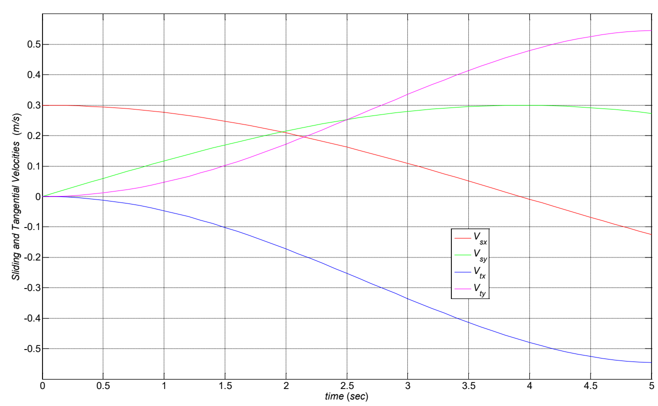


Velocity Terms

What is the relationship between these plots?



Point *P* Translational Velocity



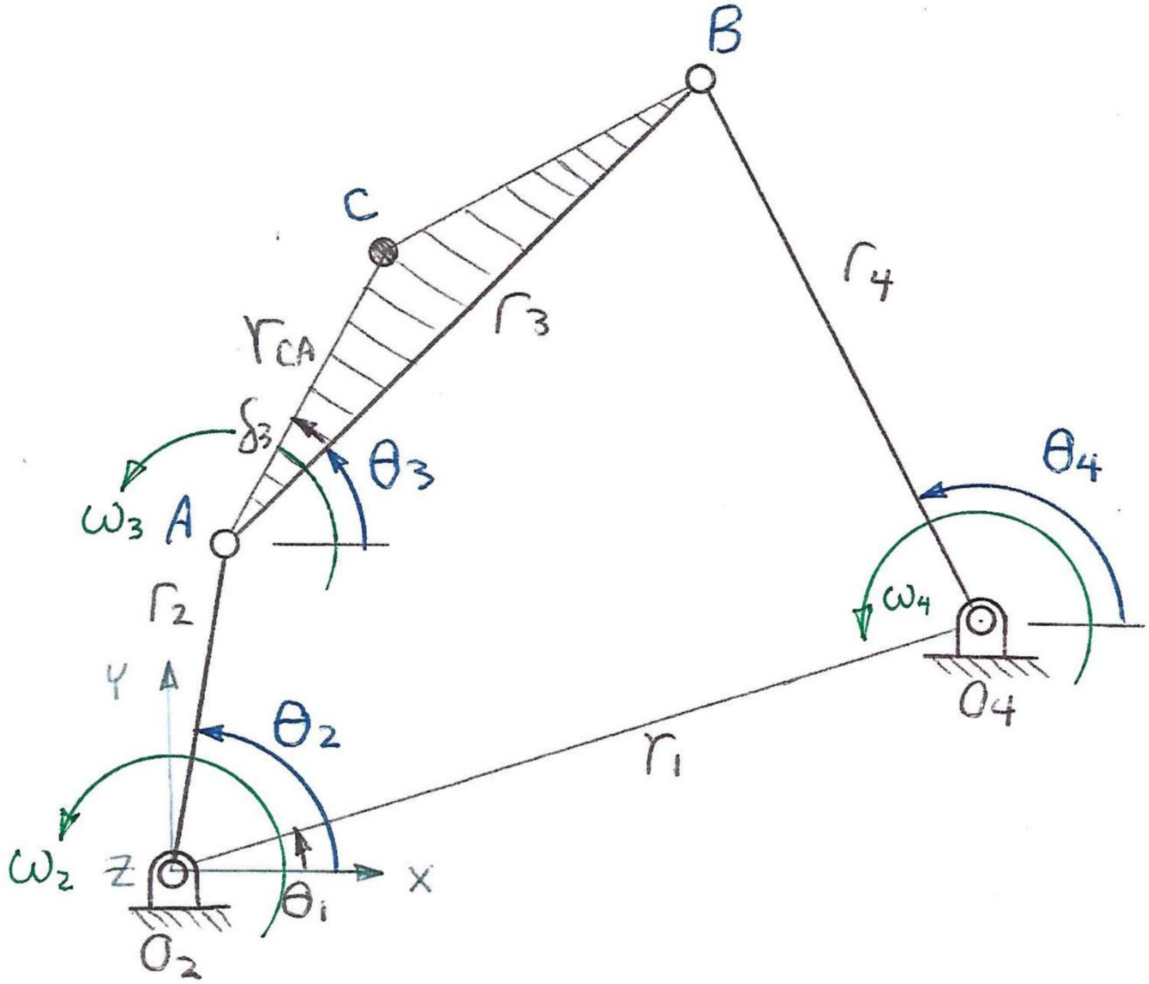
Sliding and Tangential Velocity Components

Constant velocity terms $\left\{ V_{0x} \quad V_{0y} \quad V \quad \omega \right\} = \left\{ 0.2 \quad 0.1 \quad 0.3 \quad 0.4 \right\}$ lead to non-constant point *P* velocity (due to the nonlinear position kinematics).

3.3 Four-Bar Mechanism Velocity Analysis

Step 1. The four-bar mechanism **Position Analysis** must first be complete.

Step 2. Draw the four-bar mechanism **Velocity Diagram**.



Where $\underline{\omega}_i$ (i = 2,3,4) is the absolute angular velocity of link i. $\underline{\omega}_1 = 0$ since the ground link is fixed.

Step 3. State the Problem

Step 4. Derive the velocity equations. Take the first time derivative of the vector loop closure equations from position analysis, in *XY* component form.

Four-bar mechanism position equations

VLCE $\underline{r}_2 + \underline{r}_3 = \underline{r}_1 + \underline{r}_4$ *XY* components

$$r_2c_2 + r_3c_3 = r_1c_1 + r_4c_4$$
$$r_2s_2 + r_3s_3 = r_1s_1 + r_4s_4$$

The first time derivative of the position equations requires the chain rule.

$$\begin{aligned}\frac{d}{dt}(r_i\cos\theta_i(t)) &= r_i\frac{d}{dt}(\cos\theta_i(t))\\ &= r_i\frac{d\cos\theta_i(t)}{d\theta_i}\frac{d\theta_i(t)}{dt}\\ &= r_i(-\sin\theta_i(t))\dot{\theta}_i(t)\\ &= -r_i\omega_i(t)\sin\theta_i(t)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(r_i\sin\theta_i(t)) &= r_i\frac{d}{dt}(\sin\theta_i(t))\\ &= r_i\frac{d\sin\theta_i(t)}{d\theta_i}\frac{d\theta_i(t)}{dt}\\ &= r_i(\cos\theta_i(t))\dot{\theta}_i(t)\\ &= r_i\omega_i(t)\cos\theta_i(t)\end{aligned}$$

Here we don't have to use the product rule because all $\dot{r}_i = 0$ (due to the rigid links).

The first time derivative of the *XY* position equations yields the *XY* velocity equations.

$$\begin{aligned}r_2c_2 + r_3c_3 &= r_1c_1 + r_4c_4\\ r_2s_2 + r_3s_3 &= r_1s_1 + r_4s_4\end{aligned}$$

Gathering unknowns on the LHS

Substituting simpler terms

$$\begin{aligned}a &= r_3s_3 & d &= -r_3c_3\\ b &= -r_4s_4 & e &= r_4c_4\\ c &= -r_2\omega_2s_2 & f &= r_2\omega_2c_2\end{aligned}$$

Step 5. **Solve the velocity equations** for the unknowns ω_3, ω_4. Algebra solution

$$\omega_3 = \frac{ce - bf}{ae - bd} \qquad \omega_4 = \frac{af - cd}{ae - bd}$$

Back substituting the terms $a - f$ yields the following equivalent solutions, which simplify nicely using the sum-of-angles formula $\sin(a-b) = \sin a \cos b - \cos a \sin b$ and better display the structure of the solutions.

$$\omega_3 = \frac{-r_2 \sin(\theta_4 - \theta_2)}{r_3 \sin(\theta_4 - \theta_3)}\omega_2 \qquad \omega_4 = \frac{-r_2 \sin(\theta_3 - \theta_2)}{r_4 \sin(\theta_4 - \theta_3)}\omega_2$$

Translational velocity of a point on the four-bar mechanism

The basic four-bar mechanism velocity analysis problem is now solved. Now that we know the angular velocity unknowns, we can find the **translational velocity of any point** on the mechanism, e.g. coupler point *C*. From earlier, the position vector of coupler point *C* is repeated here:

$$\mathbf{P}_C = \mathbf{r}_2 + \mathbf{r}_{CA} = \begin{Bmatrix} P_{CX} \\ P_{CY} \end{Bmatrix} = \begin{Bmatrix} r_2 c_2 + r_{CA} c\beta \\ r_2 s_2 + r_{CA} s\beta \end{Bmatrix}$$

$$\beta = \theta_3 + \delta_3$$

Four-bar mechanism velocity example – Term Example 1 continued

Given r_1 = 0.284, r_2 = 0.076, r_3 = 0.203, r_4 = 0.178, r_{CA} = 0.127 m, and $\theta_1 = 10.3^\circ, \theta_2 = 30^\circ$, $\theta_3 = 53.8^\circ, \theta_4 = 121.7^\circ, \delta_3 = 36.9^\circ$. This is the open branch of the Term Example 1 four-bar mechanism.

Snapshot Analysis

Given this mechanism position analysis plus $\omega_2 = 20$ rad/s (positive, which indicates ccw), calculate ω_3, ω_4, and $\underline{V}_C$ for this instant in motion (snapshot).

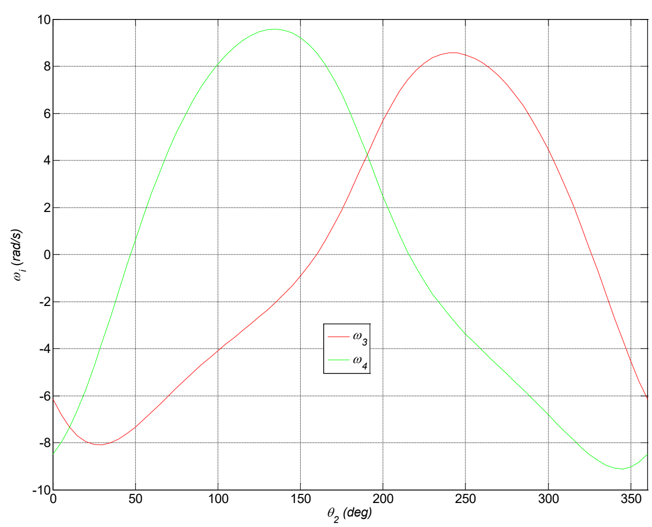
$$\begin{bmatrix} 0.164 & -0.151 \\ -0.120 & -0.094 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \omega_4 \end{Bmatrix} = \begin{Bmatrix} -0.760 \\ 1.316 \end{Bmatrix} \qquad \begin{Bmatrix} \omega_3 \\ \omega_4 \end{Bmatrix} = \begin{Bmatrix} -8.073 \\ -3.729 \end{Bmatrix}$$

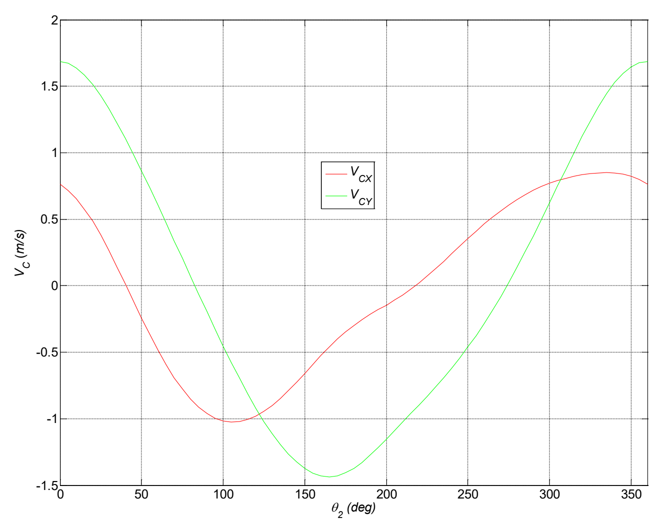
Both ω_3 and ω_4 are negative, so they are in the cw direction for this snapshot. These results are the absolute angular velocities of links 3 and 4 with respect to the ground link. The associated coupler point translational velocity vector for this snapshot is:

$$\underline{V}_C = \begin{Bmatrix} 0.265 \\ 1.330 \end{Bmatrix} \ (m/s)$$

Full-Range-Of-Motion (F.R.O.M.) Analysis: Term Example 1 continued

A more meaningful result from velocity analysis is to solve and plot the velocity analysis unknowns for the entire range of mechanism motion. The left plot below gives ω_3 (red) and ω_4 (green) (rad/s) for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, for the open branch only. For all of Term Example 1, assume the ω_2 given above is constant. Since ω_2 is constant, we can plot the velocity results vs. θ_2 (since it is related to time t via $\theta_2 = \omega_2 t$). The right plot below gives the absolute translational coupler point C velocity for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, for the open branch only.





ω_3 and ω_4 **Coupler Point Velocity**

Term Example 1 F.R.O.M. Velocity Results

Four-Bar Mechanism Velocity Analysis, Alternate Step 5

Alternate **matrix-vector solution** (this must yield the same solution since the equations are linear). The four-bar mechanism velocity equations are re-written in matrix form.

For an overview of matrices, please see the on-line Matrices and Linear Algebra Review (people.ohio.edu/williams/html/PDF/MatricesLinearAlgebra.pdf). For an overview of vectors, matrices, and linear algebra in MATLAB, please see Dr. Bob's online MATLAB Primer (people.ohio.edu/williams/html/PDF/MATLABPrimer.pdf).

Four-bar mechanism singularity condition

When does the four-bar mechanism velocity solution fail? At a mechanism singularity, when the determinant of the coefficient matrix goes to zero, the velocity solution fails. This case would require division by zero, yielding infinite ω_3, ω_4. Let's see what this means physically.

$$\left|A\right| = ae - bd = (r_3 s_3)(r_4 c_4) - (-r_4 s_4)(-r_3 c_3) = r_3 r_4 s_3 c_4 - r_3 r_4 c_3 s_4 = -r_3 r_4 \sin(\theta_4 - \theta_3)$$

Using $\sin(a-b) = \sin a \cos b - \cos a \sin b$

$\left|A\right| = 0$ when $\sin(\theta_4 - \theta_3) = 0$, or $\theta_4 - \theta_3 = 0^\circ, 180^\circ, \cdots$

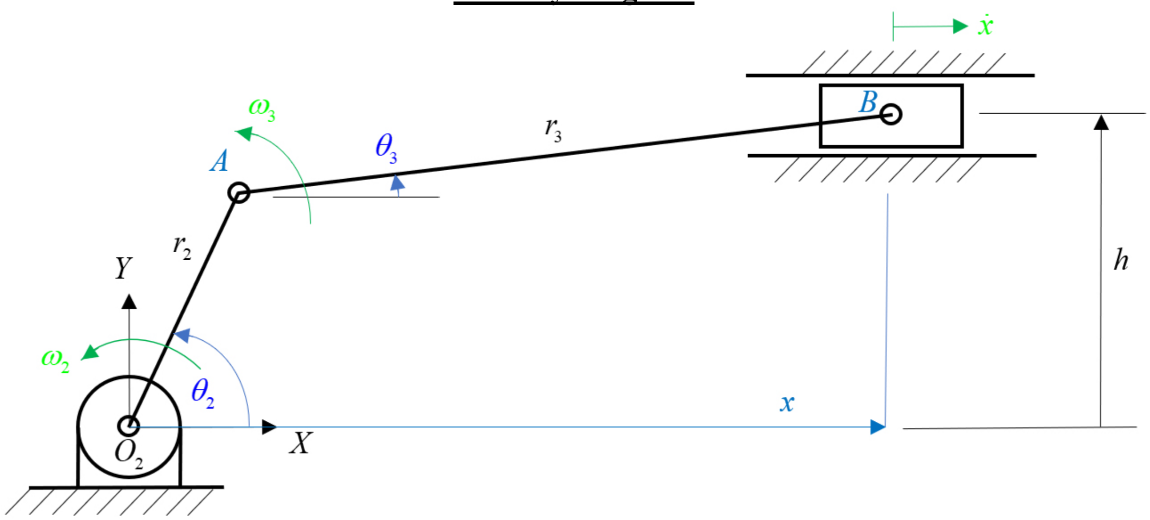
Physically, this happens when links 3 and 4 are straight out or folded on top of each other (what does this correspond to?). So we see that the four-bar mechanism velocity singularity condition is not a new problem, but it corresponds to the known problem of joint limits from position analysis.

3.4 Slider-Crank Mechanism Velocity Analysis

Again, we will solve the **air compressor** case where the crank is the input and the slider is the output. The **internal combustion engine** case (slider input/crank output) is also interesting.

Step 1. The slider-crank mechanism **Position Analysis** must first be complete.

Step 2. Draw the slider-crank mechanism **Velocity Diagram**.



where $\underline{\omega}_i$ (i = 2,3) is the absolute angular velocity of link i, $\dot{x}$ is the slider translational velocity, and $\underline{\omega}_4 = 0$ since the slider cannot rotate.

Step 3. State the Problem

Step 4. Derive the velocity equations.

Take the first time derivative of the vector loop closure equations from position analysis, in *XY* component form.

Slider-crank mechanism position equations

$$\underline{r}_2 + \underline{r}_3 = \underline{x} + \underline{h}$$

$$r_2c_2 + r_3c_3 = x$$
$$r_2s_2 + r_3s_3 = h$$

The first time derivative of the position equations yields the velocity equations.

Step 5. **Solve the velocity equations** for the unknowns $\omega_3, \dot{x}$.

These equations are decoupled so we don't need a matrix solution. First solve ω_3 from the *Y* velocity equation and then solve $\dot{x}$ from the *X* velocity equation using the ω_3 result.

Alternate matrix solution

Gathering unknowns on the LHS and writing the velocity equations in matrix-vector form:

$$\begin{bmatrix} r_3 s_3 & 1 \\ -r_3 c_3 & 0 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} -r_2 \omega_2 s_2 \\ r_2 \omega_2 c_2 \end{Bmatrix} \qquad \begin{bmatrix} a & 1 \\ d & 0 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} c \\ f \end{Bmatrix} \qquad \begin{matrix} a = r_3 s_3 & c = -r_2 \omega_2 s_2 \\ d = -r_3 c_3 & f = r_2 \omega_2 c_2 \end{matrix}$$

$$\begin{Bmatrix} \omega_3 \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} a & 1 \\ d & 0 \end{bmatrix}^{-1} \begin{Bmatrix} c \\ f \end{Bmatrix}$$

Where *a*, *c*, *d*, and *f* are identical terms from the four-bar mechanism velocity analysis. The matrix solution yields the same result.

$$\begin{Bmatrix} \omega_3 \\ \dot{x} \end{Bmatrix} = \frac{1}{-d} \begin{bmatrix} 0 & -1 \\ -d & a \end{bmatrix} \begin{Bmatrix} c \\ f \end{Bmatrix} = \begin{Bmatrix} \dfrac{f}{d} \\ \dfrac{-af + cd}{d} \end{Bmatrix}$$

$$\omega_3 = \frac{f}{d} = \frac{-r_2 \omega_2 c_2}{r_3 c_3}$$

$$\dot{x} = \frac{-af + cd}{d} = \frac{r_2 \omega_2 r_3 c_2 s_3 - r_2 \omega_2 r_3 s_2 c_3}{r_3 c_3} = \frac{r_2 \omega_2 c_2 s_3 - r_2 \omega_2 s_2 c_3}{c_3} = r_2 \omega_2 (c_2 t_3 - s_2)$$

where the determinant of the coefficient matrix is $|A| = a(0) - d(1) = -d = r_3 c_3$ and $t_3 = \tan\theta_3$.

This alternate matrix/vector solution yields identical results to the original algebra solution. ω_3 is immediately apparent, and $\dot{x}$ can be shown to be equivalent after substituting ω_3 into the algebra solution.

Slider-crank mechanism singularity condition

When does the slider-crank mechanism velocity solution fail? The slider-crank mechanism singularity occurs when the determinant of the coefficient matrix goes to zero. The result is dividing by zero, resulting in infinite $\omega_3, \dot{x}$.

$$|A| = r_3 c_3 = 0$$

$$|A| = 0 \quad \text{when} \quad \cos\theta_3 = 0\text{, or} \qquad \theta_3 = 90^\circ, 270^\circ, \cdots$$

Physically, this happens when link 3 is straight up or down ($\theta_3 = \pm 90^\circ$). This cannot happen for nominal full-rotation slider-crank mechanisms, even with offsets.

Coupler length r_3 cannot go to zero, otherwise we have a degenerate slider-crank mechanism which would be always singular.

Slider-crank mechanism singularity condition is related to the Full Rotation Condition presented earlier for the same mechanism. To avoid this condition by design, use $r_3 \geq |h| + r_2$.

Slider-crank mechanism velocity example – Term Example 2 continued

Given r_2 = 0.102, r_3 = 0.203, h = 0.076 m, and $\theta_2 = 30^\circ, \theta_3 = 7.2^\circ$, x = 0.290 m. This is the right branch of the slider-crank position example of Term Example 2.

Snapshot Analysis (one input angle)

Given this mechanism position analysis plus $\omega_2 = 15$ rad/s (+ so ccw), calculate $\omega_3, \dot{x}$ for this instant in time (snapshot).

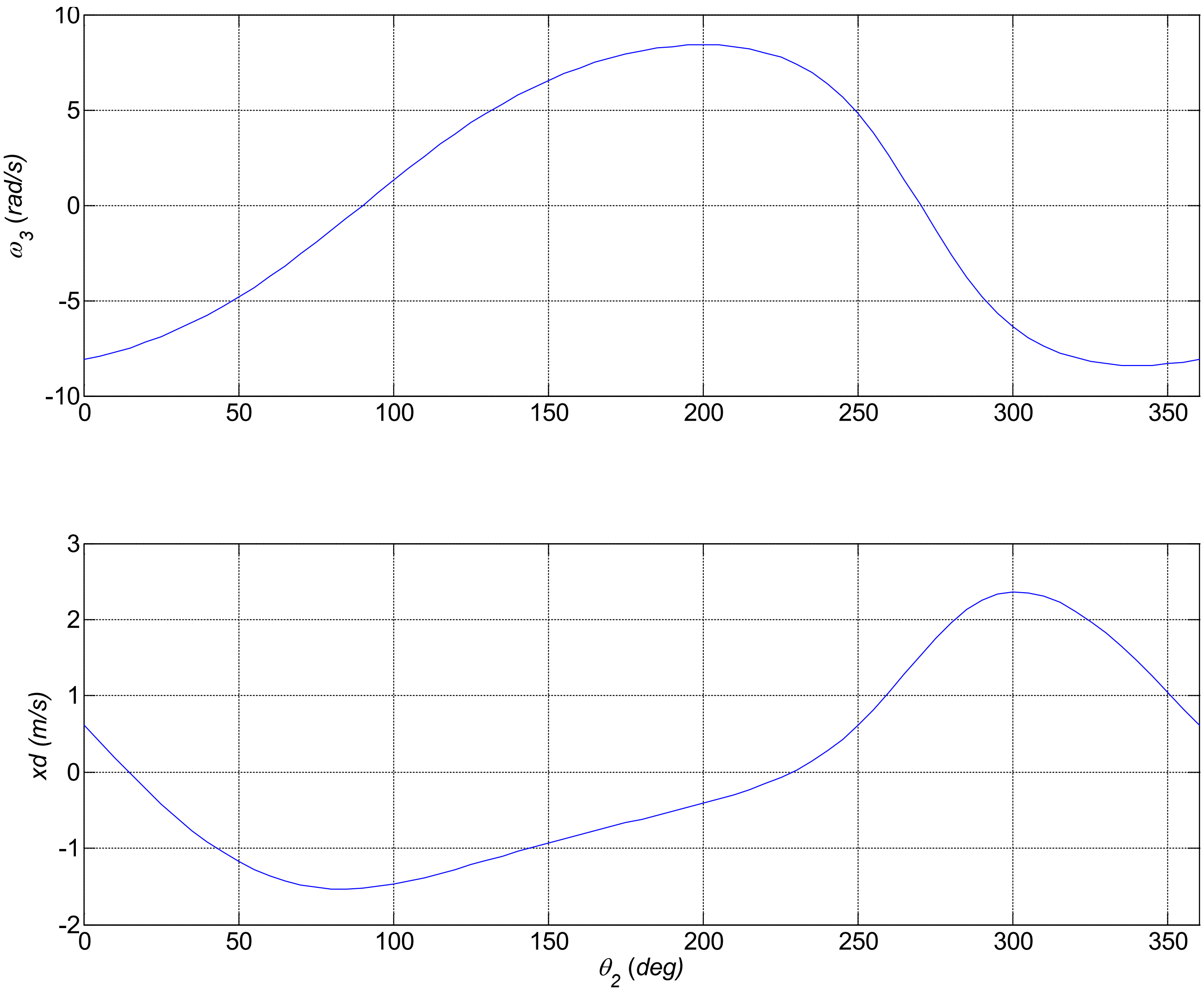
$$\begin{bmatrix} 0.025 & 1 \\ -0.202 & 0 \end{bmatrix} \begin{Bmatrix} \omega_3 \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} -0.762 \\ 1.320 \end{Bmatrix}$$

$$\begin{Bmatrix} \omega_3 \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} -6.577 \\ -0.601 \end{Bmatrix}$$

These results are the absolute rotational and translational velocities of links 3 and 4 with respect to the fixed ground link. Both are negative, so the coupler link 3 is currently rotating in the *clockwise* direction and the slider 4 is currently translating to the *left*.

Full-Range-Of-Motion (F.R.O.M.) Analysis – Term Example 2 continued

A more meaningful result from velocity analysis is to solve and plot the velocity analysis unknowns for the entire range of mechanism motion. The subplot arrangement below gives ω_3 (top, *rad*/*s*) and $\dot{x}$ (bottom, *m*/*s*), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 2, the right branch only. For all of Term Example 2, assume the ω_2 given above is constant. Since ω_2 is constant, we can plot the velocity results vs. θ_2 (since θ_2 changes linearly, as it is related to time *t* via $\theta_2 = \omega_2 t$).



Term Example 2 F.R.O.M. Velocity Results, ω_3 and $\dot{x}$

4. Acceleration Kinematics Analysis

4.1 Acceleration Kinematics Analysis Introduction

Acceleration analysis is the third step in general **kinematics** analysis. It relates the translational and angular accelerations of the links for a mechanism in motion. Acceleration is the first time derivative of the velocity and second time derivative of the position, and is also a vector quantity. Analytical acceleration analysis involves taking two time derivatives of the *XY* component equations from position analysis and solving for the unknowns. Position and velocity analyses must be complete first. Here are the general translational and rotational acceleration expressions and units:

Mechanism Acceleration Analysis

Mechanism acceleration analysis is the determination of all angular and linear accelerations of links in a mechanism in motion. It is required for complete motion analysis. It is also required for further analysis: position, velocity, ***acceleration***, dynamics, forces, and machine design. Linear equations result from the second time derivative of the position equations. There is a unique acceleration solution for each mechanism branch. For one-dof mechanisms, one acceleration input must be given.

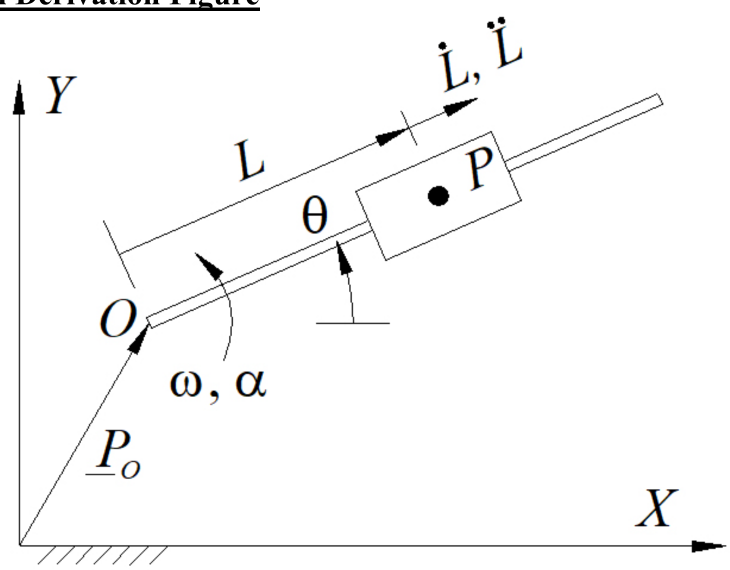
Generic Mechanism Acceleration Analysis Problem Statement

Given the mechanism, complete position and velocity analyses, and one-dof of acceleration input, calculate the acceleration unknowns.

4.2 Five-Part Acceleration Formula

In this section we will derive the five-part acceleration formula, showing the most general acceleration motions possible for planar devices.

Five-Part Acceleration Derivation Figure



This is a translating/rotating rigid rod with a slider, from the three-part velocity formula derivation. Find the total absolute translational acceleration of point P on the slider. Recall the two-part position and three-part velocity results:

$$\underline{P}_P = \underline{P}_O + \underline{L} = \begin{Bmatrix} P_{OX} + L\cos\theta \\ P_{OY} + L\sin\theta \end{Bmatrix} \qquad \underline{V}_P = \underline{V}_O + \underline{V} + \underline{\omega}\times\underline{L} = \begin{Bmatrix} V_{OX} + V\cos\theta - L\omega\sin\theta \\ V_{OY} + V\sin\theta + L\omega\cos\theta \end{Bmatrix}$$

The angular velocity is changing with time (as shown below). The sliding velocity is also changing with time.

Product and Chain Rules of Differentiation

Again, we'll need to use the product and chain rules of differentiation in acceleration analysis derivations.

Product rule

$$\frac{d}{dt}(x(t)y(t)) = \frac{dx(t)}{dt}y(t) + x(t)\frac{dy(t)}{dt}$$

x, y are both functions of time t.

Chain rule

$$\frac{d}{dt}(f(x(t))) = \frac{df(x(t))}{dx(t)}\frac{dx(t)}{dt}$$

f is a function of x, which is a function of t.

Example

$$\begin{aligned}
\frac{d^2}{dt^2}(L(t)\cos\theta(t)) &= \frac{d}{dt}(V(t)\cos\theta(t) - L(t)\omega(t)\sin\theta(t)) \\
&= A(t)\cos\theta(t) + V(t)\frac{d}{dt}(\cos\theta(t)) - V(t)\omega(t)\sin\theta(t) - L(t)\alpha(t)\sin\theta(t) - L(t)\omega(t)\frac{d}{dt}(\sin\theta(t)) \\
&= A(t)\cos\theta + V(t)\frac{d\cos\theta(t)}{d\theta(t)}\frac{d\theta(t)}{dt} - V(t)\omega(t)\sin\theta(t) - L(t)\alpha(t)\sin\theta(t) - L(t)\omega(t)\frac{d\sin\theta(t)}{d\theta(t)}\frac{d\theta(t)}{dt} \\
&= A(t)\cos\theta(t) + V(t)(-\sin\theta(t))\omega(t) - V(t)\omega(t)\sin\theta(t) - L\alpha\sin\theta(t) - L(t)\omega(t)(\cos\theta(t))\omega(t) \\
&= A(t)\cos\theta(t) - 2V(t)\omega(t)\sin\theta(t) - L(t)\alpha(t)\sin\theta(t) - L(t)\omega(t)^2\cos\theta(t)
\end{aligned}$$

where

$$V(t) = \dot{L}(t) = \frac{dL(t)}{dt} \qquad \omega(t) = \dot{\theta}(t) = \frac{d\theta(t)}{dt}$$

$$A(t) = \dot{V}(t) = \ddot{L}(t) = \frac{d^2L(t)}{dt^2} \qquad \alpha(t) = \dot{\omega}(t) = \ddot{\theta}(t) = \frac{d^2\theta(t)}{dt^2}$$

and all terms $L(t)$, $V(t)$, $A(t)$, $\theta(t)$, $\omega(t)$, and $\alpha(t)$ are functions of time.

Now we can return to the basic acceleration derivation.

First time derivative of the velocity vector (second time derivative of the position vector)

$$\underline{V}_P = \underline{V}_O + \underline{V} + \underline{\omega} \times \underline{L} = \begin{Bmatrix} V_{OX}(t) + V(t)\cos\theta(t) - L(t)\omega(t)\sin\theta(t) \\ V_{OY}(t) + V(t)\sin\theta(t) + L(t)\omega(t)\cos\theta(t) \end{Bmatrix}$$

$$\underline{A}_P = \frac{d\underline{V}_P}{dt} = \frac{d^2\underline{P}_P}{dt^2} =$$

We have just derived the **Five-Part Acceleration Equation**.

$$\underline{A}_P = \underline{A}_O + \underline{A} + 2\underline{\omega} \times \underline{V} + \underline{\alpha} \times \underline{L} + \underline{\omega} \times (\underline{\omega} \times \underline{L})$$

The terms for the **Five-Part Acceleration Equation** can be expressed in various ways as summarized in the table below.

vector	$\underline{A}_O$	$\underline{A}$	$2\underline{\omega} \times \underline{V}$	$\underline{\alpha} \times \underline{L}$	$\underline{\omega} \times (\underline{\omega} \times \underline{L})$
name	point *O* acceleration vector	sliding acceleration vector	Coriolis acceleration vector	tangential acceleration vector	centripetal acceleration vector
***XY* components**					
magnitude / direction					

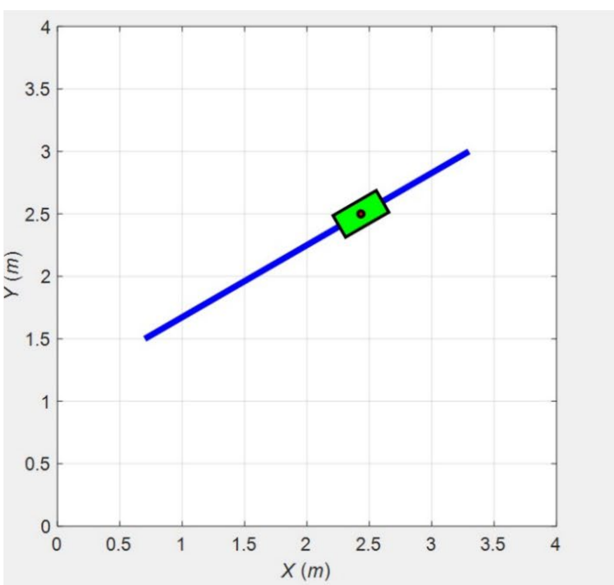
Five-Part Acceleration Snapshot Example

This is a continuation of the Three-Part Velocity Example. Given L=2 m, $\theta = 30^\circ$, ω = 1 rad/s, α = 2 rad/s^2, $|\underline{V}| = \dot{L} = 3$ m/s (outward), $\underline{V}_O = \{3 \quad 2\}^T$, $|\underline{A}| = \ddot{L} = 4$ m/s^2 (outward), $\underline{A}_O = \{1 \quad 2\}^T$, find $\underline{A}_P$.

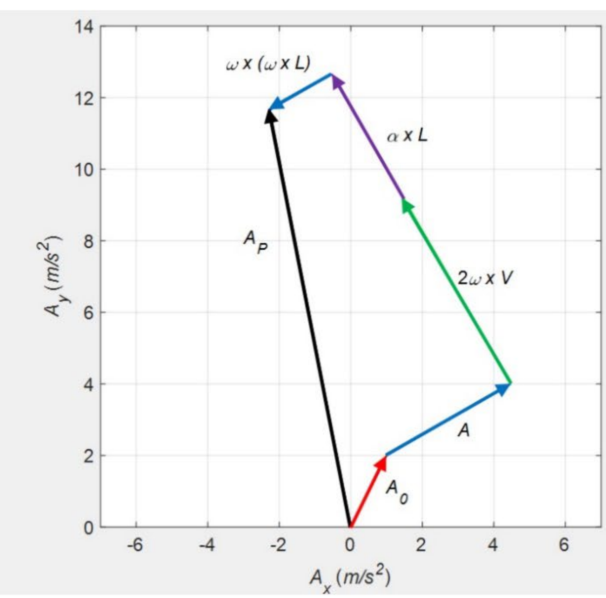
$$\underline{A}_P = \begin{Bmatrix} A_{OX} + A\cos\theta - 2V\omega\sin\theta - L\alpha\sin\theta - L\omega^2\cos\theta \\ A_{OY} + A\sin\theta + 2V\omega\cos\theta + L\alpha\cos\theta - L\omega^2\sin\theta \end{Bmatrix}$$

$$\underline{A}_P = \begin{Bmatrix} 1+3.5-3-2-1.7 \\ 2+2+5.2+3.5-1 \end{Bmatrix} = \begin{Bmatrix} -2.3 \\ 11.7 \end{Bmatrix} = 11.9 \quad @ \quad 101.0^\circ \quad \frac{m}{s^2}$$

vector	$\underline{A}_O$	$\underline{A}$	$2\underline{\omega}\times\underline{V}$	$\underline{\alpha}\times\underline{L}$	$\underline{\omega}\times(\underline{\omega}\times\underline{L})$	$\underline{A}_P$
name	point O acceleration vector	sliding acceleration vector	Coriolis acceleration vector	tangential acceleration vector	centripetal acceleration vector	total acceleration vector
***XY* components**	$\begin{Bmatrix}1\\2\end{Bmatrix}$	$\begin{Bmatrix}3.5\\2\end{Bmatrix}$	$\begin{Bmatrix}-3\\5.2\end{Bmatrix}$	$\begin{Bmatrix}-2\\3.5\end{Bmatrix}$	$\begin{Bmatrix}-1.7\\-1\end{Bmatrix}$	$\begin{Bmatrix}-2.3\\11.7\end{Bmatrix}$
magnitude / direction	2.2 $@63.4^\circ$	4 $@30^\circ$	6 $@120^\circ$	4 $@120^\circ$	2 $@210^\circ$	11.9 $@101^\circ$



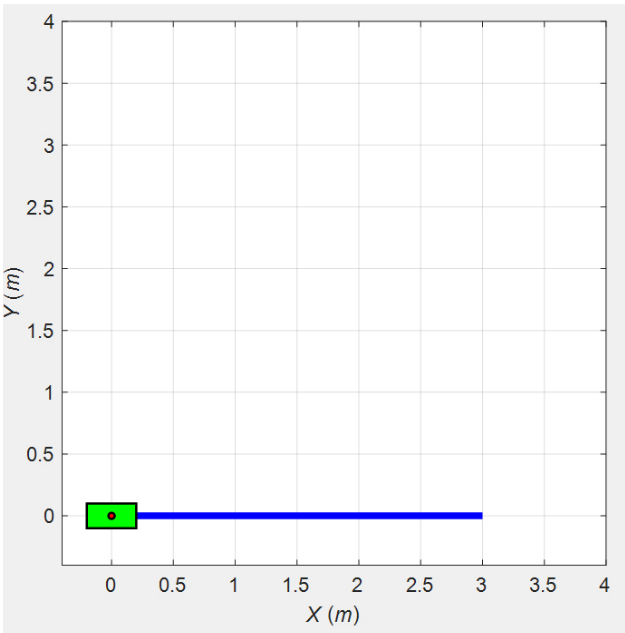
Kinematic Diagram with $\underline{P}_0 = \{0.7 \quad 1.5\}^T$



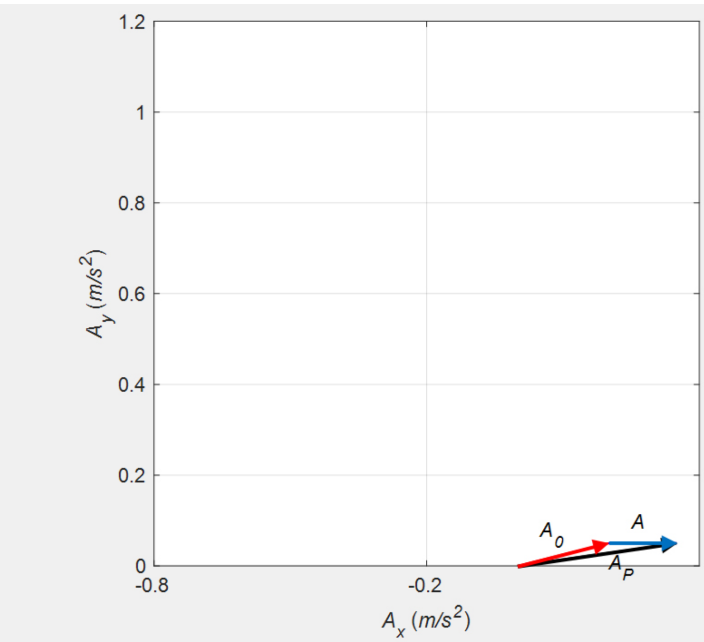
Five-Part Acceleration Diagram

Five-Part Acceleration Moving Example

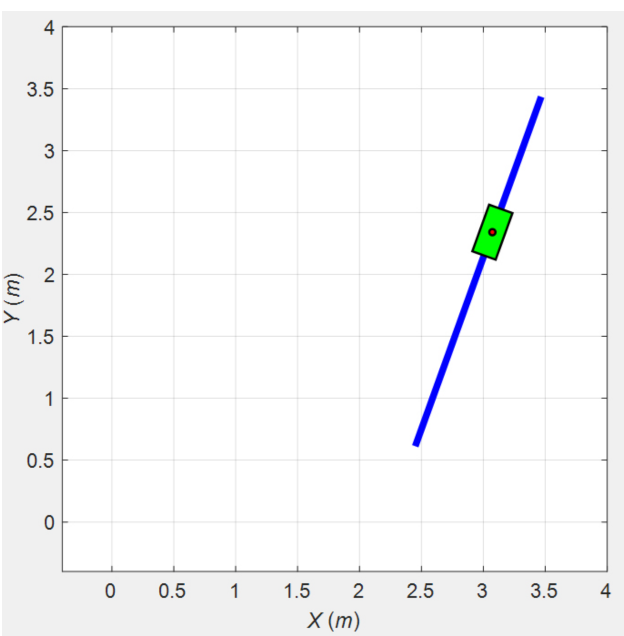
Given initial positions $\{P_{0x} \quad P_{0y} \quad L \quad \theta\} = \{0 \quad 0 \quad 0 \quad 0\}$ $(m, \; rad)$, initial velocities $\{V_{0x} \quad V_{0y} \quad V \quad \omega\} = \{0 \quad 0 \quad 0 \quad 0\}$ $(m/s, \; rad/s)$ and constant accelerations $\{A_{0x} \quad A_{0y} \quad A \quad \alpha\} = \{0.2 \quad 0.05 \quad 0.15 \quad 0.1\}$ $(m/s^2, rad/s^2)$, simulate this motion and determine $\underline{A}_P$ at each instant. $t_f = 5$ and $\Delta t = 0.1$ sec was used; the initial and final snapshots, with their five-part acceleration diagrams, are shown below.



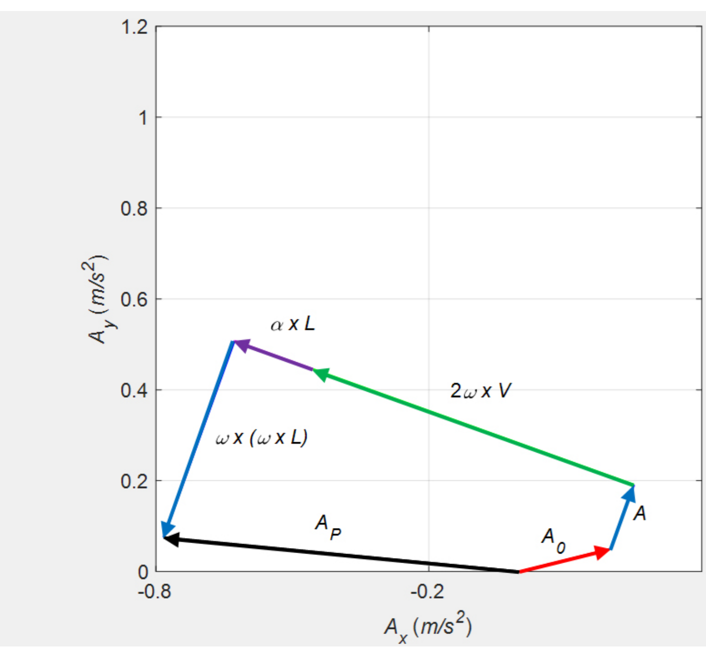
Initial Kinematic Diagram



Initial Five-Part Acceleration Diagram

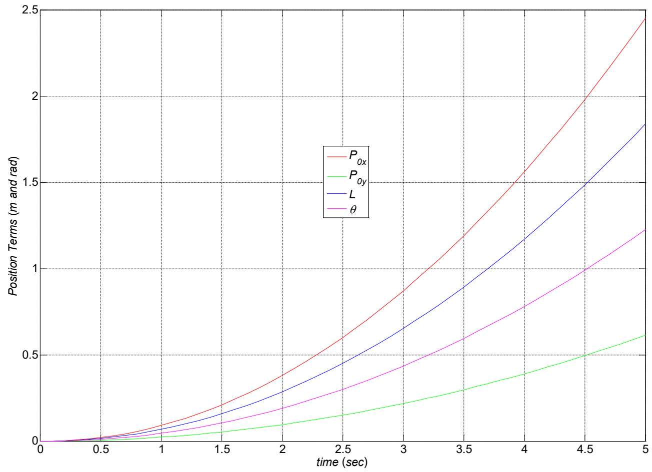


Final Kinematic Diagram

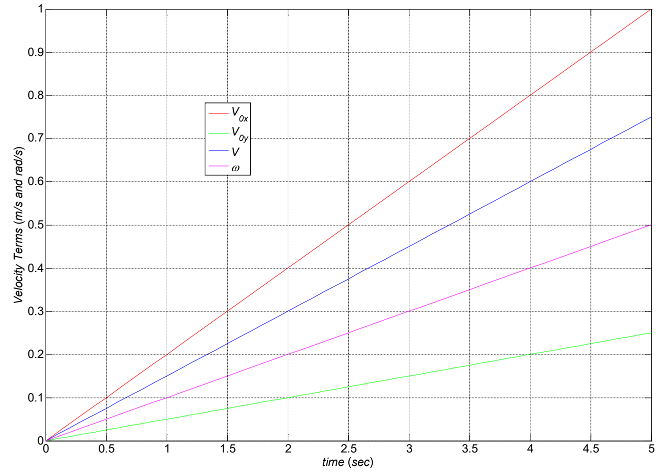


Final Five-Part Acceleration Diagram

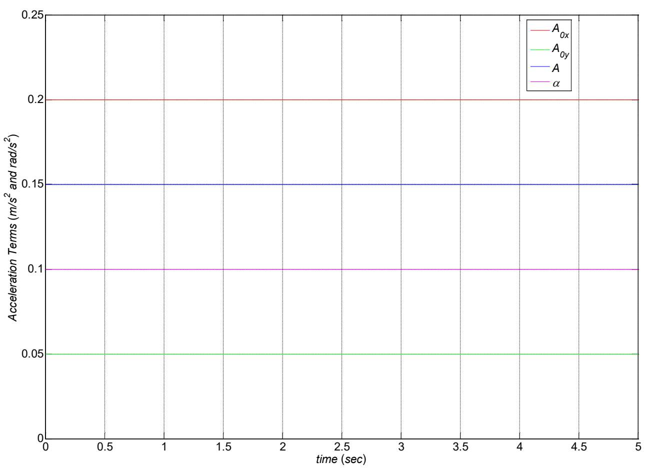
Five-Part Acceleration Moving Example Plots



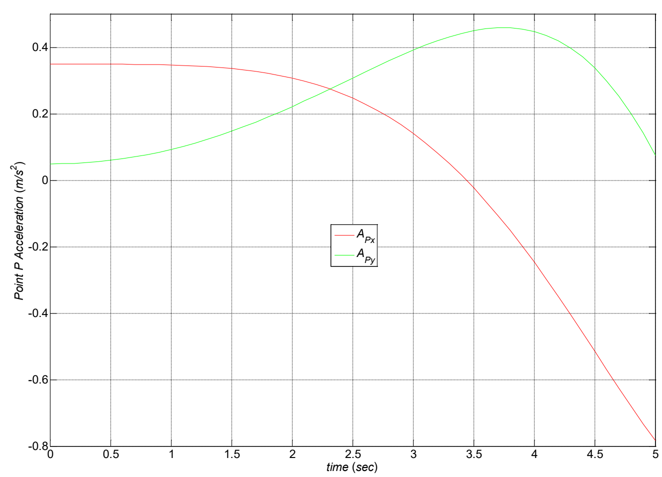
Position Terms



Velocity Terms



Acceleration Terms



Point *P* Translational Acceleration

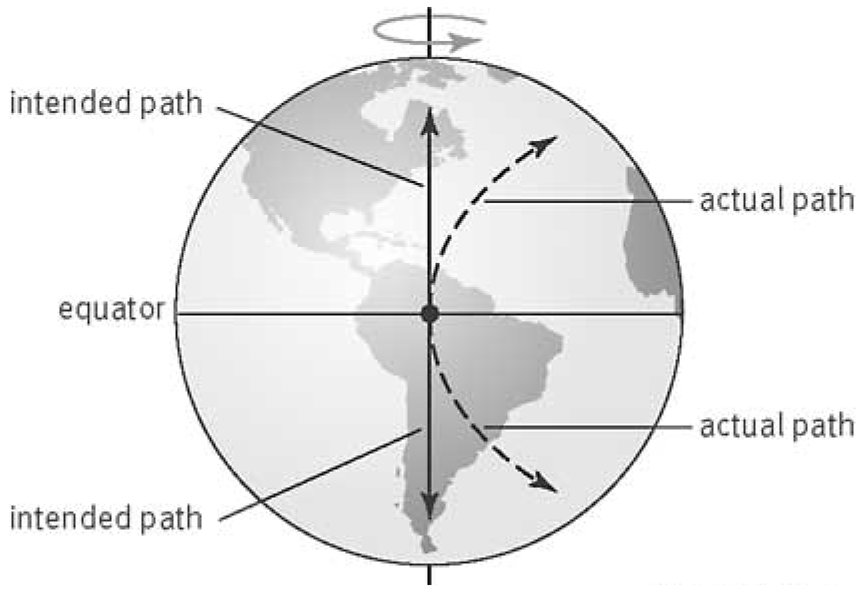
What is the relationship amongst the first three plots?

Constant acceleration terms $\{A_{0x} \quad A_{0y} \quad A \quad \alpha\} = \{0.2 \quad 0.05 \quad 0.15 \quad 0.1\}$ lead to non-constant point P acceleration (due to nonlinear position kinematics and centripetal acceleration).

Coriolis Acceleration History

Coriolis Effect definition (noun)

"The apparent deflection (**Coriolis Acceleration**) of a body in motion with respect to the earth, as seen by an observer on the earth, attributed to a fictitious force (**Coriolis Force**) but actually caused by the rotation of the earth and appearing as a deflection to the right in the Northern Hemisphere and a deflection to the left in the Southern Hemisphere."



dictionary.reference.com

"Italian scientists **Giovanni Battista Riccioli** and his assistant Francesco **Maria Grimaldi** described the **Coriolis Effect** in connection with **artillery** in the 1651 *Almagestum Novum*, writing that rotation of the Earth should cause a cannonball fired to the north to deflect to the east. The effect was described in the tidal equations of **Pierre-Simon Laplace** in 1778."

"Gaspard-Gustave Coriolis published a paper in 1835 on the energy yield of machines with rotating parts, such as **waterwheels**. That paper considered the supplementary forces that are detected in a rotating frame of reference. Coriolis divided these supplementary forces into two categories. The second category contained a force that arises from the **cross product of the angular velocity of a coordinate system and the projection of a particle's velocity** into a plane perpendicular to the system's axis of rotation. Coriolis referred to this force as the "compound centrifugal force" due to its analogies with the centrifugal force already considered in category one. The effect was known in the early 20th century as the "acceleration of Coriolis", and by 1920 as "Coriolis force"."

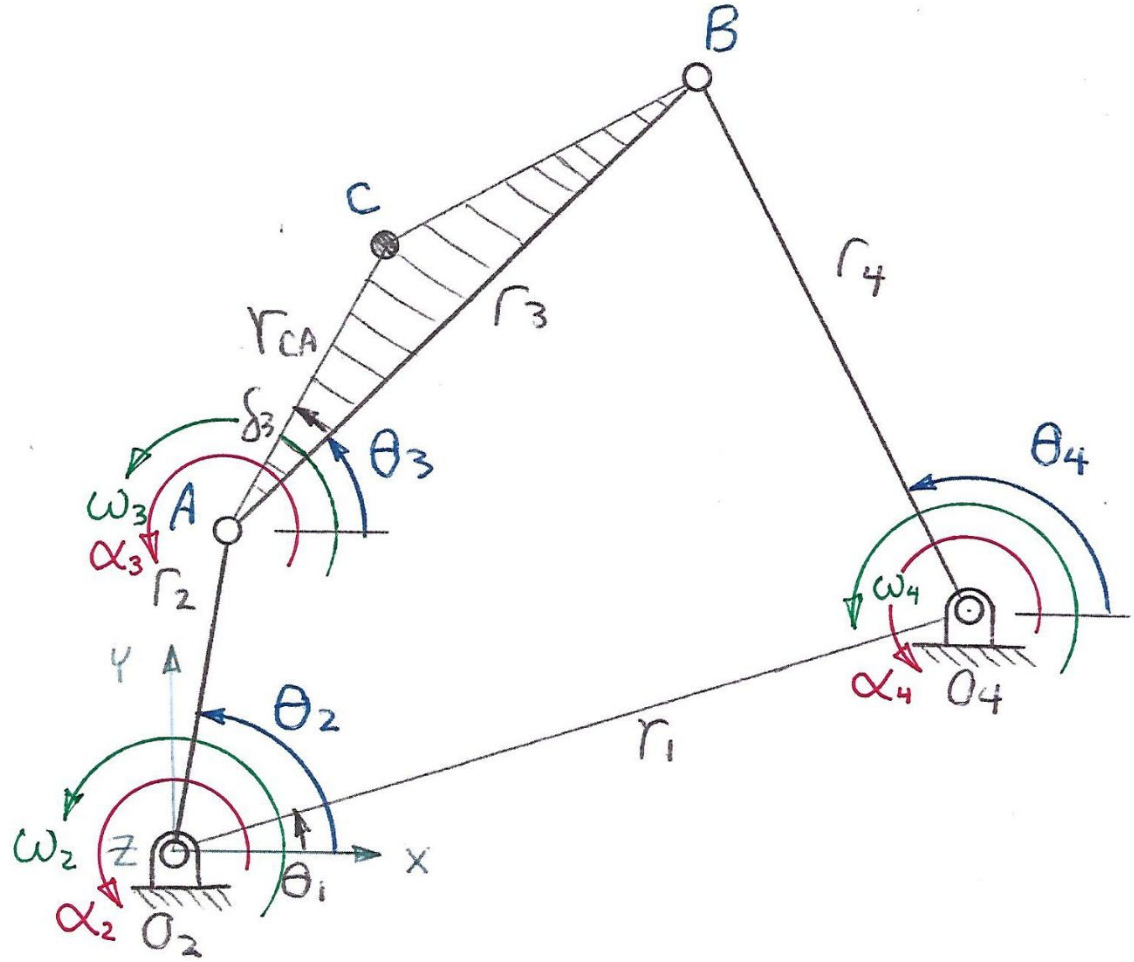
"In 1856, **William Ferrel** proposed the existence of a circulation cell in the mid-latitudes with air being deflected by the Coriolis force to create the prevailing westerly winds. Early in the 20th century, the term ***Coriolis Force*** began to be used in connection with **meteorology**."

en.wikipedia.org

4.3 Four-Bar Mechanism Acceleration Analysis

Step 1. The four-bar mechanism **Position and Velocity Analyses** must first be complete.

Step 2. Draw the four-bar mechanism **Acceleration Diagram**.



where $\underline{\alpha}_i$ ($i = 2,3,4$) is the absolute angular acceleration of link i. $\underline{\alpha}_1 = 0$ since the ground link is fixed.

Step 3. State the Problem

Step 4. **Derive the acceleration equations**. Take the first time derivative of the four-bar mechanism velocity equations from velocity analysis, in *XY* component form.

Four-bar mechanism velocity equations

$$-r_2\omega_2 s_2 - r_3\omega_3 s_3 = -r_4\omega_4 s_4$$
$$r_2\omega_2 c_2 + r_3\omega_3 c_3 = r_4\omega_4 c_4$$

The first time derivative of the velocity equations requires the product and chain rules. Note $\dot{r}_i = \ddot{r}_i = 0$ due to the rigid links.

$$\begin{aligned}\frac{d}{dt}(-r_i\omega_i \sin\theta_i) &= -r_i \frac{d}{dt}(\omega_i \sin\theta_i)\\ &= -r_i\left(\alpha_i \sin\theta_i + \omega_i \frac{d}{dt}(\sin\theta_i)\right)\\ &= -r_i\left(\alpha_i \sin\theta_i + \omega_i \frac{d\sin\theta_i}{d\theta_i}\frac{d\theta_i}{dt}\right)\\ &= -r_i(\alpha_i \sin\theta_i + \omega_i(\cos\theta_i)\omega_i)\\ &= -r_i\alpha_i \sin\theta_i - r_i\omega_i^2\cos\theta_i\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(r_i\omega_i \cos\theta_i) &= r_i \frac{d}{dt}(\omega_i \cos\theta_i)\\ &= r_i\left(\alpha_i \cos\theta_i + \omega_i \frac{d}{dt}(\cos\theta_i)\right)\\ &= r_i\left(\alpha_i \cos\theta_i + \omega_i \frac{d\cos\theta_i}{d\theta_i}\frac{d\theta_i}{dt}\right)\\ &= r_i\left(\alpha_i \cos\theta_i + \omega_i(-\sin\theta_i)\omega_i\right)\\ &= r_i\alpha_i \cos\theta_i - r_i\omega_i^2\sin\theta_i\end{aligned}$$

Where θ_i, ω_i, and α_i are all functions of time, but the r_i are constant.

The first time derivative of the velocity equations yields the acceleration equations.

$$-r_2\omega_2 s_2 - r_3\omega_3 s_3 = -r_4\omega_4 s_4$$
$$r_2\omega_2 c_2 + r_3\omega_3 c_3 = r_4\omega_4 c_4$$

Gathering unknowns on the LHS

Substituting simpler terms

$$
\begin{aligned}
a &= \ \ r_3 s_3 \\
b &= -r_4 s_4 \\
C &= -r_2\alpha_2 s_2 - r_2\omega_2^2 c_2 - r_3\omega_3^2 c_3 + r_4\omega_4^2 c_4 \\
d &= -r_3 c_3 \\
e &= \ \ r_4 c_4 \\
F &= \ \ r_2\alpha_2 c_2 - r_2\omega_2^2 s_2 - r_3\omega_3^2 s_3 + r_4\omega_4^2 s_4
\end{aligned}
$$

Written in matrix form

<u>Step 5.</u> **<u>Solve the acceleration equations</u>** for the unknowns α_3, α_4.

Algebra solution:

$$\alpha_3 = \frac{Ce - bF}{ae - bd} \qquad\qquad \alpha_4 = \frac{aF - Cd}{ae - bd}$$

Back substituting the terms a, b, C, d, e, F yields the following equivalent solutions, which simplify nicely using the sum-of-angles formulae $\sin(a-b) = \sin a \cos b - \cos a \sin b$ and $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ which better displays the structure of the solutions.

$$\alpha_3 = \frac{-r_2\alpha_2 \sin(\theta_4 - \theta_2) + r_2\omega_2^2 \cos(\theta_4 - \theta_2) + r_3\omega_3^2 \cos(\theta_4 - \theta_3) - r_4\omega_4^2}{r_3 \sin(\theta_4 - \theta_3)}$$

$$\alpha_4 = \frac{-r_2\alpha_2 \sin(\theta_3 - \theta_2) + r_2\omega_2^2 \cos(\theta_3 - \theta_2) + r_3\omega_3^2 - r_4\omega_4^2 \cos(\theta_4 - \theta_3)}{r_4 \sin(\theta_4 - \theta_3)}$$

The matrix solution must yield the same results since these are linear equations.

Four-Bar mechanism singularity condition

The acceleration problem has the same coefficient matrix *A* as the four-bar velocity problem, so the singularity condition is identical.

$$\theta_4 - \theta_3 = 0^\circ, 180^\circ, \cdots$$

This condition is the same problem for position, velocity, and acceleration. At this singularity, there is zero transmission angle μ and link 2 is at a joint limit. So we see that the four-bar mechanism acceleration singularity condition is not a new problem, but it corresponds to the known problem of joint limits from position analysis and the same singularity condition from velocity analysis.

Translational Acceleration of a Point on the Four-Bar Mechanism

The basic four-bar mechanism acceleration analysis problem is now solved. Now that we know the angular unknowns, we can find the **translational acceleration of any point** on the mechanism, e.g. coupler point *C*. From earlier, the velocity vector of coupler point *C* is repeated here.

$$\underline{V}_C = \begin{Bmatrix} V_{CX} \\ V_{CY} \end{Bmatrix} = \begin{Bmatrix} -r_2\omega_2 s_2 - r_{CA}\omega_3 s\beta \\ r_2\omega_2 c_2 + r_{CA}\omega_3 c\beta \end{Bmatrix} \qquad \beta = \theta_3 + \delta_3$$

Four-bar mechanism acceleration example – Term Example 1 continued

Given r_1 = 0.284, r_2 = 0.076, r_3 = 0.203, r_4 = 0.178, r_{CA} = 0.127 m, and $\theta_1 = 10.3^\circ, \theta_2 = 30^\circ$, $\theta_3 = 53.8^\circ, \theta_4 = 121.7^\circ, \delta_3 = 36.9^\circ$; $\omega_2 = 20$ (rad/s, constant), $\omega_3 = -8.073, \omega_4 = -3.729$ rad/s. This is the open branch of the position and velocity example (Term Example 1).

Snapshot Analysis

Given this mechanism position and velocity analysis, plus $\alpha_2 = 0$ rad/s^2, calculate α_3, α_4 for this instant in motion (snapshot).

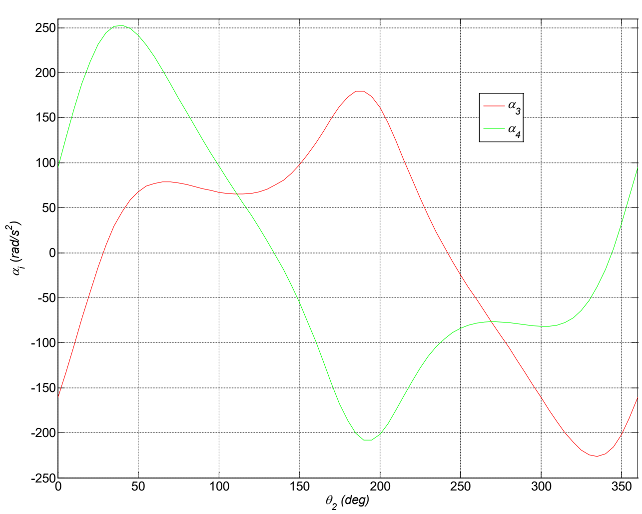
$$\begin{bmatrix} 0.164 & -0.151 \\ -0.120 & -0.093 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{Bmatrix} -35.433 \\ -23.785 \end{Bmatrix} \qquad \begin{Bmatrix} \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{Bmatrix} 7.994 \\ 243.018 \end{Bmatrix}$$

Both are positive, so they are *ccw* in direction. These results are the absolute angular accelerations of links 3 and 4 with respect to the ground link. The coupler point translational acceleration vector is:

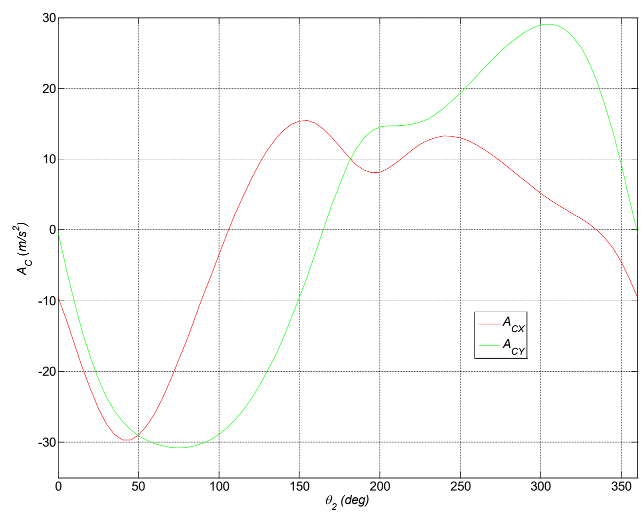
$$\underline{A}_C = \begin{Bmatrix} -27.230 \\ -23.490 \end{Bmatrix} m/s^2$$

Full-Range-Of-Motion (F.R.O.M.) Analysis – Term Example 1 continued

A more meaningful result from acceleration analysis is to solve and plot the acceleration analysis unknowns for the entire range of mechanism motion. The left plot below gives α_3 (red) and α_4 (green), (rad/s^2), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, open branch. In the Term Example 1 velocity section it was assumed that the given ω_2 is constant, which means that the given α_2 is always zero. Since ω_2 is constant, we can plot the acceleration results vs. θ_2 (since it is related to time t via $\theta_2 = \omega_2 t$). The right plot below gives the translational coupler point acceleration for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 1, the open branch only.



α_3 **and** α_4



Coupler Point Acceleration

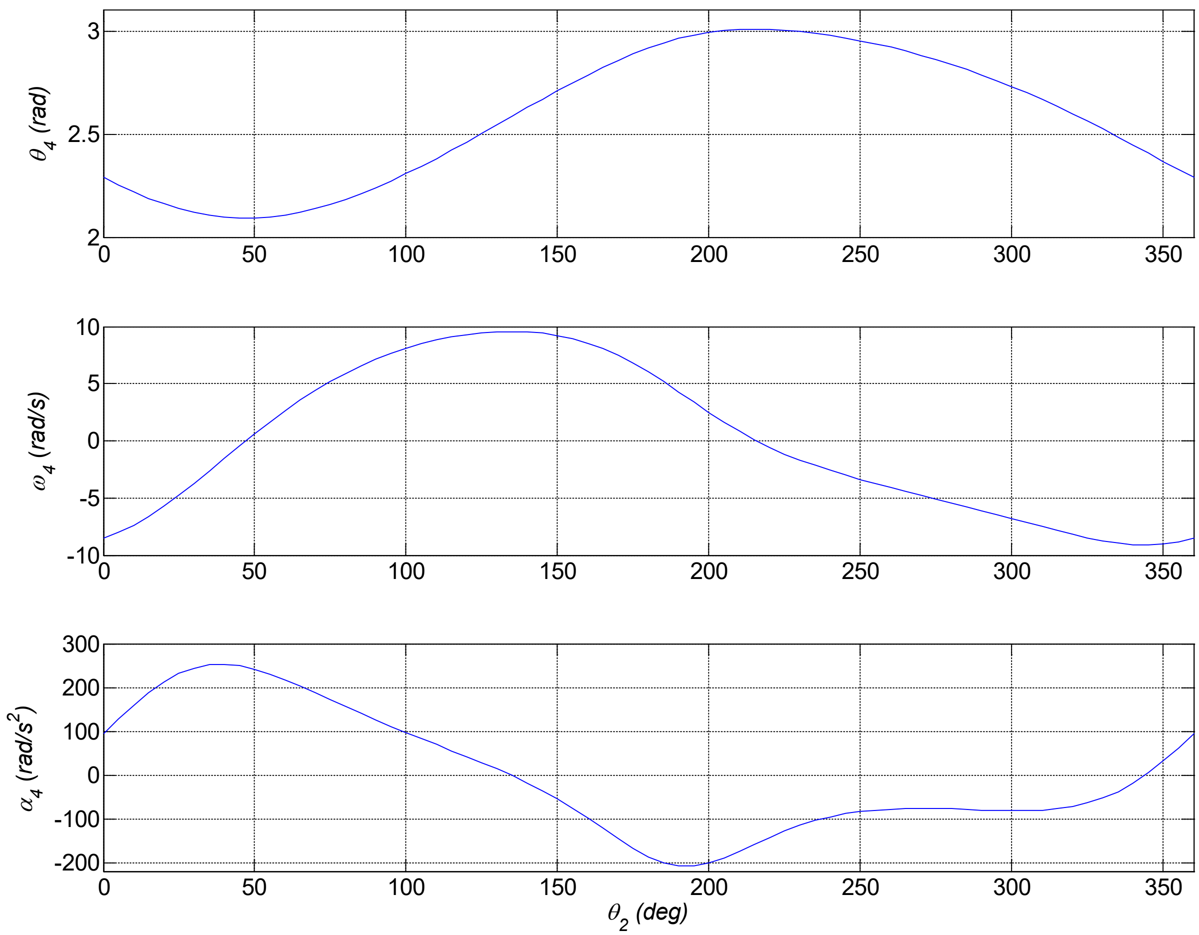
Term Example 1 F.R.O.M. Acceleration Results

Derivative/Integral Relationships

When one variable is the derivative of another, recall the relationships from calculus, e.g.:

$$\omega_4(t) = \frac{d\theta_4(t)}{dt} \qquad \theta_4(t) = \theta_{40} + \int \omega_4(t)dt$$

$$\alpha_4(t) = \frac{d\omega_4(t)}{dt} \qquad \omega_4(t) = \omega_{40} + \int \alpha_4(t)dt$$



The value of ω_4 at any point is the slope of the θ_4 curve at that point. The value of θ_4 at any point is the integral of the ω_4 curve up to that point (the value of θ_4 at any point is the area under the ω_4 curve up to that point plus the initial value θ_{40}). A similar relationship exists for α_4 and ω_4.

These graphs are plotted vs. θ_2, but the same type of relationships hold when plotted vs. time t since ω_2 is constant. This is the Term Example 1 F.R.O.M. result, but θ_4 was changed from *deg* to *rad* for better comparison. These curves should be plotted vs. time t instead of θ_2 in order to see the true slope and area values accurately.

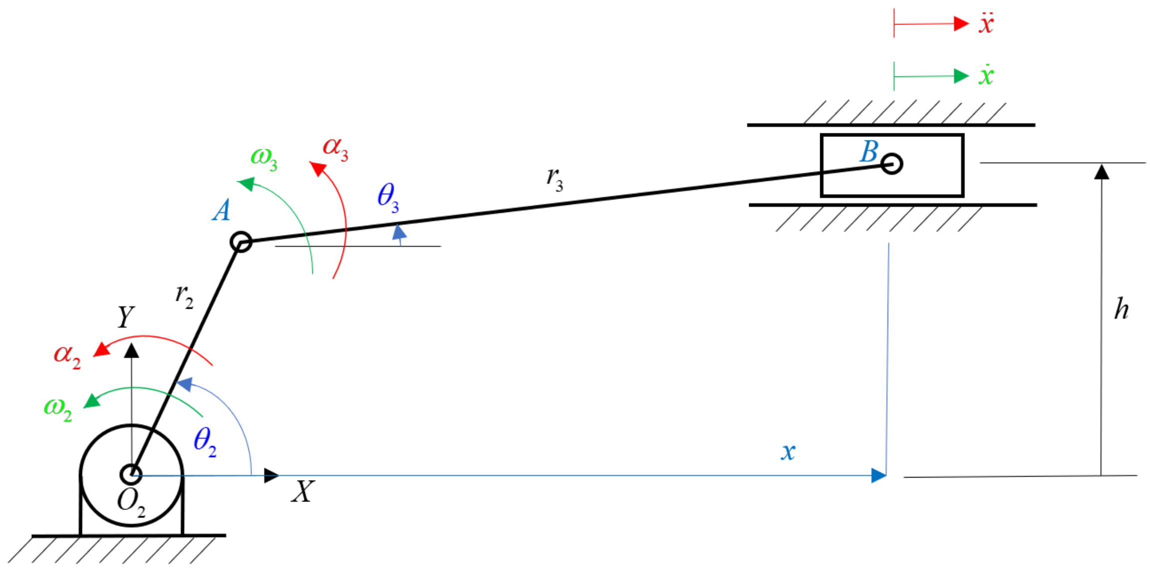
Look for: zero derivative value when the function is flat; max or min derivative value when the function is at an inflection point, for both $\omega_4(t) \leftrightarrow \theta_4(t)$ and $\alpha_4(t) \leftrightarrow \omega_4(t)$.

4.4 Slider-Crank Mechanism Acceleration Analysis

Again, we will solve the *air compressor* case with the input crank 2 and the output slider 4.

Step 1. The slider-crank mechanism **Position and Velocity Analyses** must first be complete.

Step 2. Draw the slider-crank mechanism **Acceleration Diagram**.



where $\underline{\alpha}_i$ ($i = 2,3$) is the absolute angular acceleration of link i and $\underline{\alpha}_4 = 0$ since the slider cannot rotate. The slider translational acceleration is $\ddot{x}$.

Step 3. State the Problem

Step 4. Derive the acceleration equations. Take the first time derivative of the velocity equations from slider-crank mechanism velocity analysis, in *XY* component form. Here are the slider-crank mechanism velocity equations:

$$-r_2\omega_2 s_2 - r_3\omega_3 s_3 = \dot{x}$$
$$r_2\omega_2 c_2 + r_3\omega_3 c_3 = 0$$

The first time derivative of the velocity equations yields the acceleration equations.

Gathering unknowns on the LHS and writing the slider-crank acceleration equations in matrix form.

$$\begin{bmatrix} r_3 s_3 & 1 \\ -r_3 c_3 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \ddot{x} \end{Bmatrix} = \begin{Bmatrix} -r_2\alpha_2 s_2 - r_2\omega_2^2 c_2 - r_3\omega_3^2 c_3 \\ r_2\alpha_2 c_2 - r_2\omega_2^2 s_2 - r_3\omega_3^2 s_3 \end{Bmatrix} \qquad \begin{bmatrix} a & 1 \\ d & 0 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \ddot{x} \end{Bmatrix} = \begin{Bmatrix} g \\ h \end{Bmatrix}$$

$$\begin{aligned} a &= r_3 s_3 & g &= -r_2\alpha_2 s_2 - r_2\omega_2^2 c_2 - r_3\omega_3^2 c_3 \\ d &= -r_3 c_3 & h &= r_2\alpha_2 c_2 - r_2\omega_2^2 s_2 - r_3\omega_3^2 s_3 \end{aligned}$$

Where *a* and *d* are identical terms from the four-bar mechanism velocity/acceleration analysis (and slider-crank velocity), but *g* and *h* are different than *C* and *F* (without the link 4 centripetal acceleration terms).

<u>Step 5.</u> **<u>Solve the acceleration equations</u>** for the unknowns $\alpha_3, \ddot{x}$.

These equations are decoupled so we don't need a matrix solution. First, solve α_3 from the *Y* equation:

Then solve $\ddot{x}$ from the *X* equation using the α_3 result.

The matrix solution yields the same result.

$$\begin{Bmatrix} \alpha_3 \\ \ddot{x} \end{Bmatrix} = \left[A^{-1} \right] \begin{Bmatrix} g \\ h \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_3 \\ \ddot{x} \end{Bmatrix} = \frac{1}{-d} \begin{bmatrix} 0 & -1 \\ -d & a \end{bmatrix} \begin{Bmatrix} g \\ h \end{Bmatrix} = \begin{Bmatrix} h/d \\ g - (ah/d) \end{Bmatrix}$$

$$= \begin{Bmatrix} (-r_2\alpha_2 c_2 + r_2\omega_2^2 s_2 + r_3\omega_3^2 s_3)/r_3 c_3 \\ -r_2\alpha_2 s_2 - r_2\omega_2^2 c_2 - r_3\omega_3^2 c_3 + \tan\theta_3 (r_2\alpha_2 c_2 - r_2\omega_2^2 s_2 - r_3\omega_3^2 s_3) \end{Bmatrix}$$

where the determinant of the coefficient matrix is $|A| = a(0) - d(1) = -d = r_3 c_3$.

<u>Slider-crank mechanism singularity condition</u>

The acceleration problem has the same coefficient matrix [*A*] as the velocity problem, so the singularity condition is identical (see the singularity discussion in the slider-crank velocity section – the only singularity is when link 3 is straight up or down, $\theta_3 = \pm 90^\circ$, which never happens for full-rotation slider-crank mechanisms). The related full-rotation was discussed for the slider-crank mechanism in the position analysis subsection.

Slider-crank mechanism acceleration example – Term Example 2 continued

Given r_2 = 0.102, r_3 = 0.203, h = 0.076 *m*, and $\theta_2 = 30^\circ, \theta_3 = 7.2^\circ$, x = 0.290 *m*; and $\omega_2 = 15$, $\omega_3 = -6.55$ *rad/s*, $\dot{x} = -0.60$ *m/s*. This is the right branch of the position and velocity example for the slider-crank mechanism of Term Example 2.

Snapshot Analysis (one input angle)

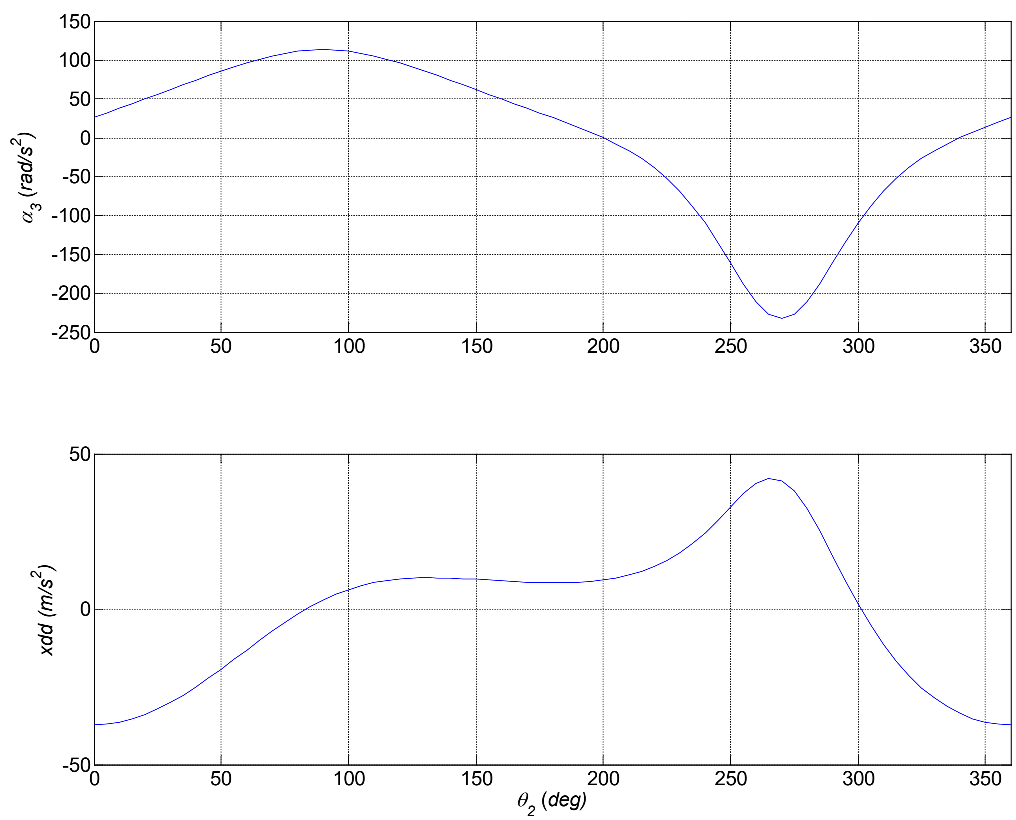
Given this mechanism position and velocity analysis plus $\alpha_2 = 0$ *rad/s*^{2}, calculate $\alpha_3, \ddot{x}$ for this snapshot in time.

$$\begin{bmatrix} 0.025 & 1 \\ -0.202 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_3 \\ \ddot{x} \end{Bmatrix} = \begin{Bmatrix} -28.590 \\ -12.557 \end{Bmatrix} \qquad \begin{Bmatrix} \alpha_3 \\ \ddot{x} \end{Bmatrix} = \begin{Bmatrix} 62.329 \\ -30.148 \end{Bmatrix}$$

These results are the absolute angular and linear accelerations of links 3 and 4 with respect to the fixed ground link.

Full-Range-Of-Motion (F.R.O.M.) Analysis – Term Example 2 continued

A more meaningful result from acceleration analysis is to solve and plot the acceleration analysis unknowns for the entire range of slider-crank mechanism motion. The top plot gives α_3 (*rad/s*^{2}) and the bottom plot gives $\ddot{x}$ (*m/s*^{2}), for all $0^\circ \le \theta_2 \le 360^\circ$, for Term Example 2, for the right branch only. In the Term Example 2 velocity section it was assumed that the given ω_2 is constant, which means that the given α_2 is always zero. Since ω_2 is constant, we can plot the velocity results vs. θ_2 (since it is related to time t via $\theta_2 = \omega_2 t$).



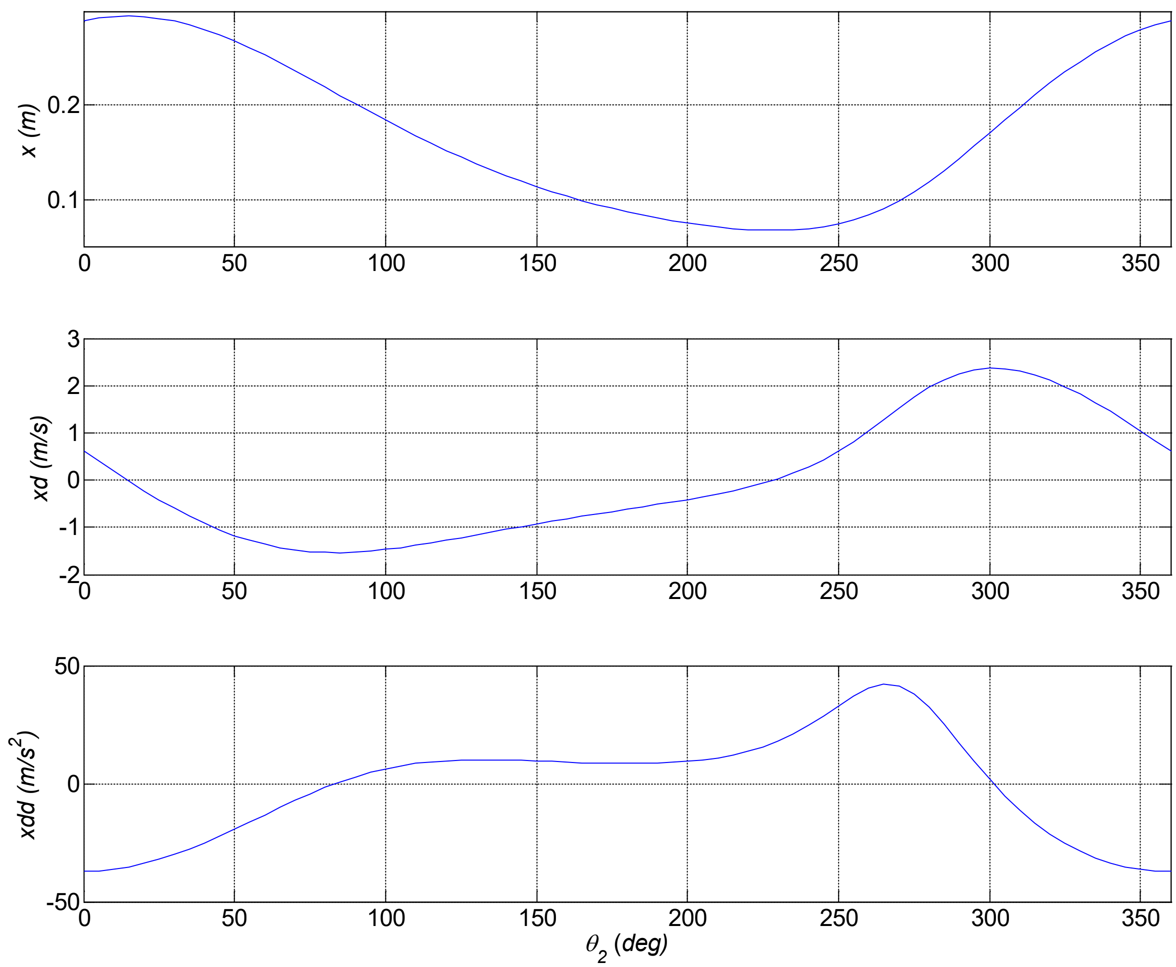
Term Example 2 F.R.O.M. Acceleration Results, α_3 and $\ddot{x}$

Derivative/Integral Relationships

When one variable is the derivative of another, recall the relationships from calculus, e.g.:

$$\dot{x}(t) = \frac{dx(t)}{dt} \qquad x(t) = x_0 + \int \dot{x}(t)dt$$

$$\ddot{x}(t) = \frac{d\dot{x}(t)}{dt} \qquad \dot{x}(t) = \dot{x}_0 + \int \ddot{x}(t)dt$$



Term Example 2 F.R.O.M. Slider Results, x, $\dot{x}$, $\ddot{x}$

The value of $\dot{x}$ at any point is the slope of the x curve at that point. The value of x at any point is the integral of the $\dot{x}$ curve up to that point (the value of x at any point is the area under the $\dot{x}$ curve up to that point plus the initial value x_0). A similar relationship exists for $\ddot{x}$ and $\dot{x}$.

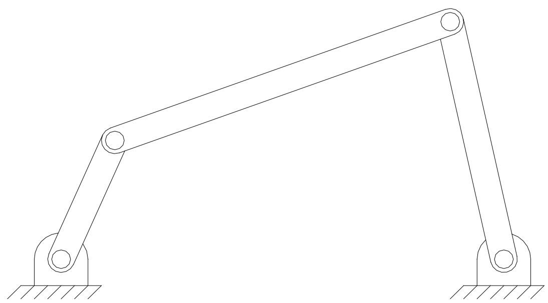
These graphs are plotted vs. θ_2, but the same type of relationships hold when plotted vs. time t since ω_2 is constant. This is the Term Example 2 F.R.O.M. result. Note these curves should be plotted vs. time t instead of θ_2 in order to see the true slope and area values accurately.

Look for: zero derivative value when the function is flat; max or min derivative value when the function is at an inflection point, for both $\dot{x}(t) \leftrightarrow x(t)$ and $\ddot{x}(t) \leftrightarrow \dot{x}(t)$.

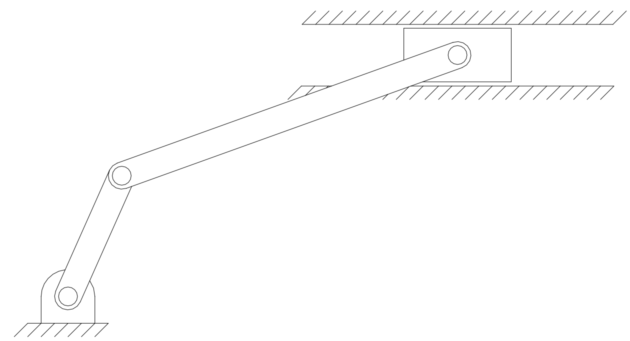
5. Other Kinematics Topics

5.1 Link Extensions Graphics

Using the methods presented thus far, we can use MATLAB to animate mechanisms for the entire range of motion. However, these methods have focused on basic mechanism models. What if your term project requires animation of mechanisms with link extensions from the existing rigid links?

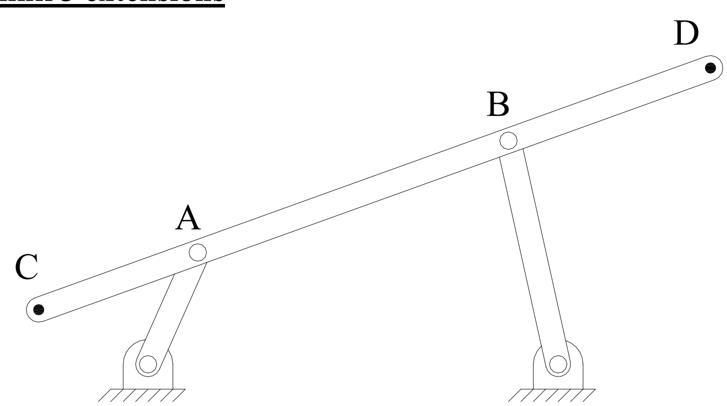


Four-Bar Mechanism



Offset Slider-Crank Mechanism

Four-bar mechanism link 3 extensions



Here are the kinematics equations (we previously presented the point C kinematics equations).

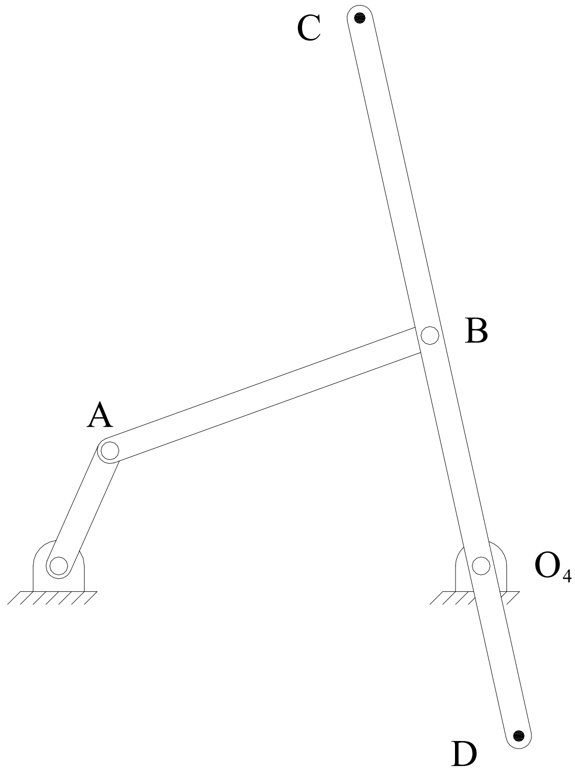
$$\underline{\mathbf{C}} = \begin{Bmatrix} c_x \\ c_y \end{Bmatrix} = \begin{Bmatrix} a_x + r_{CA}\cos(\theta_3 + \delta_{3C}) \\ a_y + r_{CA}\sin(\theta_3 + \delta_{3C}) \end{Bmatrix} \qquad \underline{\mathbf{D}} = \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} = \begin{Bmatrix} b_x + r_{DB}\cos(\theta_3 + \delta_{3D}) \\ b_y + r_{DB}\sin(\theta_3 + \delta_{3D}) \end{Bmatrix}$$

where $\underline{\mathbf{A}} = \begin{Bmatrix} a_x \\ a_y \end{Bmatrix} = \begin{Bmatrix} r_2\cos\theta_2 \\ r_2\sin\theta_2 \end{Bmatrix} \qquad \underline{\mathbf{B}} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \begin{Bmatrix} r_1\cos\theta_1 + r_4\cos\theta_4 \\ r_1\sin\theta_1 + r_4\sin\theta_4 \end{Bmatrix}$

In this simple straight-line case, use $\delta_{3C} = 180^\circ$ and $\delta_{3D} = 0^\circ$. Here is partial MATLAB code for link 3 extensions animation.

```
x2 = [0       ax(i)];    % coordinates of link 2
y2 = [0       ay(i)];
x3 = [cx(i)   dx(i)];    % coordinates of link 3
y3 = [cy(i)   dy(i)];
x4 = [r1x     bx(i)];    % coordinates of link 4
y4 = [r1y     by(i)];
figure;     plot(x2,y2,'r',x3,y3,'g',x4,y4,'b');
```

Four-bar mechanism link 4 extensions



The kinematics equations are given below.

$$\underline{\mathbf{C}} = \begin{Bmatrix} c_x \\ c_y \end{Bmatrix} = \begin{Bmatrix} b_x + r_{CB}\cos(\theta_4 + \delta_{4C}) \\ b_y + r_{CB}\sin(\theta_4 + \delta_{4C}) \end{Bmatrix} \qquad \underline{\mathbf{D}} = \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} = \begin{Bmatrix} O_{4x} + r_{DO_4}\cos(\theta_4 + \delta_{4D}) \\ O_{4y} + r_{DO_4}\sin(\theta_4 + \delta_{4D}) \end{Bmatrix}$$

In this simple straight-line case, use $\delta_{4C} = 0^\circ$ and $\delta_{4D} = 180^\circ$. b_x and b_y were given above and $\underline{\mathbf{O}_4}$ is:

$$\underline{\mathbf{O}_4} = \begin{Bmatrix} r_1\cos\theta_1 \\ r_1\sin\theta_1 \end{Bmatrix}$$

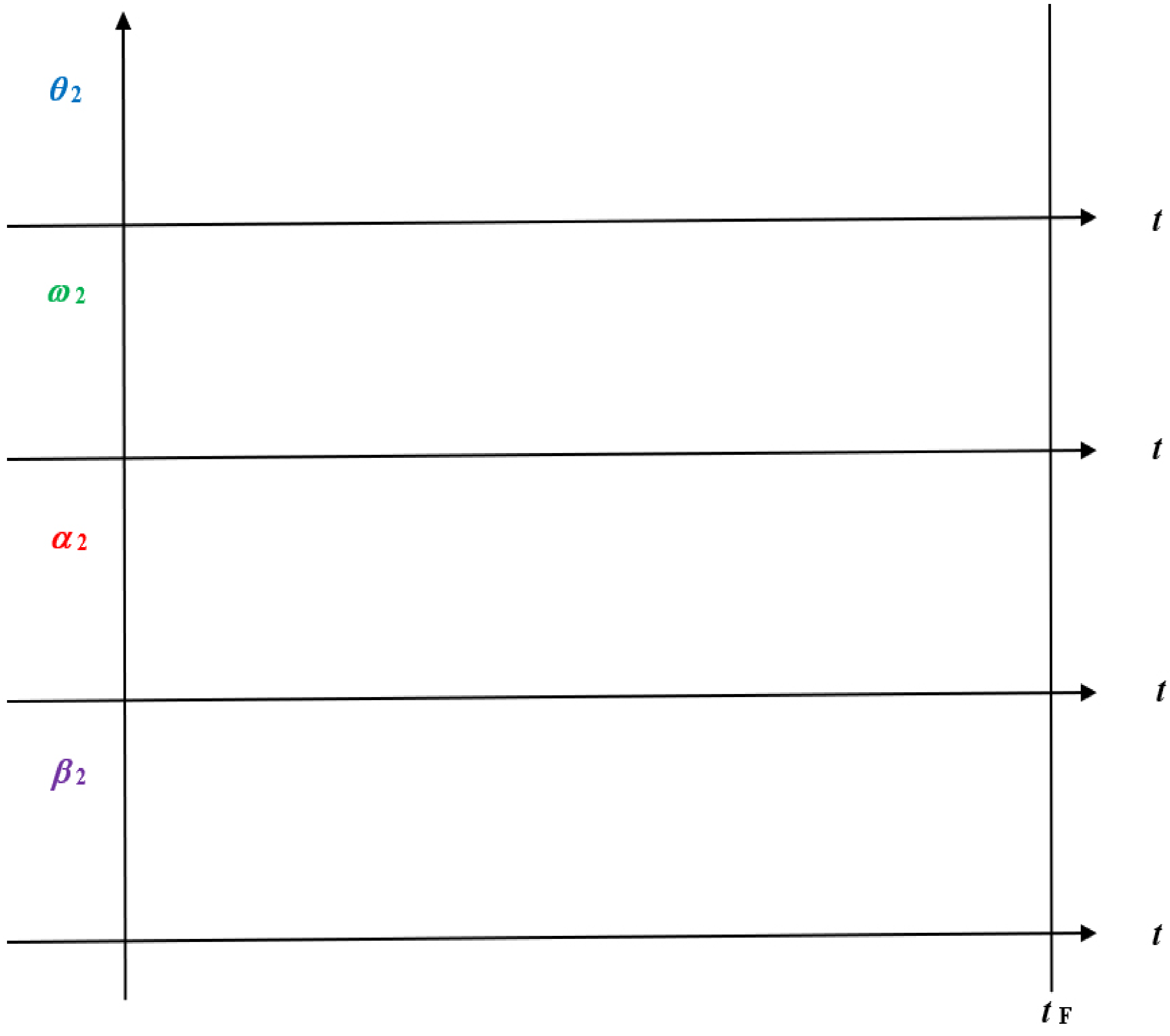
Here is partial MATLAB code for link 4 extensions animation.

```
x2 = [0      ax(i)];    % coordinates of link 2
y2 = [0      ay(i)];
x3 = [ax(i)  bx(i)];    % coordinates of link 3
y3 = [ay(i)  by(i)];
x4 = [cx(i)  dx(i)];    % coordinates of link 4
y4 = [cy(i)  dy(i)];
figure;
plot(x2,y2,'r',x3,y3,'g',x4,y4,'b');
```

Of course, one can use combinations of these graphics approaches as necessary. Also, use **`patch`** for drawing solid polygonal links rather than straight lines. You can also use **`patch`** for drawing solid circles. This method is also extendable to non-straight-line links by using the appropriate δ angles.

5.2 Input Motion Specification

Up to this point, for the full range of motion (F.R.O.M.) we have assumed that the input link rotates fully with a given **constant input angular velocity**. Further we assume that the mechanism is already in motion at a constant input velocity, i.e. not starting from rest. Also, the velocity at the end of the F.R.O.M. is still constant, i.e. not ramping down to 0. Our input motion specification has thus been $0^\circ \le \theta_2 \le 360^\circ$, ω_2 constant, and $\alpha_2 = 0$. This situation is plotted below.

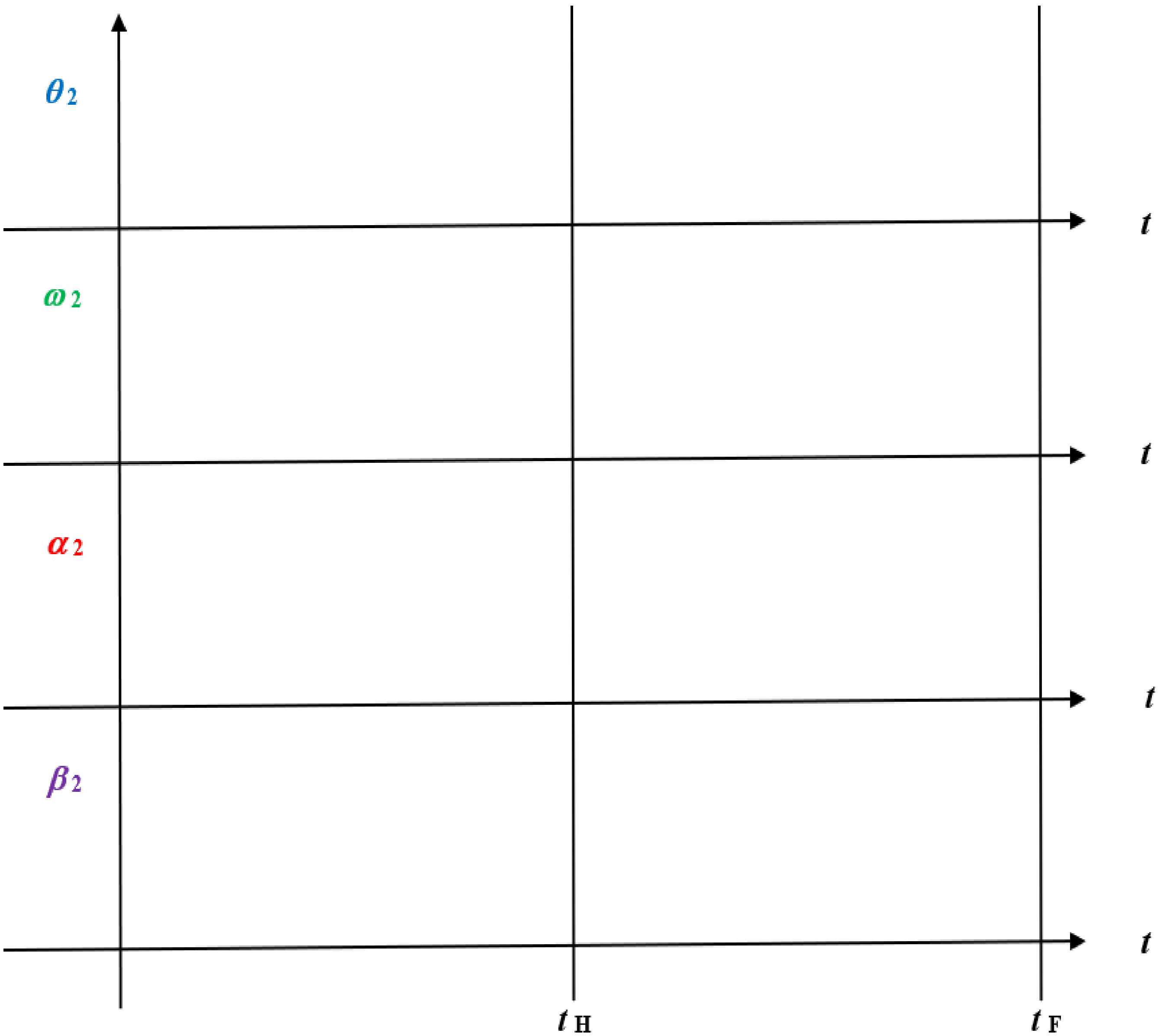


Note that we have been plotting calculated results vs. θ_2. Since ω_2 is constant, time integration yields $\theta_2 = \omega_2 t$, so we could just as well plot all results vs. time t, since both θ_2 and t increase linearly.

As mentioned above, this situation only works well for a mechanism already at constant input velocity (not starting nor stopping at zero velocity – why?).

This constant ω_2 input specification works fine for mechanisms whose input rotates fully and when considering steady-state motion only; i.e. not accelerating up or down to the constant input velocity. Many useful mechanisms have input links that do not rotate fully but travel between joint limits, starting and stopping at zero angular velocity. Why are the previous plots unacceptable in this case?

The simplest change is to specify a symmetric **linear angular velocity**, starting and stopping at zero.



Now we cannot plot the results vs. θ_2 since it is not increasing linearly. Thus we now must plot all results vs. time t.

What is the weakness of this approach? (The discontinuous acceleration function yields infinite jerk at the start, middle, and end of the time range.)

Our guiding principle here is from general mechanical design: for high-speed machinery the rule of thumb is that the position, velocity, and acceleration functions must be continuous. The jerk (third derivative of position) can be discontinuous, but must be finite for the entire motion range.

We can fix this with a symmetric **trapezoidal input acceleration profile**.

This input motion specification should be fine (trapezoidal input torque profiles are often used for industrial robots), but there are 5 different function zones to handle which is not desirable. What acceleration profile is similar but with a single function?

Full-cycloidal function input angle specification

Here are the angle and three derivative functions for the full cycloidal input specification:

$$\theta_2(t) = \theta_{20} + (\theta_{2F} - \theta_{20})\left[\frac{t}{t_F} - \frac{1}{2\pi}\sin\left(\frac{2\pi t}{t_F}\right)\right]$$

$$\omega_2(t) = \frac{(\theta_{2F} - \theta_{20})}{t_F}\left[1 - \cos\left(\frac{2\pi t}{t_F}\right)\right]$$

$$\alpha_2(t) = \frac{2\pi(\theta_{2F} - \theta_{20})}{t_F^2}\sin\left(\frac{2\pi t}{t_F}\right)$$

$$\beta_2(t) = \frac{4\pi^2(\theta_{2F} - \theta_{20})}{t_F^3}\cos\left(\frac{2\pi t}{t_F}\right)$$

Example. Generate and plot the full-cycloidal input angle θ_2, given $\theta_{20} = 45^\circ$, $\theta_{2F} = 135^\circ$, and $t_F = 4$ sec (units below are *deg*, *rad/s*, *rad/s*^{2}, and *rad/s*^{3}, respectively).

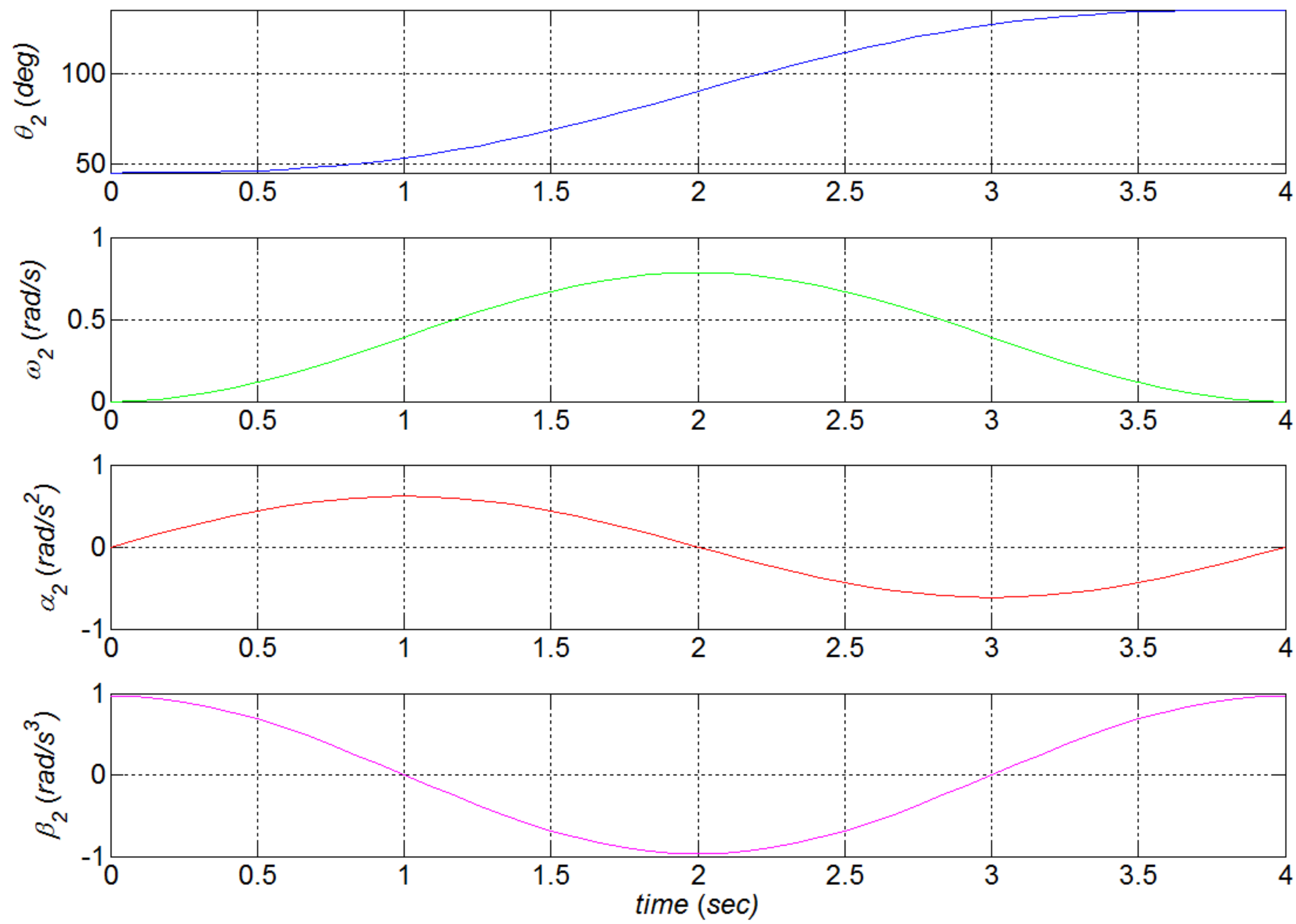
$$\theta_2(t) = 45 + 90\left[\frac{t}{4} - \frac{1}{2\pi}\sin\left(\frac{\pi t}{2}\right)\right]$$

$$\omega_2(t) = 0.393\left[1 - \cos\left(\frac{\pi t}{2}\right)\right]$$

$$\alpha_2(t) = 0.617\sin\left(\frac{\pi t}{2}\right)$$

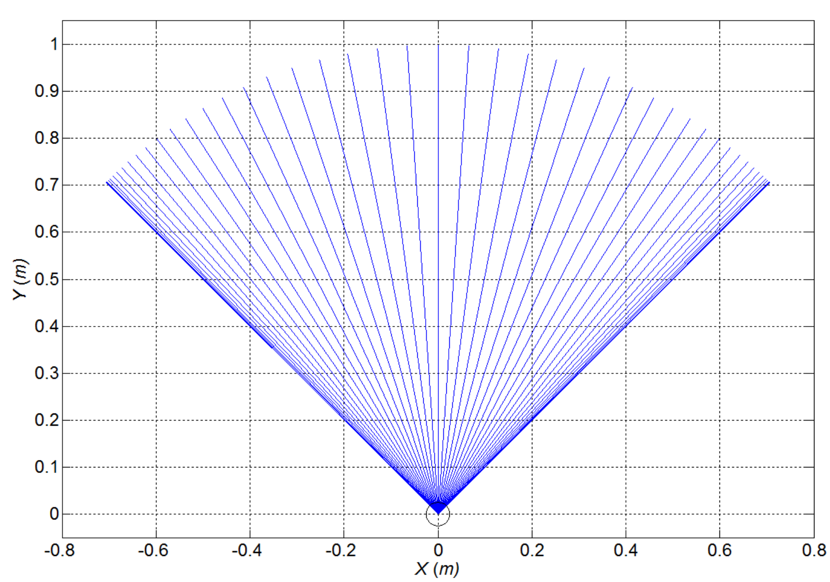
$$\beta_2(t) = 0.969\cos\left(\frac{\pi t}{2}\right)$$

The subplots below give the graphical results for this example. The motion is smooth, starting and stopping at zero angular velocity and acceleration, suitable for input link motion specification. The initial and final angular jerks are not zero, but they are finite, which is acceptable in machine design.



Input Link Cycloidal Motion – θ_2, ω_2, α_2, β_2

As seen in the MATLAB animation plot below (with input link length 1 *m*, a Dr.-Bob-called Spirograph image made with `hold on`), the motion is smooth and continuous, starting and stopping at zero angular velocity and acceleration, with the highest angular velocity ω_2 in the middle.



Cycloidal Input Link Motion – Animation

Full-Range-Of-Motion (F.R.O.M.) MATLAB m-file Revisited

In order to adapt our existing F.R.O.M. MATLAB programs (based on θ_2 as the input) to this new full-cycloidal input concept with time *t* as the input, the following steps are required.

- Use time array **`t`** in place of the **`th2`** array from before, since time is now our independent variable. Since I use **`t`** for the time array, we can no longer use it for the dummy polynomial value in calculation of θ_4, so I just use **`tt`** for that.

- Then calculate the required input link motion values **`th2`**, **`w2`**, and **`alp2`** based on the full-cycloidal equations. Two options, do one or the other:
  - I prefer to calculate these values in one shot, outside the loop, using the entire t array in one statement each – you need to use the 'dot' notation, e.g. **`.*`** to do element-by-element multiplication of two arrays.
  - Instead you can easily calculate **`th2(i)`**, **`w2(i)`**, and **`alp2(i)`** inside the **`i`** loop.

- In either case you must use **`th2(i)`**, **`w2(i)`**, and **`alp2(i)`** inside the **`i`** loop, since now ω_2 are is longer constant and α_2 is no longer zero.

- Plotting of variables is done outside the loop with the entire arrays, using **`t`**, not **`th2`**, as the independent variable.

- Below is a snippet of MATLAB code to demonstrate these ideas (with my preference for outside-the-loop calculation of **`th2`**, **`w2`**, and **`alp2`**). Caution – this is incomplete and will not run as-is.

```
%---------------------------------------------------------------------------------
%   m-file snippet for full-cycloidal input
%      Dr. Bob, ME 3011
%---------------------------------------------------------------------------------

t0 = 0;  dt = 0.1;  tf = 5;  t  = [t0:dt:tf];  % time array, independent variable
th2  = you fill it in
w2   = see above line
alp2 = ditto

N = (tf-t0)/dt + 1;  % number of times to repeat loop for F.R.O.M.

for i=1:N,                 % F.R.O.M. loop over all input t
   % Position analysis:  theta4
   E  =  2*r4*(r1*cos(th1) - r2*cos(th2(i)));
   F  =  2*r4*(r1*sin(th1) - r2*sin(th2(i)));
   G  =  r1^2 + r2^2 - r3^2 + r4^2 - 2*r1*r2*cos(th1-th2(i));
   tt = (-F - sqrt(E^2 + F^2 - G^2)) / (G-E);           % open branch only
   th4(i) = 2*atan(tt);
   % use w2(i) and alp2(i)
   etc.
end

%  Plots outside loop
figure;
plot(t,th3/DR,'r',t,th4/DR,'g',t,mu/DR,'b'); grid;
set(gca,'FontSize',18); legend('{\it\theta_3}', '{\it\theta_4}', '{\it\mu}');
xlabel('{\ittime} ({\itsec})'); ylabel('{\itAngles} ({\itdeg})');
```

5.3 Jerk Kinematics Analysis

Jerk is the time rate of change of the acceleration (and hence the second and third time rates of change of the velocity and position, respectively). Again, this jerk time rate of change may describe a change in magnitude of acceleration, a change in direction of acceleration, or both. What names have been given to the next three position derivatives after **jerk**? The answer is given somewhere in this section. **Jerk** analysis is the fourth step in **kinematics** analysis. It is not required for standard Newton-Euler dynamics analysis. However, it is useful for the following items.

- Input link motion specification
- Cam motion profiles and cam design
- Smooth motion control as in elevators

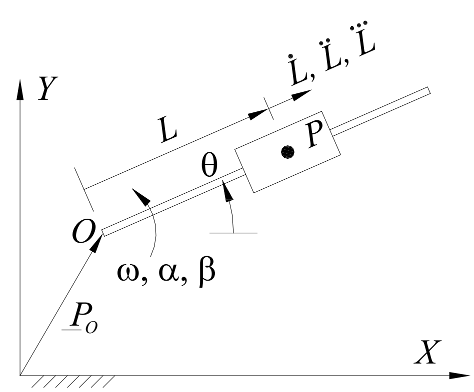
Jerk can be important for kinematic motion analysis in general. Position, velocity, and acceleration analyses must be completed first. Jerk is the first time derivative of the acceleration, the second time derivative of the velocity, and the third time derivative of the position. Like all the terms preceding it, jerk is also a vector quantity. There are both translational and rotational jerk terms.

$$\underline{J}(t)=\frac{d\,\underline{A}(t)}{dt}=\frac{d^2\underline{V}(t)}{dt^2}=\frac{d^3\underline{P}(t)}{dt^3} \qquad \text{SI units: } \frac{m}{s^3}$$

$$\underline{\beta}(t)=\frac{d\underline{\alpha}(t)}{dt}=\frac{d^2\underline{\omega}(t)}{dt^2}=\frac{d^3\theta(t)}{dt^3} \qquad \text{SI units: } \frac{rad}{s^3}$$

Now we address the n-part jerk formula, showing the most general jerk terms possible for planar devices.

***n*-part Jerk Derivation Figure**



This is a four-dof system consisting of a translating/rotating rigid rod with a slider. The same system was used for the two-part position, three-part velocity, and five-part accelerations formula derivations earlier. Find the total jerk of the point P, which is on the slider.

Start with the five-part acceleration formula from before and take another time derivate of the *XY* components. What is *n*? (Hint – clearly *n* must be greater than 5.)

$$\underline{J}_P = \frac{d\underline{A}_P}{dt} = \frac{d}{dt}\begin{Bmatrix} A_{OX}(t) + A(t)\cos\theta(t) - 2V(t)\omega(t)\sin\theta(t) - L(t)\alpha(t)\sin\theta(t) - L(t)\omega(t)^2\cos\theta(t) \\ A_{OY}(t) + A(t)\sin\theta(t) + 2V(t)\omega(t)\cos\theta(t) + L(t)\alpha(t)\cos\theta(t) - L(t)\omega(t)^2\sin\theta(t) \end{Bmatrix}$$

Recall the two-part position, three-part velocity, and five-part acceleration formula results below.

$$\begin{aligned} \underline{P}_P &= \underline{P}_O + \underline{L} \\ &= \begin{Bmatrix} P_{OX}(t) + L(t)\cos\theta(t) \\ P_{OY}(t) + L(t)\sin\theta(t) \end{Bmatrix} \end{aligned}$$

$$\begin{aligned} \underline{V}_P &= \underline{V}_O + \underline{V} + \underline{\omega}\times\underline{L} \\ &= \begin{Bmatrix} V_{OX}(t) + V(t)\cos\theta(t) - L(t)\omega(t)\sin\theta(t) \\ V_{OY}(t) + V(t)\sin\theta(t) + L(t)\omega(t)\cos\theta(t) \end{Bmatrix} \end{aligned}$$

$$\begin{aligned} \underline{A}_P &= \underline{A}_O + \underline{A} + 2\underline{\omega}\times\underline{V} + \underline{\alpha}\times\underline{L} + \underline{\omega}\times\left(\underline{\omega}\times\underline{L}\right) \\ &= \begin{Bmatrix} A_{OX}(t) + A(t)\cos\theta - 2V(t)\omega(t)\sin\theta(t) - L(t)\alpha(t)\sin\theta(t) - L(t)\omega(t)^2\cos\theta(t) \\ A_{OY}(t) + A(t)\sin\theta + 2V(t)\omega(t)\cos\theta(t) + L(t)\alpha(t)\cos\theta(t) - L(t)\omega(t)^2\sin\theta(t) \end{Bmatrix} \end{aligned}$$

The angle $\theta(t)$, angular velocity $\omega(t)$, angular acceleration $\alpha(t)$, and angular jerk $\beta(t)$ are all changing with respect to time (only the planar case is this simple; the spatial rotation case is more complicated).

$$\underline{\beta} = \frac{d\underline{\alpha}}{dt} = \frac{d^2\underline{\omega}}{dt^2} = \frac{d^3\theta}{dt^3}$$

The rod length $L(t)$, sliding velocity $V(t)$, sliding acceleration $A(t)$, and sliding jerk $J(t)$ are all changing with respect to time.

$$\underline{J} = \frac{d\underline{A}}{dt} = \frac{d^2\underline{V}}{dt^2} = \frac{d^3\underline{L}}{dt^3}$$

Here are the same relationships, using the dot notation to indicate time differentiation.

$$\underline{\dddot{\theta}} = \frac{d\underline{\ddot{\theta}}}{dt} = \frac{d^2\underline{\dot{\theta}}}{dt^2} = \frac{d^3\theta}{dt^3} \qquad\qquad \underline{\dddot{L}} = \frac{d\underline{\ddot{L}}}{dt} = \frac{d^2\underline{\dot{L}}}{dt^2} = \frac{d^3\underline{L}}{dt^3}$$

Product and Chain Rules of Differentiation

Again, we'll need to use the product and chain rules repeatedly in jerk analysis derivations.

Product rule

$$\frac{d}{dt}(x(t)y(t)) = \frac{dx(t)}{dt}y(t) + x(t)\frac{dy(t)}{dt}$$

x, *y* are both functions of time

Chain rule

$$\frac{d}{dt}(f(x(t))) = \frac{df(x(t))}{dx}\frac{dx(t)}{dt}$$

f is a function of *x*, which is an implicit function of *t*

Examples

$$\begin{aligned}\frac{d}{dt}(-2V(t)\omega(t)\sin\theta(t)) &= -2A(t)\omega(t)\sin\theta(t) - 2V(t)\alpha(t)\sin\theta(t) - 2V(t)\omega(t)\frac{d}{dt}(\sin\theta(t))\\ &= -2A(t)\omega(t)\sin\theta(t) - 2V(t)\alpha(t)\sin\theta(t) - 2V(t)\omega(t)\frac{d(\sin\theta(t))}{d\theta(t)}\frac{d\theta(t)}{dt}\\ &= -2A(t)\omega(t)\sin\theta(t) - 2V(t)\alpha(t)\sin\theta(t) - 2V(t)\omega(t)^2\cos\theta(t)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(-L(t)\omega(t)^2\cos\theta(t)) &= -V(t)\omega(t)^2\cos\theta(t) - L(t)\frac{d}{dt}(\omega(t)^2)\cos\theta(t) - L(t)\omega(t)^2\frac{d}{dt}(\cos\theta(t))\\ &= -V(t)\omega(t)^2\cos\theta(t) - L(t)\frac{d(\omega(t)^2)}{d\omega(t)}\frac{d\omega(t)}{dt}\cos\theta(t) - L(t)\omega(t)^2\frac{d(\cos\theta(t))}{d\theta(t)}\frac{d\theta(t)}{dt}\\ &= -V(t)\omega(t)^2\cos\theta(t) - 2L(t)\omega(t)\alpha(t)\cos\theta(t) + L(t)\omega(t)^3\sin\theta(t)\end{aligned}$$

Many terms will combine (like in the Coriolis acceleration case). Check the resulting units of all components to check your results. The full *n*-part jerk derivation is left to the interested student.

Generic Mechanism Jerk Analysis Problem Statement

Given the mechanism, complete position, velocity, and acceleration analyses, and one-dof of jerk input, calculate the jerk unknowns.

For a given branch of a known mechanism, this will yield linear equations so a matrix-vector approach may be used to obtain the unique solution (assuming no singularity).

Since jerk kinematics is not required for dynamics analysis or machine design (generally), this solution is beyond the scope of the class.

For our jerk needs, we may use the time differentiation approach presented earlier for Velocity and Acceleration Analyses.

Snap, what a happy sound
Snap is the happiest sound I found
You may clap, rap, tap, slap,
But Snap makes the world go round
Snap, crackle, pop – Rice Krispies!

I say it's **Crackle**, the crispy sound
You gotta have Crackle or the clock's not wound
Geese cackle, feathers tickle, belts buckle, beets pickle,
But Crackle makes the world go round
Snap, crackle, pop – Rice Krispies!

I insist that Pop's the sound
The best is missed unless Pop's around
You can't stop hoppin' when the cereal's poppin'
Pop makes the world go round
Snap, crackle, pop – Rice Krispies!

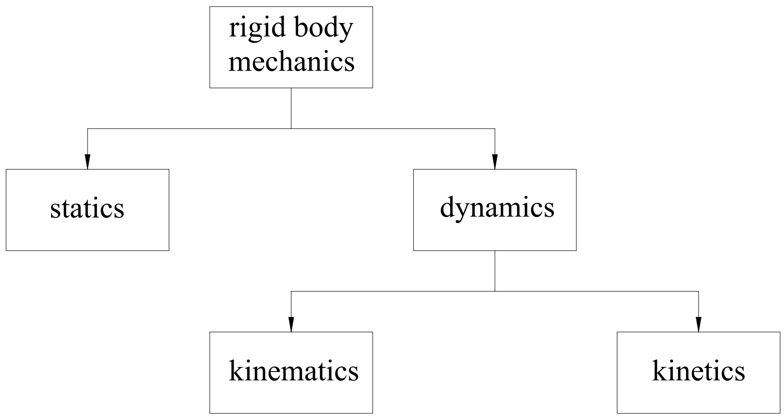
– Old Kellogg's Advertisement

6. Inverse Dynamics Analysis

Dynamics analysis is concerned with relating the kinematic motion (translational and rotational **position**, **velocity**, and **acceleration**) with forces and torques. For inverse dynamics analysis the complete kinematics problems must be solved first.

Dynamics is the study of motion with regard to forces/torques.

6.1 Dynamics Introduction

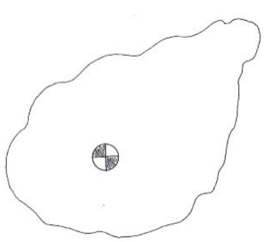


Dynamics of a single rigid body in the plane

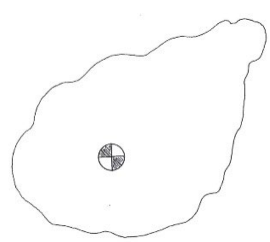
Assume a rigid body is acted on by a system of forces and moments to produce planar motion. What is the first step in dynamics analysis? Draw the **Free-Body Diagram**.

Free-Body Diagram (FBD)

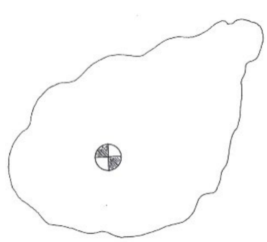
Isolate each rigid body and show all internal and external forces and moments acting. This contains all the information needed to write **Newton's Second Law** and **Euler's Rotational Dynamics Equation** on the following page.



Free-Body Diagram (FBD)



Simplified FBD



MAD (Mass-Acceleration Diagram)

Write dynamics equations for each simplified FBD

Newton's Second Law (translational)

Euler's Rotational Dynamics Equation (rotational)

$\underline{A}_G$ is the absolute translational acceleration of the link center of gravity – it must be in the same direction as the vector resultant force $\underline{R}$. Different points in rigid body have different translational accelerations. $\underline{\alpha}$ is the absolute angular acceleration of the rigid body. The entire rigid body experiences the same $\underline{\alpha}$.

Two Types of Dynamics Problems:

1. Generic Mechanism Forward Dynamics Analysis Problem Statement

Given: the mechanism, external forces and moments, and the applied driving force (or torque).

Find: the resulting mechanism motion and internal joint forces.

This problem requires the solution of coupled nonlinear (transcendental) differential equations, which many commercial dynamics software programs can perform.

Four-Bar Mechanism Forward Dynamics Analysis Problem Statement

Given: the mechanism ($r_1, \theta_1, r_2, r_3, r_4, m_2, m_3, m_4, CG_2, CG_3, CG_4, I_{GZ2}, I_{GZ3}, I_{GZ4}$), driving torque $\underline{\tau}_2$, and external forces/moments $\underline{F}_{E3}, \underline{F}_{E4}$ and $\underline{M}_{E3}, \underline{M}_{E4}$.

Find: the kinematic motion $\theta_2, \theta_3, \theta_4, \omega_2, \omega_3, \omega_4, \alpha_2, \alpha_3, \alpha_4, \underline{A}_{G2}, \underline{A}_{G3}, \underline{A}_{G4}$ and internal joint forces $\underline{F}_{21}, \underline{F}_{32}, \underline{F}_{43}, \underline{F}_{14}$.

2. Generic Mechanism <u>Inverse Dynamics</u> Analysis Problem Statement

Given: the mechanism, external forces and moments, and the desired mechanism motion.

Find: the required driving force (or torque) and internal joint forces.

The Matrix Method is well-suited to solve this problem since there are *n* linear equations in *n* unknowns.

Four-Bar Mechanism Inverse Dynamics Analysis Problem Statement

Given: the mechanism ($r_1, \theta_1, r_2, r_3, r_4, m_2, m_3, m_4, CG_2, CG_3, CG_4, I_{GZ2}, I_{GZ3}, I_{GZ4}$), kinematic motion $\theta_2, \theta_3, \theta_4, \omega_2, \omega_3, \omega_4, \alpha_2, \alpha_3, \alpha_4, \underline{A}_{G2}, \underline{A}_{G3}, \underline{A}_{G4}$, and external forces/moments $\underline{F}_{E3}, \underline{F}_{E4}$ and $\underline{M}_{E3}, \underline{M}_{E4}$.

Find: the driving torque $\underline{\tau}_2$ and internal joint forces $\underline{F}_{21}, \underline{F}_{32}, \underline{F}_{43}, \underline{F}_{14}$.

Newton's Second Law and **Euler's Rotational Dynamics Equation** require the following terms.

	m	***P_G***	***I_{GZ}***
translational	mass	center of gravity	
rotational		center of gravity	mass moment of inertia

We will next review these important dynamics parameters.

6.2 Mass, Center of Gravity, and Mass Moment of Inertia

A thorough review of mass, center of gravity, and mass moment of inertia is given in this section. These important terms were not required for kinematics analysis, but are required for translational and rotational dynamics equations, for each FBD.

$$\sum \underline{F} = m\underline{A}_G \qquad \sum \underline{M}_G = I_{GZ}\underline{\alpha}$$

Newton's Second Law — Euler's Rotational Dynamics Equation

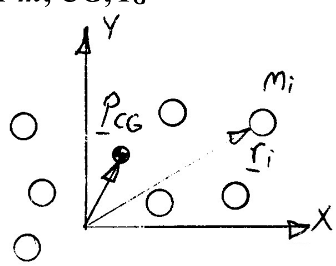
Newton's Second Law requires m and CG, Euler's Rotational Dynamics Equation requires CG and I_{GZ}.

	m	P_{CG}	I_{GZ}
translational	mass	center of gravity	
rotational		center of gravity	mass moment of inertia

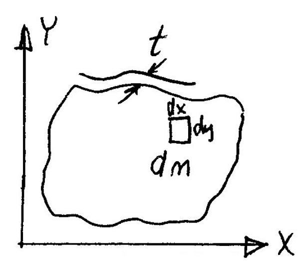
<u>Mass</u>

In Newton's Second Law $\sum \underline{F} = m\underline{A}_G$, the mass m is the proportionality constant. Mass is measure of translational inertia – a resistance to change in motion according to Newton's First Law. Mass is also a measure of the storage of translational kinetic energy $KE_T = \frac{1}{2}mv^2$ and the units are kg.

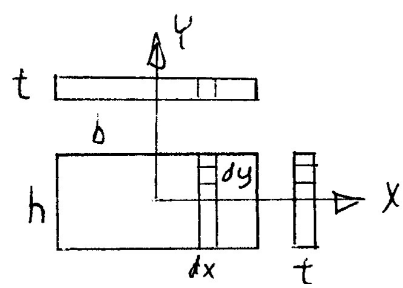
Examples for *m*, *CG*, I_G



System of particles



General rigid body



Rectangular rigid body

Mass calculation

System of particles

$$m = \sum_{i=1}^{N} m_i$$

General rigid body

$$m = \int_{body} dm$$

Rectangular rigid body

using $\rho = \dfrac{m}{V}$, $dm = \rho dV$, so $m = \int_{body} dm = \rho \int_{body} dV$

$$m = \rho \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} t dx dy = \rho t b h$$

$m = \rho V$ (an obvious result)

Center of Gravity (*CG*, *G*)

The *CG* is the point at which a body is balanced with respect to gravity. It is also the point at which the body weight acts. The *CG* is also called the mass center, center of mass, and centroid. It is a vector quantity and the units are length units, m.

***CG* calculation**

System of particles

$$\underline{P}_{CG} = \frac{\sum m_i \underline{r}_i}{\sum m_i}$$

Cartesian components

$$X_{CG} = \overline{X} = \frac{\sum m_i x_i}{\sum m_i}$$

$$Y_{CG} = \overline{Y} = \frac{\sum m_i y_i}{\sum m_i}$$

General rigid body

$$\underline{P}_{CG} = \frac{\int_{body} \underline{r} dm}{\int dm}$$

Cartesian components

$$X_{CG} = \overline{X} = \frac{\int_x x dm}{\int dm}$$

$$Y_{CG} = \overline{Y} = \frac{\int_y y dm}{\int dm}$$

Rectangular rigid body

Using an *XY* coordinate frame at the geometric center, the *CG* is calculated below.

$$\bar{X} = \frac{\int_x x\,dm}{\int dm} = \frac{\rho}{m}\int_x x\,dV = \frac{\rho}{m}\int_{-b/2}^{b/2} xth\,dx = \frac{\rho th}{m}\int_{-b/2}^{b/2} x\,dx = \frac{\rho th}{m}\left.\frac{x^2}{2}\right|_{-b/2}^{b/2} = \frac{\rho th}{2m}\left(\frac{b^2}{4}-\frac{b^2}{4}\right) = 0$$

$$\bar{Y} = \frac{\int_y y\,dm}{\int dm} = \frac{\rho}{m}\int_y y\,dV = \frac{\rho}{m}\int_{-h/2}^{h/2} ytb\,dy = \frac{\rho tb}{m}\int_{-h/2}^{h/2} y\,dy = \frac{\rho tb}{m}\left.\frac{y^2}{2}\right|_{-h/2}^{h/2} = \frac{\rho tb}{2m}\left(\frac{h^2}{4}-\frac{h^2}{4}\right) = 0$$

$$\underline{P}_{CG} = \begin{Bmatrix} \bar{X} \\ \bar{Y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For a homogeneous, regular geometric body of uniform thickness, the *CG* is the geometric center.

Mass Moment of Inertia I_G

This is ***not*** the same as **area moment of inertia** (I_A) for beam bending, which is recalled below.

$$I_{Ax} = \int_y y^2 dA \qquad I_{Ay} = \int_x x^2 dA \qquad \text{units: } I_A \equiv m^4$$

In Euler's rotational dynamics equation $\sum \underline{M}_G = I_{GZ}\underline{\alpha}$, the mass moment of inertia I_{GZ} is the proportionality constant. I_{GZ} is also a measure of rotational inertia, i.e. the resistance to change in rotational motion according to Newton's First Law. Also, it is a measure of how hard it is to accelerate about certain axes in rotation. I_{GZ} is also a measure of the storage of rotational kinetic energy, $KE_R = \frac{1}{2} I_{GZ}\omega^2$, and its units are $kg\text{-}m^2$.

What is the mass moment of inertia, a scalar, vector, matrix, or something else? Answer – it is a tensor.

Mass Moment of Inertia I_G calculation

System of particles

$$I_{axis} = \sum m_i r_i^2$$

where r_i is the scalar perpendicular distance from the *axis* to the i^{th} particle. With squaring, all terms will be positive, so there can be no canceling like for the *CG*. If the first moment (*CG*) is balanced, the second moment (I_{GZ}) terms do not cancel since the squared terms are all positive.

General rigid body inertia tensor (symmetric)

$$I_{axis} = \begin{bmatrix} I_{XX} & I_{XY} & I_{XZ} \\ I_{XY} & I_{YY} & I_{YZ} \\ I_{XZ} & I_{YZ} & I_{ZZ} \end{bmatrix}$$

$$I_{XX} = \int_{body} (y^2 + z^2) dm \qquad I_{YY} = \int_{body} (x^2 + z^2) dm \qquad I_{ZZ} = \int_{body} (x^2 + y^2) dm$$

What is the only term that matters for *XY* planar motion? Answer – I_{ZZ}.

In the yardstick example:

$$I_{GZ} > I_{GY} > I_{GX} \qquad \text{also} \qquad I_{OZ} > I_{GZ}$$

Rectangular rigid body

Using an XY coordinate frame at the CG, I_{GZ} is calculated below.

$$I_{GZ} = \int_{body} (x^2 + y^2)dm = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} (x^2 + y^2)\rho t dx dy$$

$$= \rho t \int_{-b/2}^{b/2} \left(x^2 y + \frac{y^3}{3} \bigg|_{-h/2}^{h/2} \right) dx$$

$$= \rho t \int_{-b/2}^{b/2} \left(x^2 \left(\frac{h}{2} - \frac{-h}{2} \right) + \frac{1}{3}\left(\frac{h^3}{8} - \frac{-h^3}{8} \right) \right) dx$$

$$= \rho t \int_{-b/2}^{b/2} \left(hx^2 + \frac{h^3}{12} \right) dx = \rho t \left(\frac{hx^3}{3} + \frac{h^3 x}{12} \bigg|_{-b/2}^{b/2} \right)$$

$$= \rho t \left(\frac{h}{3}\left(\frac{b^3}{8} - \frac{-b^3}{8} \right) + \frac{h^3}{12}\left(\frac{b}{2} - \frac{-b}{2} \right) \right)$$

$$= \rho t \left(\frac{b^3 h}{12} + \frac{bh^3}{12} \right) = \frac{\rho t b h}{12}(b^2 + h^2)$$

$$I_{GZ} = \frac{m}{12}(b^2 + h^2) \qquad \text{(using } m = \rho V = \rho t b h \text{)}$$

units: mass times distance squared, $kg\text{-}m^2$

This formula agrees with the result given in tables.

How do we find mass, center of gravity, and mass moment of inertia in the real world?

- From tables – for example, see the three tables at the end of this section.
- CAD packages (such as SolidWorks or AutoCAD) calculate m, CG, and I_{GZ} automatically for each link drawn, once material is associated with the 3D model.

Parallel Axis Theorem

The mass moment of inertia through the *CG* is related to mass moments of inertia of parallel axes through different points as follows.

$$I_{ZZO} = I_{ZZG} + md^2$$

where d is the scalar distance separating the axis of interest O from the axis through the *CG*. Notice I_{ZZG} is as small as possible. Any I_{ZZO} must be greater, due to the term md^2 which is always positive.

Parallel axis theorem example

Rectangular rigid body (where axis *O* is the corner)

$$\begin{aligned} I_{ZZO} &= I_{ZZG} + md^2 \\ &= \frac{m}{12}(b^2+h^2) + m\left(\frac{b^2}{4}+\frac{h^2}{4}\right) \\ &= m\left(\frac{b^2}{12}+\frac{b^2}{4}+\frac{h^2}{12}+\frac{h^2}{4}\right) \\ &= m\left(\frac{b^2}{3}+\frac{h^2}{3}\right) \\ &= \frac{m}{3}(b^2+h^2) \end{aligned}$$

Combining multiple rigid bodies into a rigid link

To combine any number of bodies n of known material, shape, size, and location into one rigid body, use the following equations for mass, center of gravity, and mass moment of inertia.

$$m_T = m_1 + m_2 + \cdots + m_n = \sum_{i=1}^{n} m_i$$

$$\underline{P}_{CGT} = \begin{Bmatrix} \overline{X}_T \\ \overline{Y}_T \end{Bmatrix} = \begin{Bmatrix} \dfrac{m_1\overline{x}_1 + m_2\overline{x}_2 + \cdots + m_n\overline{x}_n}{m_T} \\ \dfrac{m_1\overline{y}_1 + m_2\overline{y}_2 + \cdots + m_n\overline{y}_n}{m_T} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\sum_{i=1}^{n} m_i\overline{x}_i}{m_T} \\ \dfrac{\sum_{i=1}^{n} m_i\overline{y}_i}{m_T} \end{Bmatrix}$$

$$I_{GZT} = I_{GZ1} + m_1 d_1^2 + I_{GZ2} + m_2 d_2^2 + \cdots + I_{GZn} + m_n d_n^2 = \sum_{i=1}^{n} I_{GZi} + m_i d_i^2$$

The subscript 'T' indicates total (or combined) and d_i is the distance between the combined *CG* ($\underline{P}_{CG_T}$) and the *CG* of body i ($\underline{P}_{CG_i} = \begin{bmatrix} \overline{x}_i & \overline{y}_i \end{bmatrix}^T$). These equations are obtained from the mass, *CG*, and I_{GZ} equations for particles, where now each particle is instead a rigid body.

Example 1. Two rectangles joined as shown below

Given $L_1 = 2.2$ x $h_1 = 0.1$ x $t_1 = 0.005$ (m)
$L_2 = 1.0$ x $h_2 = 0.8$ x $t_2 = 0.005$ (m)
the material is steel with a mass density of $\rho = 7850$ kg/m^3

The equations are:

$$m_T = m_1 + m_2 \qquad \underline{P}_{CGT} = \begin{Bmatrix} \overline{X}_T \\ \overline{Y}_T \end{Bmatrix} = \begin{Bmatrix} \dfrac{m_1\overline{x}_1 + m_2\overline{x}_2}{m_T} \\ \dfrac{m_1\overline{y}_1 + m_2\overline{y}_2}{m_T} \end{Bmatrix} \qquad I_{GZT} = I_{GZ1} + m_1 d_1^2 + I_{GZ2} + m_2 d_2^2$$

where d_i is the distance from CG_i to the combined CG $\underline{P}_{CGT}$.

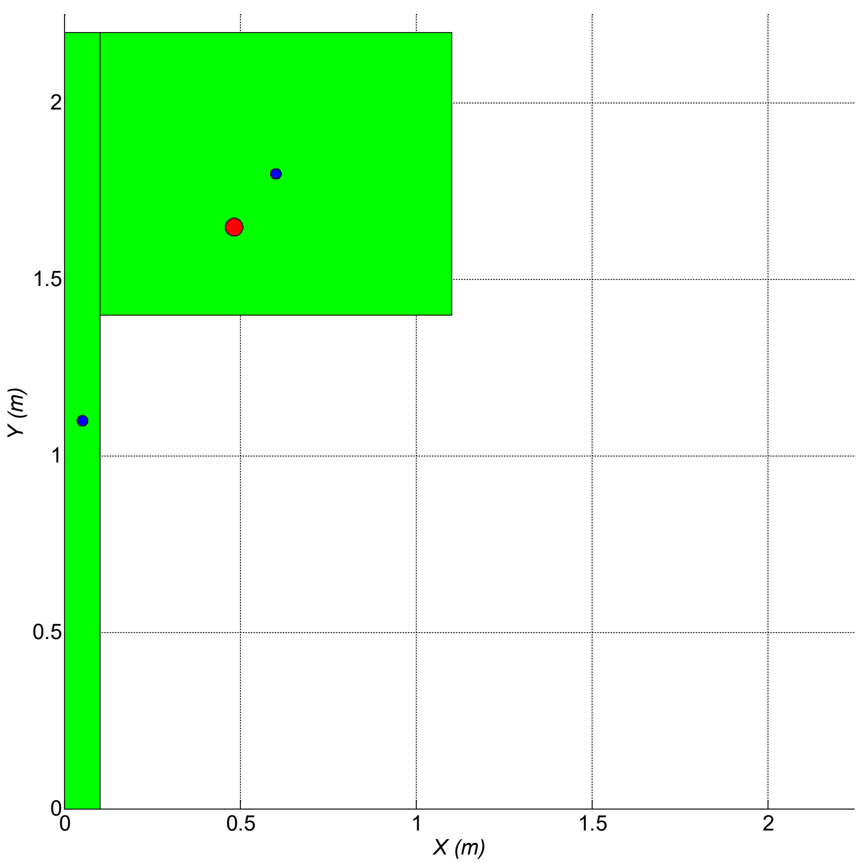
The results are:

$m_1 =$ 8.64 $m_2 =$ 31.40 $m_T =$ 40.04 (kg)

$$\underline{P}_{CG1} = \begin{Bmatrix} 0.05 \\ 1.10 \end{Bmatrix} \qquad \underline{P}_{CG2} = \begin{Bmatrix} 0.60 \\ 1.80 \end{Bmatrix} \qquad \underline{P}_{CGT} = \begin{Bmatrix} 0.48 \\ 1.65 \end{Bmatrix} \ (m)$$

$d_1 =$ 0.70 $d_2 =$ 0.19 (m)

$$I_{GZ1} = \frac{m_1}{12}\left(L_1^2 + h_1^2\right) = 3.49 \qquad I_{GZ2} = \frac{m_2}{12}\left(L_2^2 + h_2^2\right) = 4.29 \qquad I_{GZT} = 13.15 \ (kgm^2)$$



Example 1. Two Unequal Steel Rectangles (here L_1 is vertical and L_2 is horizontal)

Example 2. Two equal rectangles joined in the same manner

Given $L = 2.0$ x $h = 0.2$ x $t = 0.005$ (m) (twice)
$\rho = 7850\ kg/m^3$

The analytical equations for this special case are:

$$m_T = 2m \qquad \underline{P}_{CGT} = \begin{Bmatrix} \overline{X}_T \\ \overline{Y}_T \end{Bmatrix} = \begin{Bmatrix} \dfrac{L+3h}{4} \\ \dfrac{3L-h}{4} \end{Bmatrix} \qquad I_{GZT} = \frac{5}{12} m (L^2 + h^2)$$

where $d_1 = d_2 = \sqrt{\dfrac{L^2 + h^2}{8}}$ is the distance from CG_i to $\underline{P}_{CGT}$.

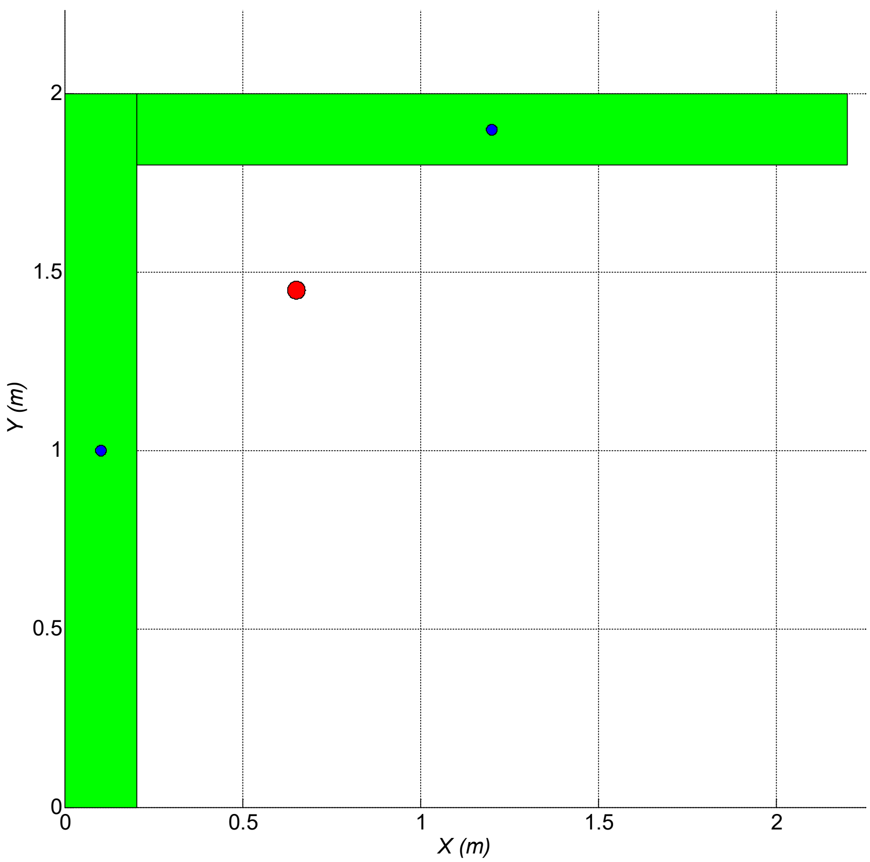
The results are:

$m_1 = m_2 =$ 15.70 $\qquad m_T =$ 31.40 (kg)

$$\underline{P}_{CG1} = \begin{Bmatrix} 0.10 \\ 1.00 \end{Bmatrix} \qquad \underline{P}_{CG2} = \begin{Bmatrix} 1.20 \\ 1.90 \end{Bmatrix} \qquad \underline{P}_{CGT} = \begin{Bmatrix} 0.65 \\ 1.45 \end{Bmatrix} \ (m)$$

$d_1 = d_2 = 0.71$ (m)

$I_{GZ1} = I_{GZ2} = \dfrac{m}{12}(L^2 + h^2) = 5.29 \qquad I_{GZT} = 26.43$ (kgm^2)



Example 2. Two Equal Steel Rectangles

(again, L_1 is vertical and L_2 is horizontal, with $L_1 = L_2 = L$ here)

Tables of Mass, Center of Gravity, and Mass Moment of Inertia

The tables below present the mass (*kg*), center of gravity (*m*), and mass moment of inertia (*kg*-m^2) for some common planar link shapes.

Note that mass moment of inertia (*kg*-m^2) is not the same as area moment of inertia for beam bending (m^4). The former represents rotational inertia while the latter is a measure of resistance to beam bending.

We assume that all link materials are homogeneous and uniformly distributed, with mass density ρ (kg/m^3), all links have regular geometry, and all links have a constant thickness *t* in the *Z* direction.

The general equations for mass, center of gravity, and mass moment of inertia are given below for general rigid bodies.

$$m = \int_{body} dm$$

$$\underline{P}_{CG} = \frac{\int_{body} \underline{r}dm}{\int dm}$$

$$I_{ZZ} = \int_{body} (x^2 + y^2)dm$$

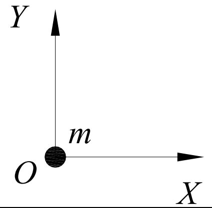
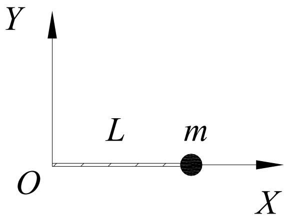
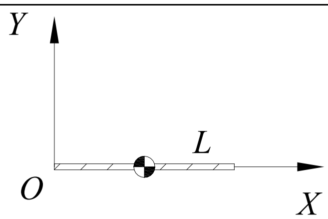
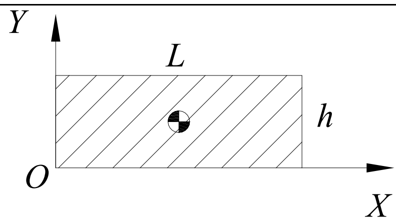
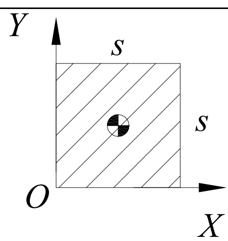
All of these terms require double integrals over the rigid body in the *XY* plane. Both center of gravity and mass moment of inertia depend on the origin of the chosen *XYZ* Cartesian coordinate system.

For general 3D rigid bodies, mass moment of inertia is a 3x3 tensor. For planar mechanism dynamics we only need one scalar term out of these 9 terms, I_{ZZ}.

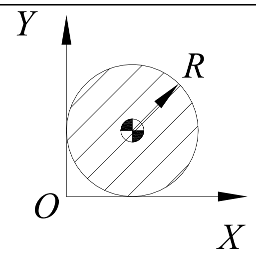
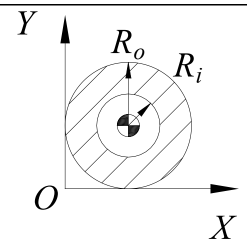
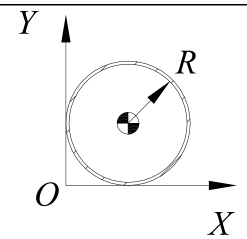
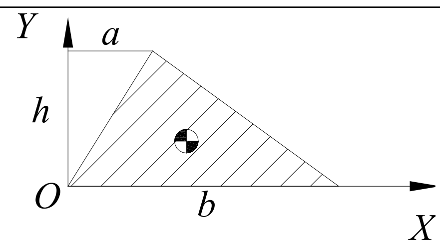
In the drawings below, the planar reference Cartesian coordinate system is shown, with origin *O*, and the standard symbol is used for center of gravity, denoted as point *G*. The mass moment of inertia about axis *O* is related to the mass moment of inertia about axis *G* is via the parallel axis theorem, where *d* is the scalar distance (vector length) between axes *G* and *O* in the *XY* plane:

$$I_{OZ} = I_{GZ} + md^2$$

Mass Properties for Planar Links

Name	Model	Mass (kg)	Center of Gravity (m)	Mass Moment of Inertia (kg-m^2)
point mass		m	$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$	0
point mass on massless rod		m	$\begin{Bmatrix} L \\ 0 \end{Bmatrix}$	$I_{GZ} = 0$ $I_{OZ} = mL^2$
slender rod		m	$\begin{Bmatrix} \frac{L}{2} \\ 0 \end{Bmatrix}$	$I_{GZ} = \frac{mL^2}{12}$ $I_{OZ} = \frac{mL^2}{3}$
rectangular parallelepiped		ρLht	$\begin{Bmatrix} \frac{L}{2} \\ \frac{h}{2} \end{Bmatrix}$	$I_{GZ} = \frac{m(L^2 + h^2)}{12}$ $I_{OZ} = \frac{m(L^2 + h^2)}{3}$
square		$\rho s^2 t$	$\begin{Bmatrix} \frac{s}{2} \\ \frac{s}{2} \end{Bmatrix}$	$I_{GZ} = \frac{ms^2}{6}$ $I_{OZ} = \frac{2ms^2}{3}$

Mass Properties for Planar Links (continued)

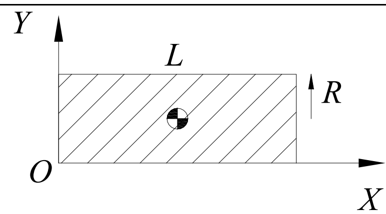
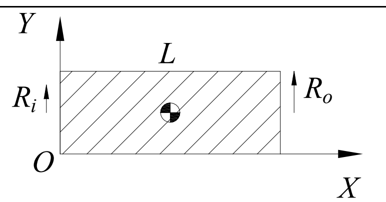
Name	Model	Mass (*kg*)	Center of Gravity (*m*)	Mass Moment of Inertia (*kg-m²*)
cylinder		$\rho\pi R^2 t$	$\begin{Bmatrix} R \\ R \end{Bmatrix}$	$I_{GZ} = \frac{mR^2}{2}$ $I_{OZ} = \frac{5mR^2}{2}$
hollow cylinder		$\rho\pi\left(R_o^2 - R_i^2\right)t$	$\begin{Bmatrix} R_0 \\ R_0 \end{Bmatrix}$	$I_{GZ} = \frac{m(R_o^2 + R_i^2)}{2}$ $I_{OZ} = \frac{m(5R_o^2 + R_i^2)}{2}$
thin ring		m	$\begin{Bmatrix} R \\ R \end{Bmatrix}$	$I_{GZ} = mR^2$ $I_{OZ} = 3mR^2$
triangle		$\frac{\rho bht}{2}$	$\begin{Bmatrix} \frac{a+b}{3} \\ \frac{h}{3} \end{Bmatrix}$	$I_{GZ} = \frac{m(a^2 - ab + b^2 + h^2)}{18}$ $I_{OZ} = \frac{m(a^2 + ab + b^2 + h^2)}{6}$

The formula for the mass moment inertia of a triangle was derived via double integral over the body by Ohio University Ph.D. student Elvedin Kljuno – it could not be found in any sophomore-level dynamics book, or in any other textbook, nor in any Internet search.

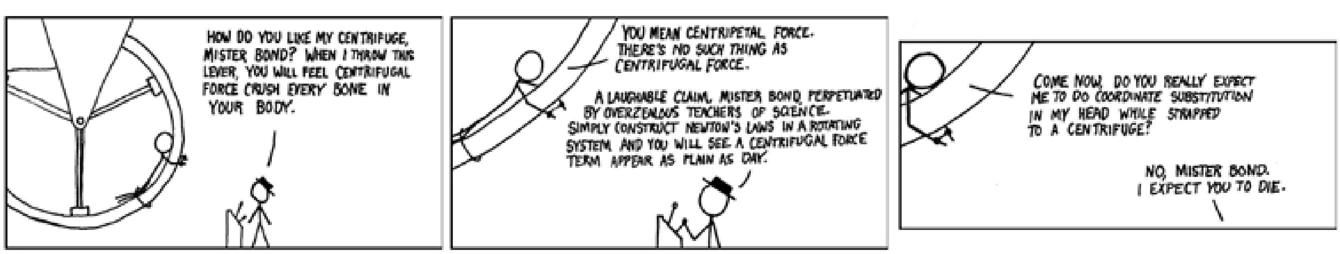
Mass Properties for Cylindrical Links

Now, the previous nine shapes for which the mass properties were summarized are all planar shapes, with a constant thickness t in the Z direction (except for the point mass, point mass on massless rod, and slender rod, whose Z dimensions are unimportant). Here we give one more shape, a cylinder that is 3D but useful for planar slider-crank mechanisms and other planar mechanisms with a prismatic joint and sliding cylindrical piston. The cylinder given on the previous page was arranged with the circle in the XY plane. Now we need to rotate this so the rectangular projection of the cylinder is the XY plane. The mass moment of inertia is quite different from that of the rectangular parallelepiped, due to the effect of the radius R in this case, as opposed to the constant thickness t in the rectangular parallelepiped case.

Not all pistons are solid, so we also include a similar model for the hollow piston cylinder, with outer radius R_o and inner radius R_i.

Name	Model	Mass (kg)	Center of Gravity (m)	Mass Moment of Inertia (kg-m^2)
piston cylinder		$\rho\pi R^2 L$	$\begin{Bmatrix} \frac{L}{2} \\ R \end{Bmatrix}$	$I_{GZ} = \frac{m(L^2+3R^2)}{12}$ $I_{OZ} = \frac{m(4L^2+15R^2)}{12}$
hollow piston cylinder		$\rho\pi\left(R_o^2 - R_i^2\right)L$	$\begin{Bmatrix} \frac{L}{2} \\ R \end{Bmatrix}$	$I_{GZ} = \frac{m(L^2+3R_o^2+3R_i^2)}{12}$ $I_{OZ} = \frac{m(4L^2+15R_o^2+3R_i^2)}{12}$

Dynamics Humor from xkcd



xkcd.com

English Units for Mass

ME 3011 uses SI units exclusively. However, many of you perform your capstone project work using English units, which is fine, since we live in the U.S. In the 1970s the U.S. government mandated a change to the SI system – this failed spectacularly (why?).

One big benefit of the SI system is seen in the units for Newton's Second Law, $F = ma$. Using standard SI units, this equation uses all ones (1s):

1 Newton accelerates 1 kg 1 m/s^2

Sadly the English units DO NOT behave with ones in Newton's Second Law:

1 lb_f DOES NOT accelerate 1 lb_m 1 ft/s^2

1 lb_f DOES NOT accelerate 1 lb_m 1 in/s^2

Further, the English system has another confusion which does not exist for the SI system. The same unit, pound (lb), applies both to force (lb_f) and mass (lb_m), depending on the context. Please always use the correct subscript for clarity. Happily, a mass of 1 lb_m does weigh 1 lb_f at standard gravity (g = 32.2 ft/s^2 or 386.1 in/s^2).

Now we present the standard English mass units; there are two, depending on if you use feet or inches for the length unit.

1 lb_f accelerates 1 **slug** 1 ft/s^2

1 lb_f accelerates 1 **blob** 1 in/s^2

WTF?!? **slug**? **blob**? I promise you I am not making this up. A **slug** is a rather large mass, equivalent to 32.2 lb_m (14.6 kg). A **blob** is even larger, equivalent to 12 **slugs**, 386.1 lb_m (175.1 kg).

In conclusion, do not use lb_m in dynamics equations for your project. Instead use slugs if you are using feet or blobs if you are using inches. If you have estimated your masses in lb_m, simply divide by 32.2 to get slugs, or divide by 386.1 to get blobs.

Finally, from the above we have the following units equivalents:

$$1 \;\; lb_f \;=\; 1 \; \frac{slug \cdot ft}{s^2} \qquad \text{so} \qquad 1 \;\; slug \;=\; 1 \; \frac{lb_f \cdot s^2}{ft}$$

$$1 \;\; lb_f \;=\; 1 \; \frac{blob \cdot in}{s^2} \qquad \text{so} \qquad 1 \;\; blob \;=\; 1 \; \frac{lb_f \cdot s^2}{in}$$

6.3 Single Rotating Link Inverse Dynamics Analysis

Generic Mechanism Inverse Dynamics Analysis Problem Statement

Given: the mechanism, external forces and moments, and the desired mechanism motion.

Find: the required driving force (or torque) and internal joint forces.

Step 1. The single rotating link **Position, Velocity, and Acceleration Analyses** must first be complete.

Step 2. Draw the single rotating link diagrams

Physical dynamics diagram

Free body diagram (FBD)

$\underline{F}_{ij}$ unknown vector internal joint force of link *i* acting on link *j*.

$\underline{r}_{ij}$ known moment arm vector pointing to the joint connection with link *i* from the *CG* of link *j*.

Step 3. State the Problem

Step 4. Derive the Newton-Euler Dynamics Equations

Newton's Second Law

Euler's rotational dynamics equation

Count the number of unknowns and the number of equations.

Step 5. Derive the *XYZ* scalar equations from the vector dynamics equations

Here is the general formula for the cross product of two planar vectors.

$$\underline{r} \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_x & r_y & 0 \\ F_x & F_y & 0 \end{vmatrix} = \hat{i} \begin{vmatrix} r_y & 0 \\ F_y & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} r_x & 0 \\ F_x & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} r_x & r_y \\ F_x & F_y \end{vmatrix}$$

$$= \hat{i}(r_y(0) - F_y(0)) - \hat{j}(r_x(0) - F_x(0)) + \hat{k}(r_x(F_y) - F_x(r_y))$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(r_x F_y - r_y F_x) = \begin{Bmatrix} 0 \\ 0 \\ r_x F_y - r_y F_x \end{Bmatrix}$$

Here are the three linear dynamics equations written in matrix/vector form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_{12Y} & r_{12X} & 1 \end{bmatrix} \begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{Bmatrix} mA_{GX} - F_{EX} \\ m(A_{GY} + g) - F_{EY} \\ I_{GZ}\alpha - r_{EX}F_{EY} + r_{EY}F_{EX} - M_E \end{Bmatrix}$$

Step 6. Solve for the unknowns

We don't need a matrix solution; the first two equations are decoupled and the solution is:

Alternate Step 6. **Solve for the unknowns using a matrix-vector approach**

From the previous page, the matrix-vector equations for the single rotating link inverse dynamics problem are repeated below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_{12Y} & r_{12X} & 1 \end{bmatrix} \begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{Bmatrix} mA_{GX} - F_{EX} \\ m(A_{GY} + g) - F_{EY} \\ I_{GZ}\alpha - r_{EX}F_{EY} + r_{EY}F_{EX} - M_E \end{Bmatrix}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

The solution is:

$$\{\mathbf{x}\} = [\mathbf{A}]^{-1}\{\mathbf{b}\}$$

$$\begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_{12Y} & r_{12X} & 1 \end{bmatrix}^{-1} \begin{Bmatrix} mA_{GX} - F_{EX} \\ m(A_{GY} + g) - F_{EY} \\ I_{GZ}\alpha - r_{EX}F_{EY} + r_{EY}F_{EX} - M_E \end{Bmatrix}$$

$$\begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_{12Y} & -r_{12X} & 1 \end{bmatrix} \begin{Bmatrix} mA_{GX} - F_{EX} \\ m(A_{GY} + g) - F_{EY} \\ I_{GZ}\alpha - r_{EX}F_{EY} + r_{EY}F_{EX} - M_E \end{Bmatrix}$$

$$\begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{Bmatrix} mA_{GX} - F_{EX} \\ m(A_{GY} + g) - F_{EY} \\ I_{GZ}\alpha + r_{12Y}(mA_{GX} - F_{EX}) - r_{12X}(m(A_{GY} + g) - F_{EY}) - r_{EX}F_{EY} + r_{EY}F_{EX} - M_E \end{Bmatrix}$$

The τ solution simplifies by substituting the known equalities for F_{12X} and F_{12Y}:

$$F_{12X} = mA_{GX} - F_{EX}$$
$$F_{12Y} = m(A_{GY} + g) - F_{EY}$$
$$\tau = I_{GZ}\alpha + r_{12Y}F_{12X} - r_{12X}F_{12Y} - r_{EX}F_{EY} + r_{EY}F_{EX} - M_E$$

This solution agrees with the previously-presented algebra solution, which is expected since linear equations yield a unique solution in the absence of singularities.

Note that the inverse of the coefficient matrix **[A]** above is very simple, since this matrix is nearly decoupled. Further, as proven on the following page, the determinant of the coefficient matrix **[A]** is 1.

Single Rotating Link Inverse Dynamics Singularity Condition

Expanding along the first row, the 3x3 coefficient matrix determinant is:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -r_{12Y} & r_{12X} & 1 \end{vmatrix} = +1\begin{vmatrix} 1 & 0 \\ r_{12X} & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ -r_{12Y} & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ -r_{12Y} & r_{12X} \end{vmatrix} = 1(1(1) - r_{12X}(0)) - 0(0) + 0(r_{12Y}) = 1$$

Since the determinant of the coefficient matrix is always 1 and not dependent on the system variables, this solution can never be singular. This result is validated by the three scalar solutions – nothing could possibly go wrong with these solutions mathematically.

Terms for the inverse dynamics equations

The inverse dynamics problem has been solved analytically for the single rotating link. How do we calculate the various terms that appear in the dynamics equations? These all must be derived from given information.

$$\underline{A}_G = \begin{Bmatrix} A_{GX} \\ A_{GY} \end{Bmatrix} =$$

$$\underline{F}_E = \begin{Bmatrix} F_{EX} \\ F_{EY} \end{Bmatrix} =$$

$$\underline{r}_{12} = \begin{Bmatrix} r_{12X} \\ r_{12Y} \end{Bmatrix} = \qquad \underline{r}_E = \begin{Bmatrix} r_{EX} \\ r_{EY} \end{Bmatrix} =$$

$$I_{GZ} =$$

Step 7. Calculate Shaking Force and Moment

After the inverse dynamics problem is solved, we can calculate the vector shaking force and moment, which is the force/moment reaction on the ground link due to the mechanism inertia and weight, kinematic motion, driving torque (or force), and external forces/moments.

Single rotating link inverse dynamics example

Given: $L = 1$, $h = 0.1$ m, $m = 2$ kg, $\omega = 100$ rad/s, $\alpha = 0$, $F_E = 150$ N, $\phi_E = 0$, $M_E = 0$ Nm.

Calculated terms $\quad \|r_{12}\| = \|r_E\| = 0.5 \text{ m} \qquad I_{GZ} = 0.17 \text{ kgm}^2$

Snapshot inverse dynamics analysis

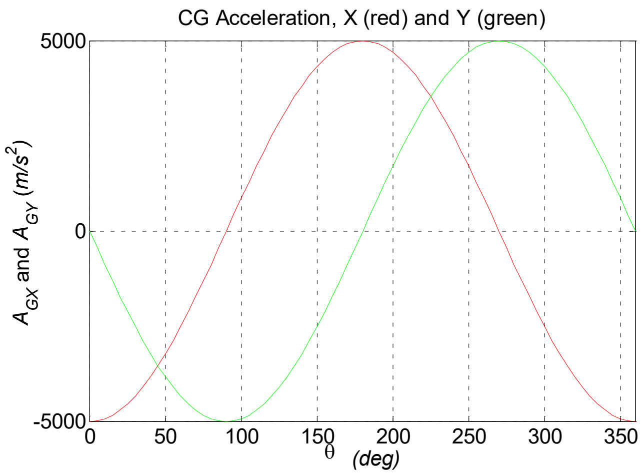
At $\theta = 150^\circ$, given this link, motion, and external force, calculate F_{12X}, F_{12Y}, τ and $\underline{F}_S, \underline{M}_S$.

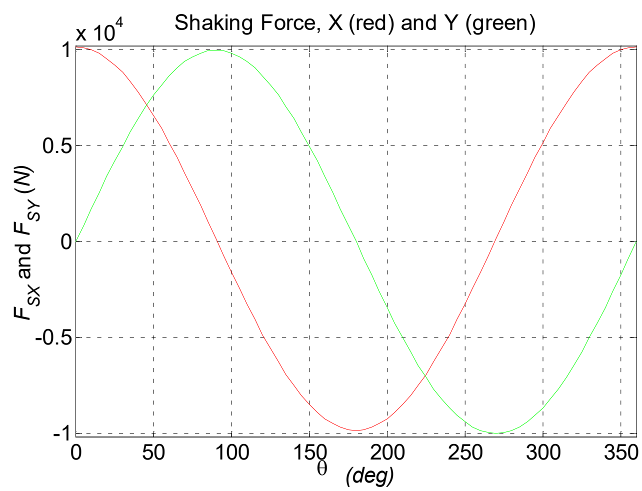
$$\begin{aligned} A_{Gx} &= 4330 \\ A_{Gy} &= -2500 \end{aligned} \quad \frac{\text{m}}{\text{s}^2} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.250 & 0.433 & 1 \end{bmatrix} \begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{Bmatrix} 8510 \\ -4980 \\ 37.5 \end{Bmatrix} \qquad \begin{Bmatrix} F_{12X} \\ F_{12Y} \\ \tau \end{Bmatrix} = \begin{Bmatrix} 8510 \\ -4980 \\ 66.5 \end{Bmatrix} \text{ N, Nm}$$

$$\underline{F}_S = \underline{F}_{21} = -\underline{F}_{12} = \begin{Bmatrix} -8510 \\ 4980 \end{Bmatrix} \text{ N} \qquad \underline{M}_S = -\underline{\tau} = -66.5\hat{k} \text{ Nm}$$

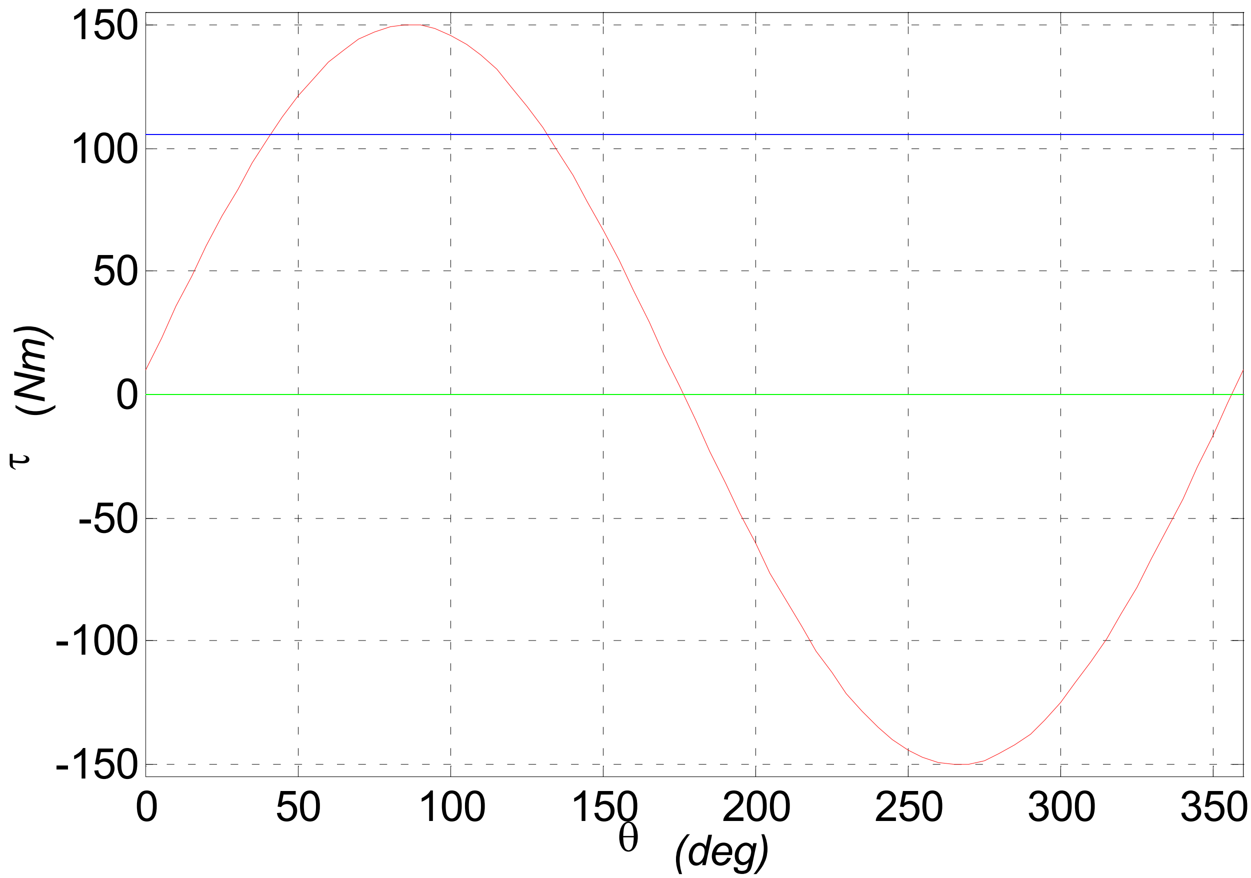
Full-Range-Of-Motion (F.R.O.M.) Analysis

A more meaningful result from inverse dynamics analysis is to solve and plot the dynamics unknowns for the entire range of mechanism motion. For the same example as the snapshot we specify that the given ω is constant. Prior to solving the inverse dynamics problem, the *CG* translational acceleration results for all $0^\circ \le \theta \le 360^\circ$ are given in the left plot below. The *X* components are red and the *Y* green. Is the static link weight (*mg*) significant in this problem? The right plot above gives the Shaking Force $\underline{F}_S$ for all $0^\circ \le \theta \le 360^\circ$. The *X* component is red and the *Y* green. The Shaking Moment $\underline{M}_S$ is simply the negative of the driving torque $\underline{\tau}$ plot shown next.





The plot below gives the required driving torque $\underline{\tau}$ (Nm, red) for all $0^\circ \le \theta \le 360^\circ$, assuming the given ω is constant, for the same example as the snapshot. This shows the torque that must be supplied by an external DC servomotor to cause the specified motion. Also plotted is the average torque (green) $\tau_{AVG} = 0$ and the root-mean-square (RMS) torque value (blue) $\tau_{RMS} = 106.1$ Nm.



Single Rotating Link Torque τ Results with average and RMS

Here are the calculations for the average and root-mean-square torques.

$$\tau_{AVG} = \frac{\tau_0 + \tau_1 + \tau_2 + \cdots + \tau_k}{k+1} \qquad \tau_{RMS} = \sqrt{\frac{\tau_0^2 + \tau_1^2 + \tau_2^2 + \cdots + \tau_k^2}{k+1}}$$

where k+1 is the total number of elements in the τ array (since the counting index k starts at zero). MATLAB can be used to calculate and plot the average and root-mean-square torques on the plot of τ for easy comparison.

```
tauAVG = mean(tau);                     % after the for loop
tauRMS = norm(tau)/sqrt(k+1);
wuns   = ones(1,length(th));            % to plot a constant line
plot(th/DR,tau,'r',th/DR,tauAVG*wuns,'g',th/DR,tauRMS*wuns,'b');
```

6.4 Four-Bar Mechanism Inverse Dynamics Analysis

Generic Mechanism Inverse Dynamics Analysis Problem Statement

Given: the mechanism, external forces and moments, and the desired mechanism motion.

Find: the required driving force (or torque) and internal joint forces.

We will apply the **Matrix Method** to solve the inverse dynamics problem for the four-bar mechanism.

Step 1. The four-bar mechanism **Position, Velocity, and Acceleration Analyses** must first be complete.

Step 2. Draw the four-bar mechanism diagrams

Physical dynamics diagram. Generally there are no external forces/moments on the input link 2.

Free body diagrams (FBDs)

$\underline{F}_{ij}$ unknown vector internal joint force of link *i* acting on link *j*.

$\underline{r}_{ij}$ known moment arm vector pointing to the joint connection with link *i* from the *CG* of link *j*.

<u>Step 3.</u> <u>State the Problem:</u> Four-Bar Mechanism Inverse Dynamics Analysis Problem Statement

Given: the mechanism $(r_1, \theta_1, r_2, r_3, r_4, m_2, m_3, m_4, CG_2, CG_3, CG_4, I_{GZ2}, I_{GZ3}, I_{GZ4})$, kinematic motion $\theta_2, \theta_3, \theta_4, \omega_2, \omega_3, \omega_4, \alpha_2, \alpha_3, \alpha_4, \underline{A}_{G2}, \underline{A}_{G3}, \underline{A}_{G4}$, and external forces/moments $\underline{F}_{E3}, \underline{F}_{E4}$ and $\underline{M}_{E3}, \underline{M}_{E4}$.

Find: the driving torque $\underline{\tau}_2$ and internal joint forces $\underline{F}_{21}, \underline{F}_{32}, \underline{F}_{43}, \underline{F}_{14}$.

First, can we simplify and solve the problem link-by-link, like the single rotating link inverse dynamics? Count the number of scalar unknowns and the number of scalar equations.

<u>Step 4.</u> <u>Derive the Newton-Euler Dynamics Equations</u>

Newton's Second Law

<u>Link 2</u>

$$\sum \underline{F}_2 = \underline{F}_{32} - \underline{F}_{21} + \underline{W}_2 = m_2 \underline{A}_{G2}$$

<u>Link 3</u>

$$\sum \underline{F}_3 \Rightarrow$$

<u>Link 4</u>

$$\sum \underline{F}_4 = \underline{F}_{14} - \underline{F}_{43} + \underline{W}_4 + \underline{F}_{E4} = m_4 \underline{A}_{G4}$$

Euler's Rotational Dynamics Equation

<u>Link 2</u>

$$\sum \underline{M}_{G2} = \underline{\tau}_2 + \underline{r}_{32} \times \underline{F}_{32} - \underline{r}_{12} \times \underline{F}_{21} = I_{G2Z} \underline{\alpha}_2$$

<u>Link 3</u>

$$\sum \underline{M}_{G3} \Rightarrow$$

<u>Link 4</u>

$$\sum \underline{M}_{G4} = \underline{r}_{14} \times \underline{F}_{14} - \underline{r}_{34} \times \underline{F}_{43} + \underline{M}_{E4} + \underline{r}_{E4} \times \underline{F}_{E4} = I_{G4Z} \underline{\alpha}_4$$

Step 5. **Derive the *XYZ* scalar dynamics equations** from the vector dynamics equations.

For each moving link we obtain

- Two *XY* force component equations from Newton's Second Law
- One *Z* moment equation from Euler's Rotational Dynamics Equation

Link 2

$$F_{32X} - F_{21X} = m_2 A_{G2X}$$

$$F_{32Y} - F_{21Y} = m_2 (A_{G2Y} + g)$$

$$\tau_2 + (r_{32X} F_{32Y} - r_{32Y} F_{32X}) - (r_{12X} F_{21Y} - r_{12Y} F_{21X}) = I_{G2Z} \alpha_2$$

Link 3

Link 4

$$F_{14X} - F_{43X} = m_4 A_{G4X} - F_{E4X}$$

$$F_{14Y} - F_{43Y} = m_4 (A_{G4Y} + g) - F_{E4Y}$$

$$(r_{14X} F_{14Y} - r_{14Y} F_{14X}) - (r_{34X} F_{43Y} - r_{34Y} F_{43X}) = I_{G4Z} \alpha_4 - M_{E4} - r_{E4X} F_{E4Y} + r_{E4Y} F_{E4X}$$

Step 5. Derive the *XYZ* scalar dynamics equations (cont.)

Write these *XYZ* scalar equations in matrix/vector form.

Four-bar mechanism inverse dynamics matrix equation

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ r_{12Y} & -r_{12X} & -r_{32Y} & r_{32X} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & r_{23Y} & -r_{23X} & -r_{43Y} & r_{43X} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & r_{34Y} & -r_{34X} & -r_{14Y} & r_{14X} & 0 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14X} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} m_2 A_{G2X} \\ m_2 (A_{G2Y} + g) \\ I_{G2Z}\alpha_2 \\ m_3 A_{G3X} - F_{E3X} \\ m_3 (A_{G3Y} + g) - F_{E3Y} \\ I_{G3Z}\alpha_3 - r_{E3X}F_{E3Y} + r_{E3Y}F_{E3X} - M_{E3} \\ m_4 A_{G4X} - F_{E4X} \\ m_4 (A_{G4Y} + g) - F_{E4Y} \\ I_{G4Z}\alpha_4 - r_{E4X}F_{E4Y} + r_{E4Y}F_{E4X} - M_{E4} \end{Bmatrix}$$

$$[A]\{v\} = \{b\}$$

Step 6. Solve for the unknowns

The coefficient matrix $[A]$ is dependent on geometry (through the moment arms, which are dependent on the angles from kinematics solutions). The known vector $\{b\}$ is dependent on inertial terms, gravity, and the given external forces and moments. $\{v\}$ is the vector of unknowns.

Solution by matrix inversion $\{v\} = [A]^{-1}\{b\}$

MATLAB
```
v = inv(A)*b;  % Solution via matrix inverse
```

Using Gaussian elimination is more efficient and robust to solve for **v**.

MATLAB
```
v = A\b;       % Solution via Gaussian elimination
```

The solution to the unknown internal forces and input torque are contained in the components of **v**. To save these values for plotting later, use the following MATLAB code, inside the **for i** loop.

```
F21x(i) = v(1);
F21y(i) = v(2);
   :         :
   :         :
tau2(i) = v(9);
```

See the on-line ME 3011 Supplement for an alternate, more efficient solution method (people.ohio.edu/williams/html/PDF/Supplement3011.pdf) for the four-bar mechanism inverse dynamics equations.

Step 7. Calculate Shaking Force and Moment

After the basic four-bar mechanism inverse dynamics problem is solved, we can calculate the vector shaking force and moment, which is the force/moment reaction on the ground link due to the four-bar mechanism motion, weight, inertia, driving torque (or force), and external forces/moments.

Details for the general four-bar mechanism model

The inverse dynamics problem has been derived analytically and solved numerically for the four-bar mechanism. Now, how do we calculate the various terms that appear in the dynamics equations? These all must be derived from given information prior to the Matrix Method solution. Here is a general link 3 diagram for these derivations (see the following page for complete link 2 and 4 information).

Link 3 details

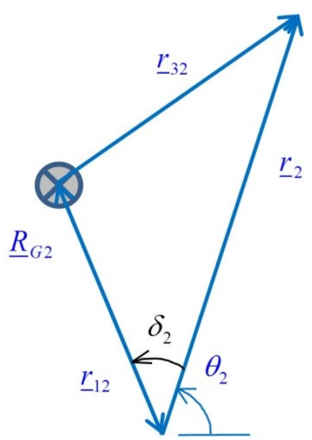
$$\underline{r}_{23} = \begin{Bmatrix} r_{23X} \\ r_{23Y} \end{Bmatrix} = \qquad\qquad \underline{r}_{43} = \begin{Bmatrix} r_{43X} \\ r_{43Y} \end{Bmatrix} =$$

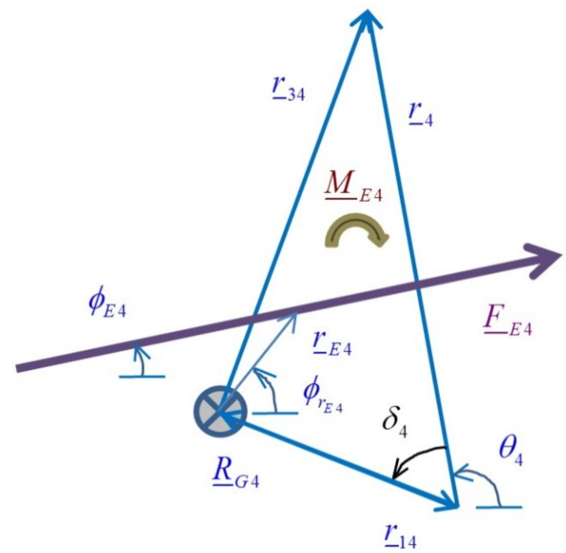
$$\underline{A}_{G3} = \begin{Bmatrix} A_{G3X} \\ A_{G3Y} \end{Bmatrix} = \begin{Bmatrix} -r_2\alpha_2 \sin\theta_2 - r_2\omega_2^2 \cos\theta_2 - R_{G3}\alpha_3 \sin(\theta_3+\delta_3) - R_{G3}\omega_3^2 \cos(\theta_3+\delta_3) \\ r_2\alpha_2 \cos\theta_2 - r_2\omega_2^2 \sin\theta_2 + R_{G3}\alpha_3 \cos(\theta_3+\delta_3) - R_{G3}\omega_3^2 \sin(\theta_3+\delta_3) \end{Bmatrix}$$

$$\underline{F}_{E3} = \begin{Bmatrix} F_{E3X} \\ F_{E3Y} \end{Bmatrix} = \begin{Bmatrix} F_{E3}\cos\phi_{E3} \\ F_{E3}\sin\phi_{E3} \end{Bmatrix} \quad \underline{r}_{E3} = \begin{Bmatrix} r_{E3X} \\ r_{E3Y} \end{Bmatrix} = \begin{Bmatrix} r_{E3}\cos\phi_{r_{E3}} \\ r_{E3}\sin\phi_{r_{E3}} \end{Bmatrix} \qquad \underline{M}_{E3} = M_{E3}\hat{k} \quad \text{(given)}$$

Here are the link 2 and 4 details for the four-bar inverse dynamics matrix given above.

Link 2 details **Link 4 details**





$$\underline{r}_{12} = \begin{Bmatrix} r_{12X} \\ r_{12Y} \end{Bmatrix} = \begin{Bmatrix} -R_{G2}\cos(\theta_2+\delta_2) \\ -R_{G2}\sin(\theta_2+\delta_2) \end{Bmatrix}$$

$$\underline{r}_{14} = \begin{Bmatrix} r_{14X} \\ r_{14Y} \end{Bmatrix} = \begin{Bmatrix} -R_{G4}\cos(\theta_4+\delta_4) \\ -R_{G4}\sin(\theta_4+\delta_4) \end{Bmatrix}$$

$$\underline{r}_{32} = \underline{r}_{12} + \underline{r}_2 = \begin{Bmatrix} r_{12X} + r_2\cos\theta_2 \\ r_{12Y} + r_2\sin\theta_2 \end{Bmatrix}$$

$$\underline{r}_{34} = \underline{r}_{14} + \underline{r}_4 = \begin{Bmatrix} r_{14X} + r_4\cos\theta_4 \\ r_{14Y} + r_4\sin\theta_4 \end{Bmatrix}$$

$$\underline{A}_{G2} = \begin{Bmatrix} A_{G2X} \\ A_{G2Y} \end{Bmatrix} = \begin{Bmatrix} -R_{G2}\alpha_2\sin(\theta_2+\delta_2) - R_{G2}\omega_2^2\cos(\theta_2+\delta_2) \\ R_{G2}\alpha_2\cos(\theta_2+\delta_2) - R_{G2}\omega_2^2\sin(\theta_2+\delta_2) \end{Bmatrix}$$

$$\underline{A}_{G4} = \begin{Bmatrix} A_{G4X} \\ A_{G4Y} \end{Bmatrix} = \begin{Bmatrix} -R_{G4}\alpha_4\sin(\theta_4+\delta_4) - R_{G4}\omega_4^2\cos(\theta_4+\delta_4) \\ R_{G4}\alpha_4\cos(\theta_4+\delta_4) - R_{G4}\omega_4^2\sin(\theta_4+\delta_4) \end{Bmatrix}$$

$$\underline{F}_{E4} = \begin{Bmatrix} F_{E4X} \\ F_{E4Y} \end{Bmatrix} = \begin{Bmatrix} F_{E4}\cos\phi_{E4} \\ F_{E4}\sin\phi_{E4} \end{Bmatrix}$$

$$\underline{r}_{E4} = \begin{Bmatrix} r_{E4X} \\ r_{E4Y} \end{Bmatrix} = \begin{Bmatrix} r_{E4}\cos\phi_{r_{E4}} \\ r_{E4}\sin\phi_{r_{E4}} \end{Bmatrix}$$

$$\underline{M}_{E4} = M_{E4}\hat{k}$$

Four-Bar mechanism inverse dynamics singularity condition

The same kinematic singularity condition from the four-bar mechanism position, velocity, and acceleration kinematics problems, when $\theta_4 - \theta_3 = 0^\circ, 180^\circ, \ldots$, is also a problem for inverse dynamics, causing a singular dynamics coefficient matrix [A]. This case corresponds to zero transmission angle μ and a link 2 joint limit. This condition should be avoided in the real world. This singularity condition never occurs when the input link is a crank.

If your 4-bar mechanism has a crank input but your MATLAB inverse dynamics results still experience singularity (**`matrix is singular to working precision`**), try the following:

1. Double-check the [A] matrix to ensure all entries agree with the matrix/vector equation given earlier. If you have an entire column (or row) of zeros, that matrix will always be singular.

2. Check all of your moment arm terms $\underline{r}_{ij} = \{ r_{ijX} \quad r_{ijY} \}^T$ – do a snapshot drawing and compare your MATLAB snapshot moment arm terms with the drawing.

3. Check your snapshot result with the four-bar mechanism dynamics example given below (**Term Example 1** continued).

If your 4-bar mechanism has a rocker input with two joint limits, your MATLAB inverse dynamics results will experience singularity when the input is a either of the joint limits. To fix this problem, simply do not simulate the mechanism all the way into these limits, but back off slightly, remaining in the valid input angle region.

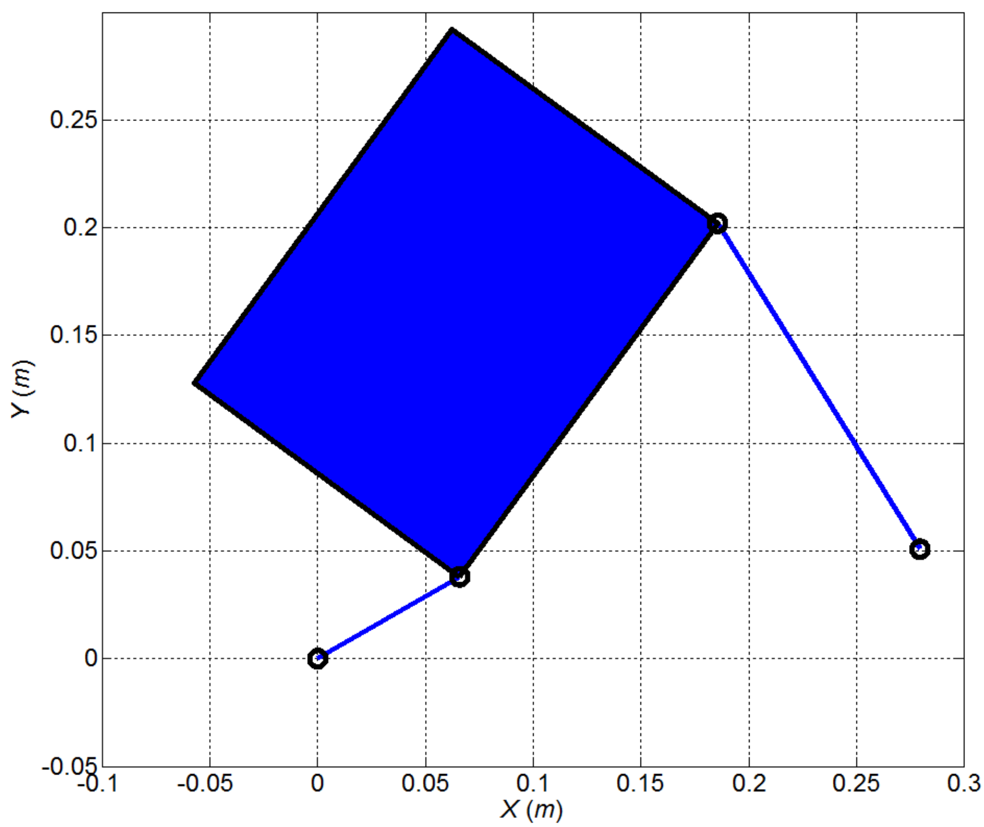
Four-bar mechanism inverse dynamics example – Term Example 1 continued

This is the mechanism from Term Example 1 (open branch), with kinematics solutions as presented before. Given $r_1 = 0.284$, $r_2 = 0.076$, $r_3 = 0.203$, $r_4 = 0.178$, $\theta_1 = 10.3^\circ, \theta_2 = 30^\circ, \theta_3 = 53.8^\circ, \theta_4 = 121.7^\circ$, $R_{G2} = 0.038$, $R_{G3} = 0.127$, $R_{G4} = 0.089$ (m), $\delta_2 = 0, \delta_3 = 36.9^\circ, \delta_4 = 0$, $\omega_2 = 20, \omega_3 = -8.09, \omega_4 = -3.73$ (rad/s), and $\alpha_2 = 0, \alpha_3 = 8.65, \alpha_4 = 244.4$ (rad/s^2).

All moving links are wood, with mass density $\rho = 830.4$ kg/m^3. Links 2 and 4 have rectangular dimensions r_i x 0.019 x 0.013 thick (m; $i = 2,4$); link 3 has rectangular dimensions 0.203 x 0.152 x 0.013 thick (m), as shown on the previous page. The calculated mass and inertia parameters are $m_2 = 0.015, m_3 = 0.327, m_4 = 0.036$ (kg) and $I_{G2Z} = 7.9\times10^{-6}, I_{G3Z} = 1.8\times10^{-3}, I_{G4Z} = 9.5\times10^{-5}$ (kgm^2). All external forces and moments are zero but gravity, $g = 9.81$ m/s^2, is included.

Figure for Term Example 1 Inverse Dynamics

The coupler link 3 is a rectangle of dimensions 0.203 x 0.152 (m). The triangle tip we have been using all along in Term Example 1 (previously called point C) is the CG of the rectangular link shown below for inverse dynamics.



Snapshot Analysis (one input angle)

At $\theta_2 = 30^\circ$, given this mechanism and motion, calculate the four vector internal joint forces, the driving torque $\underline{\tau}_2$, and the shaking force and moment $\underline{F}_S, \underline{M}_S$ for this snapshot.

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.019 & 0.033 & -0.019 & 0.033 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.127 & -0.002 & -0.037 & 0.122 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.076 & 0.047 & 0.076 & 0.047 & 0 \end{bmatrix} \begin{Bmatrix} F_{21x} \\ F_{21y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} -0.202 \\ 0.034 \\ 0 \\ -8.955 \\ -4.497 \\ 0.015 \\ -0.638 \\ -0.095 \\ 0.023 \end{Bmatrix}$$

The solution is accomplished by Gaussian elimination, or $\{v\} = [A]^{-1}\{b\}$, or by the reduced 6x6 plus decoupled link 2 method (see the on-line ME 3011 Supplement). All methods yield the same results. Snapshot answers:

$$\{v\} = \begin{Bmatrix} F_{21x} \\ F_{21y} \\ F_{32x} \\ F_{32y} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} 6.20 \\ 10.08 \\ 5.99 \\ 10.11 \\ -2.96 \\ 5.61 \\ -3.60 \\ 5.52 \\ -0.43 \end{Bmatrix} \; (N, Nm) \qquad \underline{F}_S = \begin{Bmatrix} 9.80 \\ 4.56 \end{Bmatrix} \; (N) \qquad \underline{M}_S = -1.68\hat{k} \; (Nm)$$

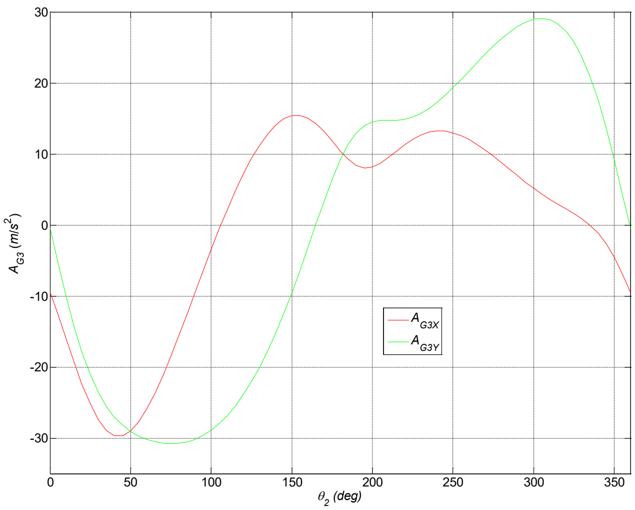
Full-Range-Of-Motion (F.R.O.M.) Analysis – Term Example 1 continued

A more meaningful result from inverse dynamics analysis is to solve and plot the dynamics unknowns for the entire range of four-bar mechanism motion. Prior to solving the inverse dynamics problem, the left plot below shows the *CG* translational acceleration results for link three for all $0^\circ \le \theta_2 \le 360^\circ$. The *X* component is red and the *Y* component is green.

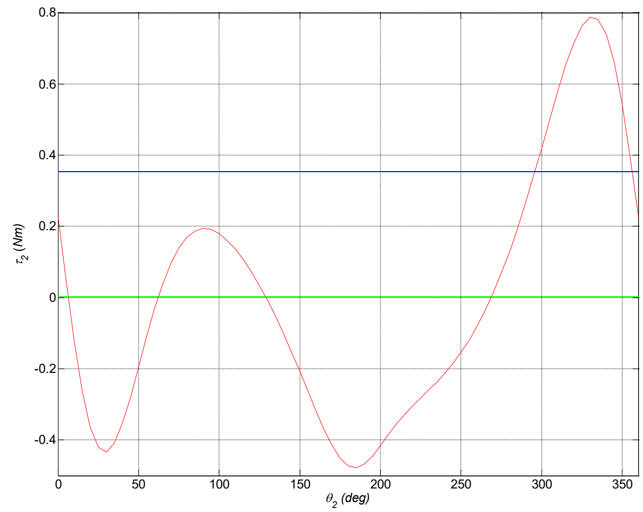
The right plot below gives the required driving torque $\underline{\tau}_2$ (*Nm*) for all $0^\circ \le \theta_2 \le 360^\circ$, for the Term Example 1 mechanism, assuming the given $\omega_2 = 20$ *rad/s* is constant. This plot shows the torque (red) that must be supplied in all configurations by an external DC servomotor to cause the specified motion. Also plotted is the average torque (green) $\tau_{2AVG} = 0.003$ and the root-mean-square torque value (blue)

$\tau_{2RMS} = 0.354$ Nm. The root-mean-square (RMS) torque is more meaningful than the average torque since its terms do not cancel each other ($k + 1$ is the number of elements in the τ_2 array).

$$\tau_{2RMS} = \sqrt{\frac{\tau_{2_0}^2 + \tau_{2_1}^2 + \tau_{2_2}^2 + \cdots + \tau_{2_k}^2}{k+1}}$$



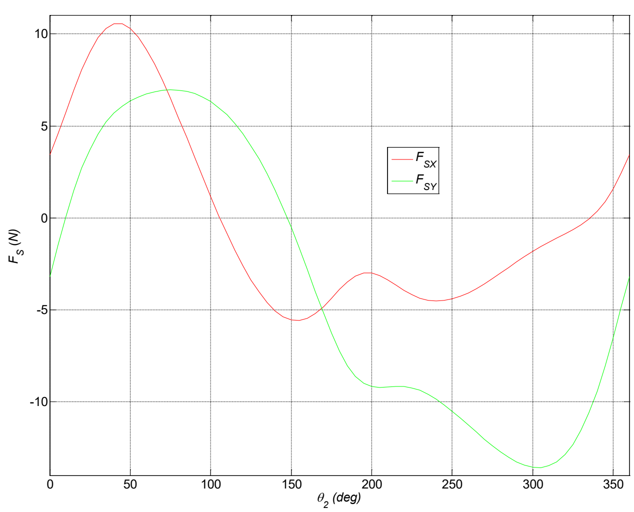
CG3 Translational Acceleration



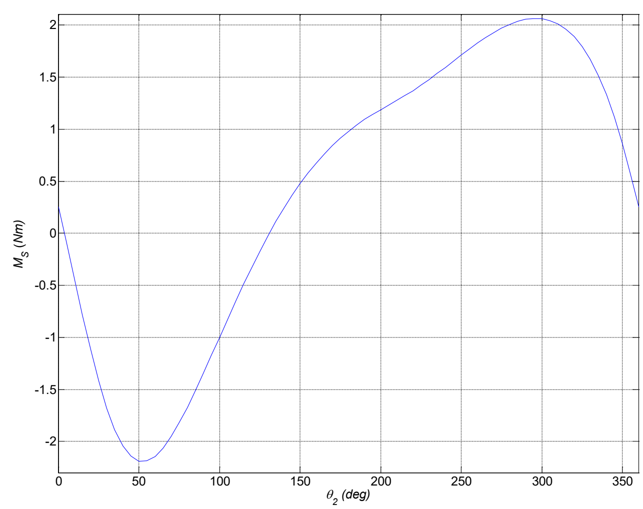
Input Torque

Term Example 1 Inverse Dynamics Results

The left plot below gives the shaking force $\underline{F}_S$ (N) results for all $0^\circ \le \theta_2 \le 360^\circ$. The X component is red and the Y component is green. The right plot below gives the shaking moment $\underline{M}_S$ (Nm) results for all $0^\circ \le \theta_2 \le 360^\circ$. There is only the Z component since a planar moment is a $\hat{k}$ vector.



Shaking Force



Shaking Moment

Term Example 1 Inverse Dynamics Results

6.5 Slider-Crank Mechanism Inverse Dynamics Analysis

Again we will address the air compressor problem. This problem is very similar to the four-bar mechanism inverse dynamics problem. In fact, links 2 and 3 are handled identically. We will again apply the **Matrix Method** to solve the inverse dynamics problem for the slider-crank mechanism.

<u>Step 1.</u> The slider-crank mechanism **<u>Position, Velocity, and Acceleration Analyses</u>** must be complete.

<u>Step 2.</u> **<u>Draw the slider-crank mechanism link 4 FBD</u>**

Link 4 free body diagram (the link 2 and 3 FBDs are identical to the four-bar mechanism):

There are two kinematic constraints on the slider, link 4:

<u>Step 3.</u> **<u>State the Problem:</u> Slider-Crank Mechanism Inverse Dynamics Analysis Problem Statement**

Given: the mechanism ($r_2, r_3, h, m_2, m_3, m_4, CG_2, CG_3, CG_4, I_{GZ2}, I_{GZ3}, I_{GZ4}$), kinematic motion $\theta_2, \theta_3, x, \omega_2, \omega_3, \dot{x}, \alpha_2, \alpha_3, \ddot{x}, \underline{A}_{G2}, \underline{A}_{G3}, \underline{A}_{G4}$, and external forces/moments $\underline{F}_{E3}, \underline{F}_{E4}$ and $\underline{M}_{E3}, \underline{M}_{E4}$.

Find: the driving torque $\underline{\tau}_2$ and internal joint forces $\underline{F}_{21}, \underline{F}_{32}, \underline{F}_{43}, \underline{F}_{14}$.

Step 4. Derive the Newton-Euler Dynamics Equations

Again, links 2 and 3 are identical to the four-bar mechanism.

Newton's Second Law

Link 2

$$\sum \underline{F}_2 = \underline{F}_{32} - \underline{F}_{21} + \underline{W}_2 = m_2 \underline{A}_{G2}$$

Link 3

$$\sum \underline{F}_3 = \underline{F}_{43} - \underline{F}_{32} + \underline{W}_3 + \underline{F}_{E3} = m_3 \underline{A}_{G3}$$

Link 4

$$\sum \underline{F}_4 \Rightarrow$$

Euler's Rotational Dynamics Equation

Link 2

$$\sum \underline{M}_{G2} = \underline{\tau}_2 + \underline{r}_{32} \times \underline{F}_{32} - \underline{r}_{12} \times \underline{F}_{21} = I_{G2Z} \underline{\alpha}_2$$

Link 3

$$\sum \underline{M}_{G3} = \underline{r}_{43} \times \underline{F}_{43} - \underline{r}_{23} \times \underline{F}_{32} + \underline{M}_{E3} + \underline{r}_{E3} \times \underline{F}_{E3} = I_{G3Z} \underline{\alpha}_3$$

Link 4

$$\sum \underline{M}_{G4} \Rightarrow$$

Step 5. Derive the *XYZ* scalar dynamics equations

Links 2 and 3 are identical to the four-bar mechanism inverse dynamics equations.

Link 2

$$F_{32X} - F_{21X} = m_2 A_{G2X}$$

$$F_{32Y} - F_{21Y} = m_2 (A_{G2Y} + g)$$

$$\tau_2 + (r_{32X} F_{32Y} - r_{32Y} F_{32X}) - (r_{12X} F_{21Y} - r_{12Y} F_{21X}) = I_{G2Z} \alpha_2$$

Link 3

$$F_{43X} - F_{32X} = m_3 A_{G3X} - F_{E3X}$$

$$F_{43Y} - F_{32Y} = m_3 (A_{G3Y} + g) - F_{E3Y}$$

$$(r_{43X} F_{43Y} - r_{43Y} F_{43X}) - (r_{23X} F_{32Y} - r_{23Y} F_{32X}) = I_{G3Z} \alpha_3 - M_{E3} - r_{E3X} F_{E3Y} + r_{E3Y} F_{E3X}$$

Link 4

Count the number of scalar unknowns and the number of scalar equations (therefore, we need an additional equation (or one less unknown):

Step 5. Derive the *XYZ* scalar dynamics equations (cont.)

Write these *XYZ* scalar equations in matrix/vector form. Substitute the friction constraint to eliminate one unknown (F_{14X}). Also eliminate one equation ($\sum \underline{M}_{G4} = I_{G4Z}\underline{\alpha}_4$).

Slider-crank mechanism inverse dynamics matrix equation

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ r_{12Y} & -r_{12X} & -r_{32Y} & r_{32X} & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & r_{23Y} & -r_{23X} & -r_{43Y} & r_{43X} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \mp\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} m_2 A_{G2X} \\ m_2(A_{G2Y}+g) \\ I_{G2Z}\alpha_2 \\ m_3 A_{G3X} - F_{E3X} \\ m_3(A_{G3Y}+g) - F_{E3Y} \\ I_{G3Z}\alpha_3 - r_{E3X}F_{E3Y} + r_{E3Y}F_{E3X} - M_{E3} \\ m_4 A_{G4X} - F_{E4X} \\ m_4 g - F_{E4Y} \end{Bmatrix}$$

$$[A]\{v\} = \{b\}$$

Step 6. Solve for the unknowns

The coefficient matrix $[A]$ is dependent on geometry (via the moment arms, which are dependent on the moving angles from kinematics). Always choose the proper sign of μ to be opposite to the current $\dot{x}$ direction: **`A(7,7) = -sign(xd(i))*mu`**. The known vector $\{b\}$ is dependent on inertial terms, gravity, and the given external forces and moments. $\{v\}$ is the vector of unknowns.

Solution by matrix inversion $\{v\} = [A]^{-1}\{b\}$

MATLAB
```
v = inv(A)*b;   % Solution via matrix inverse
```

Using Gaussian elimination is more efficient and robust to solve for **v**.

MATLAB
```
v = A\b;        % Solution via Gaussian elimination
```

The solution to the unknown internal forces and input torque are contained in the components of **v**. To save these values for later plotting, use the following MATLAB code, inside the **for i** loop.

```
F21x(i) = v(1);
F21y(i) = v(2);
   ⋮        ⋮
tau2(i) = v(8);
```

After **F14y** (the seventh term of **v**) is solved from the 8x8 matrix/vector system of linear equations, the remaining unknown **F14x** is found from the Coulomb friction constraint, inside the **for** loop.

```
F14x(i) = -sign(xd(i))*mu*F14y(i)
```

See the on-line ME 3011 Supplement for an alternate, more efficient solution method (people.ohio.edu/williams/html/PDF/Supplement3011.pdf) for the slider-crank mechanism inverse dynamics equations.

Step 7. Calculate Shaking Force and Moment

After the basic inverse dynamics problem is solved, we can calculate the vector shaking force and moment, which is the force/moment reaction on the ground link due to the slider-crank mechanism motion, weight, inertia, driving torque (or force), and external forces/moments.

Slider-crank mechanism singularity condition

This is the same kinematic singularity condition from the slider-crank mechanism position, velocity, and acceleration problems. The singularity condition $\theta_3 = 90^\circ, 270^\circ, \cdots$ is also a problem for dynamics, causing a singular dynamics coefficient matrix $[A]$. This case does not exist for standard full-rotation slider-crank mechanisms.

Details for the general slider-crank mechanism model

The inverse dynamics problem has been derived analytically for the slider-crank mechanism. Now, how do we calculate the various terms that appear in the dynamics equations? These all must be derived from given information prior to the Matrix Method numerical solution. The links 2 and 3 terms are identical to those of the four-bar presented earlier. The link 4 terms are easy since all moment arms are zero, there is no external moment, and the external force is generally aligned with the piston.

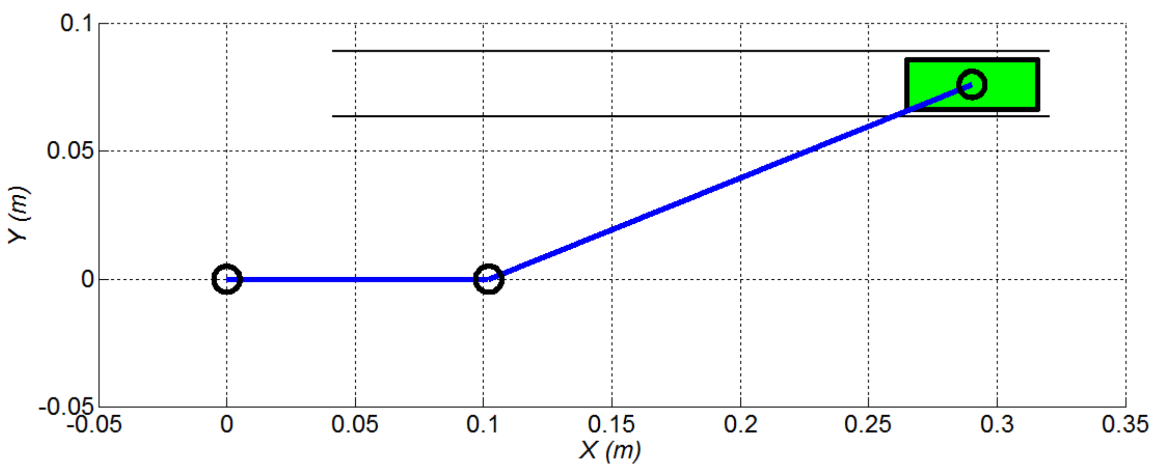
Slider-crank mechanism inverse dynamics example – Term Example 2 continued

This is the mechanism from Term Example 2 (right branch), with kinematics solutions as presented before. Given $r_2 = 0.102$, $r_3 = 0.203$, $h = 0.076\ m$, $\theta_2 = 30^{\circ}, \theta_3 = 7.1^{\circ}$, $x = 0.29\ m$, $\omega_2 = 15\ rad/s$ (constant), $\omega_3 = -6.58\ rad/s$, $\dot{x} = -0.60\ m/s$, $\alpha_2 = 0, \alpha_3 = 62.33\ rad/s^2$, and $\ddot{x} = -30.15\ m/s^2$.

All moving links are wood, with mass density $\rho = 830.4\ kg/m^3$. Links 2 and 3 have rectangular dimensions r_i x 0.019 x 0.013 thick (m; i=2,3); link 4 has rectangular dimensions 0.076 x 0.019 x 0.013 thick (m), as shown on the previous page. The calculated inertia parameters are $m_2 = 0.020$, $m_3 = 0.041$, $m_4 = 0.015$ (kg) and $I_{G2Z} = 1.819\text{e-}005$, $I_{G3Z} = 1.418\text{e-}004$ (kgm^2). The CGs all lie at their respective link centers. There is a constant external force of 1 N acting at the center of the piston end, directed horizontally to the left; gravity is included but all other external forces and moments are zero. We assume the coefficient of kinetic friction between the piston and the fixed wall is $\mu = 0.2$.

Figure for Term Example 2 Inverse Dynamics

The Term Example 2 slider-crank mechanism is shown below at the starting (or ending) position, with zero (or 360°) input angle θ_2.



Snapshot Analysis (one input angle)

At $\theta_2 = 30^\circ$, given this mechanism and motion, calculate the four vector internal joint forces, the driving torque $\underline{\tau}_2$, and the shaking force and moment $\underline{F}_S, \underline{M}_S$ for this instant (snapshot).

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.025 & 0.044 & -0.025 & 0.044 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.013 & 0.101 & -0.013 & 0.101 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} -0.202 \\ 0.084 \\ 0 \\ -1.018 \\ 0.168 \\ 0.009 \\ 0.540 \\ 0.150 \end{Bmatrix}$$

The solution is accomplished by Gaussian elimination, or $\{v\} = [A]^{-1}\{b\}$, or by the reduced 5 x 5 plus decoupled link 2 method (see the on-line ME 3011 Supplement). All methods yield the same results. Snapshot answers:

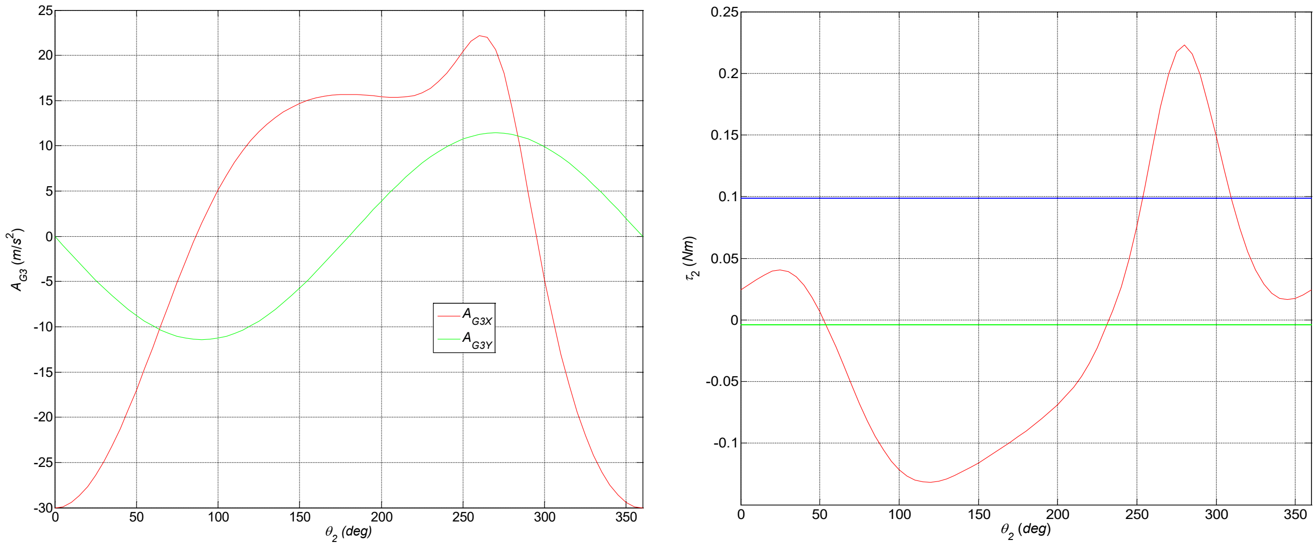
$$\{v\} = \begin{Bmatrix} F_{21X} \\ F_{21Y} \\ F_{32X} \\ F_{32Y} \\ F_{43X} \\ F_{43Y} \\ F_{14Y} \\ \tau_2 \end{Bmatrix} = \begin{Bmatrix} 0.736 \\ -0.121 \\ 0.534 \\ -0.037 \\ -0.484 \\ 0.131 \\ 0.281 \\ 0.039 \end{Bmatrix} \ (N, Nm) \qquad \underline{F}_S = \begin{Bmatrix} 0.680 \\ -0.401 \end{Bmatrix} \ (N) \qquad \underline{M}_S = -0.116\hat{k} \ (Nm)$$

Full-Range-Of-Motion (F.R.O.M.) Analysis – Term Example 2 continued

A more meaningful result from inverse dynamics analysis is to solve and plot the dynamics unknowns for the entire range of mechanism motion. Prior to solving the inverse dynamics problem, the left plot below gives the *CG* translational acceleration results for link 3 for all $0^\circ \le \theta_2 \le 360^\circ$. Here CG_3 is taken as the midpoint of link 3. The *X* component is red and the *Y* component is green.

The right plot below gives the required driving torque τ_2 (*Nm*) for all $0^\circ \le \theta_2 \le 360^\circ$, for the Term Example 2 slider-crank mechanism, right branch only, assuming the given $\omega_2 = 15$ *rad/s* is constant. This plot shows the torque (red) that must be supplied in all configurations by an external DC servomotor to cause the specified motion. Also plotted is the average torque (green) $\tau_{2AVG} = -0.004$ and the root-mean-square torque value (blue) $\tau_{2RMS} = 0.099$ *Nm*. The root-mean-square (RMS) torque is more meaningful than the average torque since its terms do not cancel each other (k+1 is the number of elements in the τ_2 array).

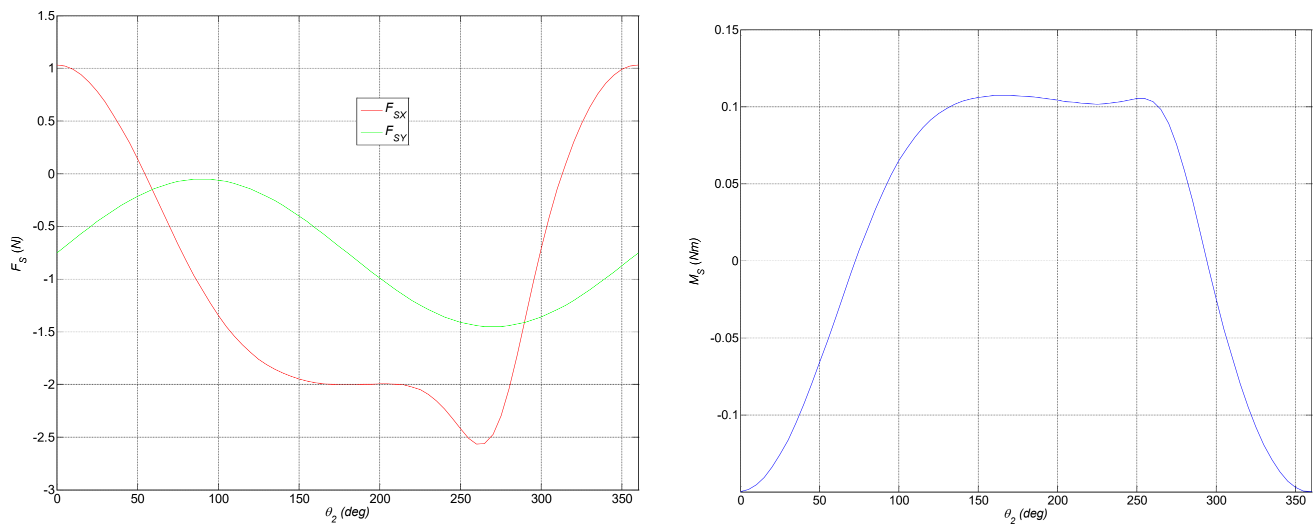
$$\tau_{2RMS} = \sqrt{\frac{\tau_{2_0}^2 + \tau_{2_1}^2 + \tau_{2_2}^2 + \cdots + \tau_{2_k}^2}{k+1}}$$



CG3 Translational Acceleration **Input Torque**

Term Example 2 Inverse Dynamics Results

The left plot below gives the shaking force $\underline{F}_S$ (N) results for all $0^\circ \le \theta_2 \le 360^\circ$. The X component is red and the Y component is green. The right plot below gives the shaking moment $\underline{M}_S$ (Nm) result for all $0^\circ \le \theta_2 \le 360^\circ$. There is only the Z component since a planar moment is a $\hat{k}$ vector.



Shaking Force **Shaking Moment**

Term Example 2 Inverse Dynamics Results

7. Gears and Cams

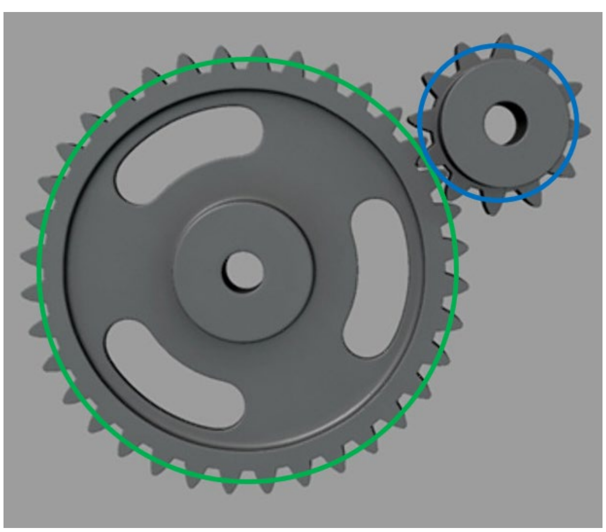
7.1 Gears

7.1.1 Gear Introduction

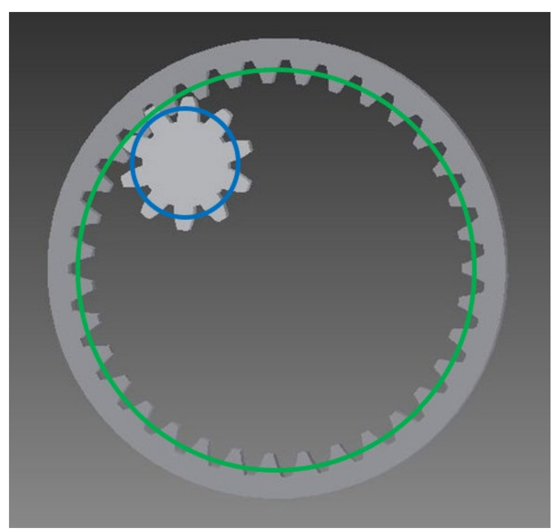
Gears are used to transfer motion between rotating shafts in machinery, mechanisms, robots, vehicles, toys, and other electromechanical systems. Gears cause changes in angular velocity, torque, and direction. Gears are used in various applications, from can openers to aircraft carriers. Belt and chain drives are related to gear mechanisms.

Robot joint example

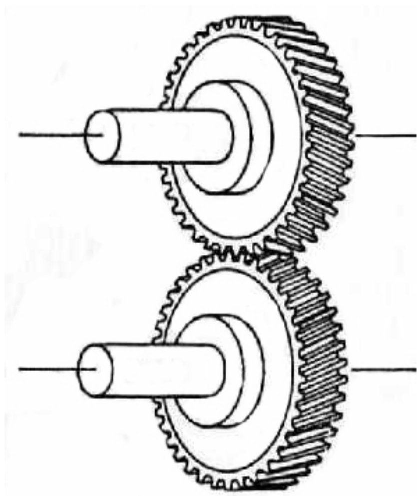
Gear Classification



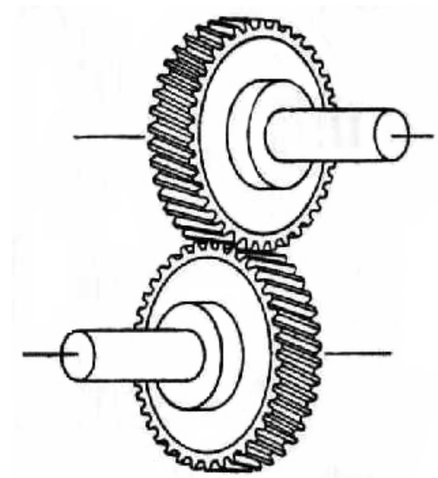
Externally-meshing Spur Gears



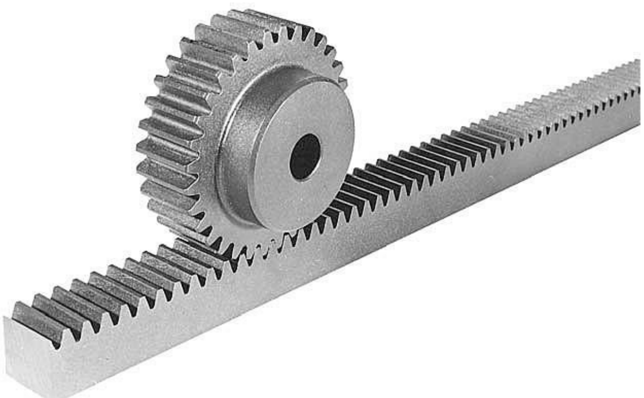
Internally-meshing Spur Gears



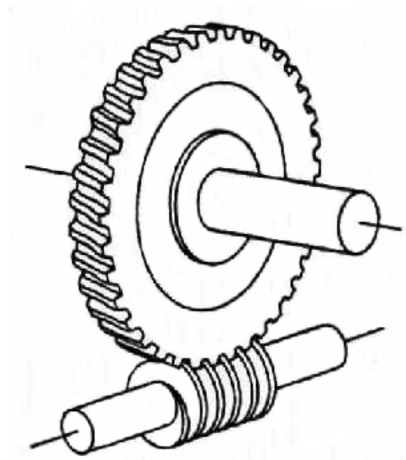
Helical (Parallel Shaft)



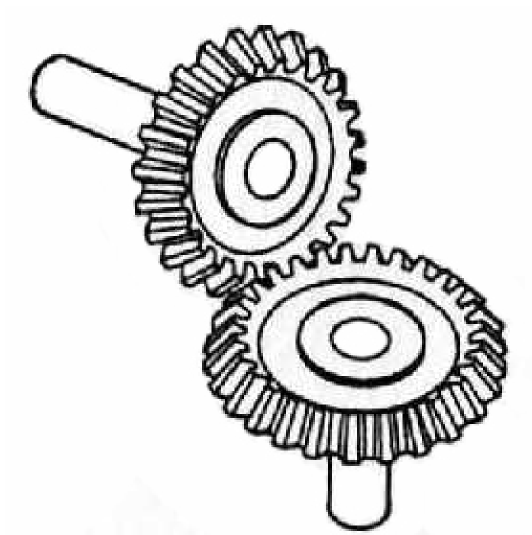
Helical (Crossed Shaft)



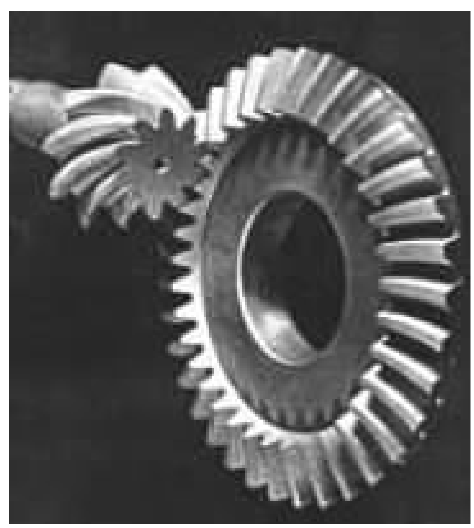
Rack & Pinion



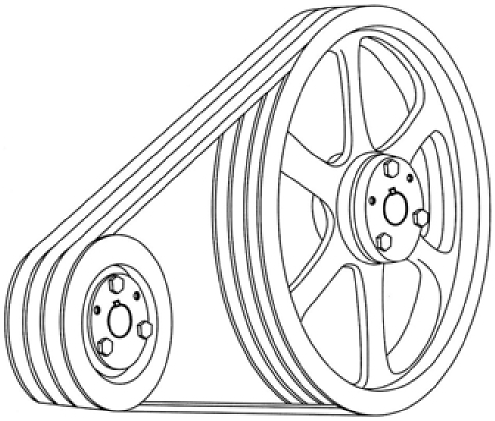
Worm and Gear



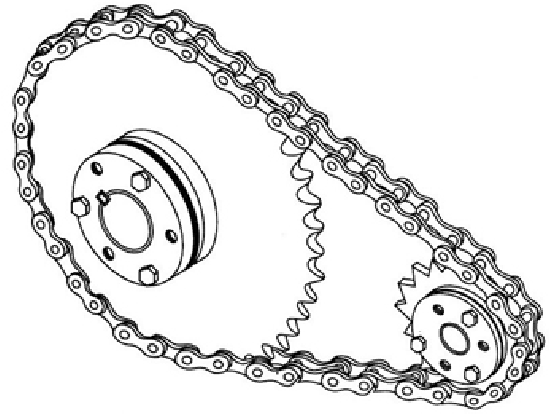
Straight Bevel Gears



Spiral Bevel Gears
(Automotive Hypoid Gears)

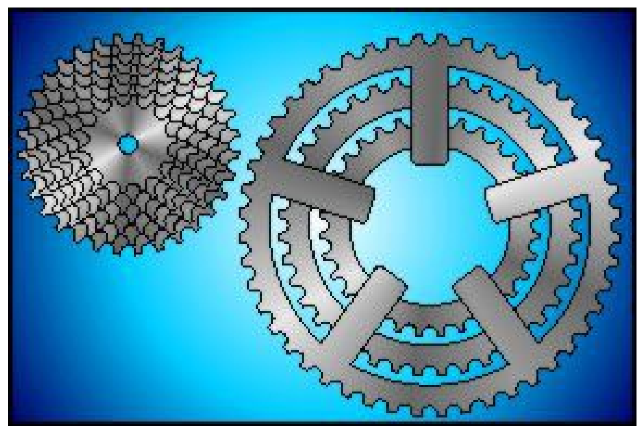


V-Belt Drive

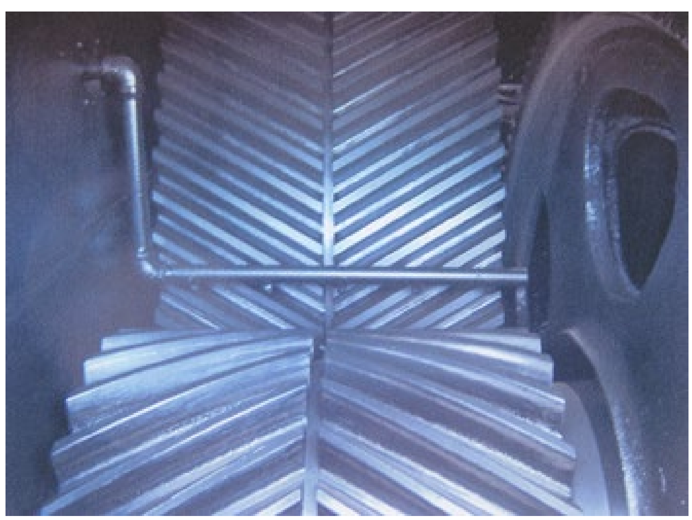


Chain Drive

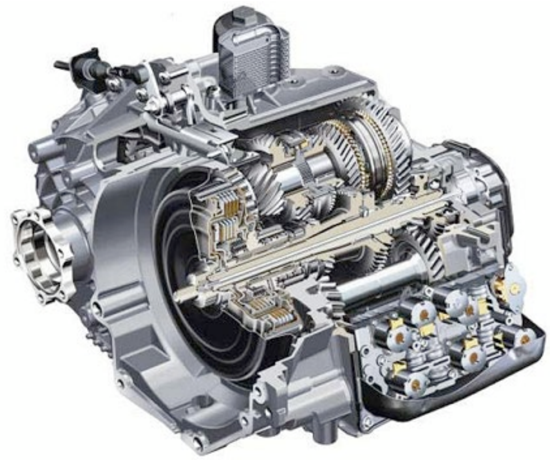
Toothed Belt Drive



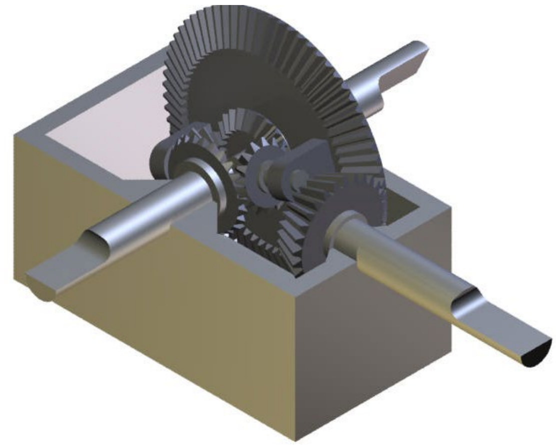
Bicycle Sprockets



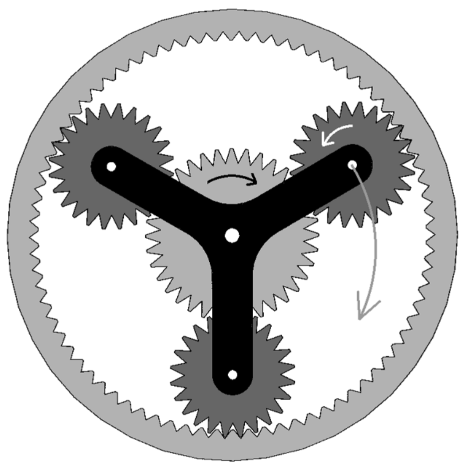
Herringbone Gears



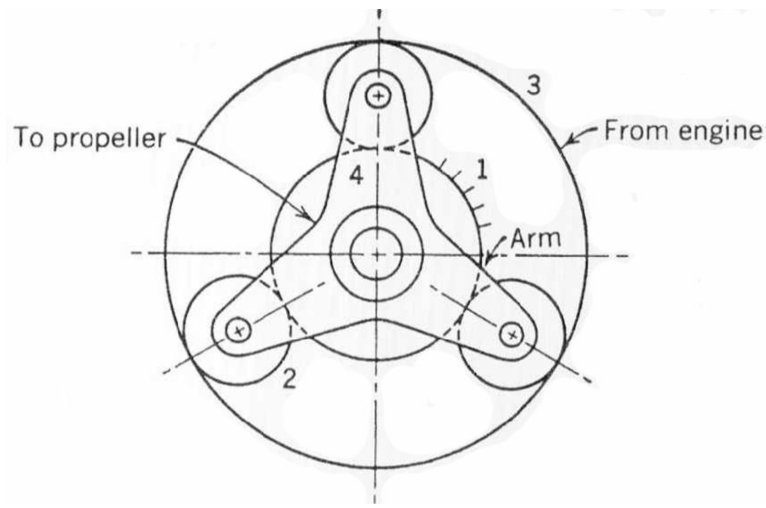
Automotive Gear Train



Automotive Differential



Planetary Gear Train



Planetary Gear Train – Aircraft

Planetary Gear Train Mobility Calculation

Planetary gear trains have two degrees-of-freedom (dof). Here is the Mobility calculation for the left planetary gear train above.

$$M = 3(7-1) - 2(6) - 1(6) = 0$$

Here we have another case where the Kutzbach Mobility equation fails – it knows nothing about the special geometry of the planetary gearing arrangement. Therefore, we can calculate the correct Mobility using only one planetary gear instead of the three as above.

$$M = 3(5-1) - 2(4) - 1(2) = 2$$

Since we generally want to use the planetary gear train as a mechanism with 1-dof, often the sun gear is fixed to the ground link (see the right figure above), yielding the following Mobility calculation:

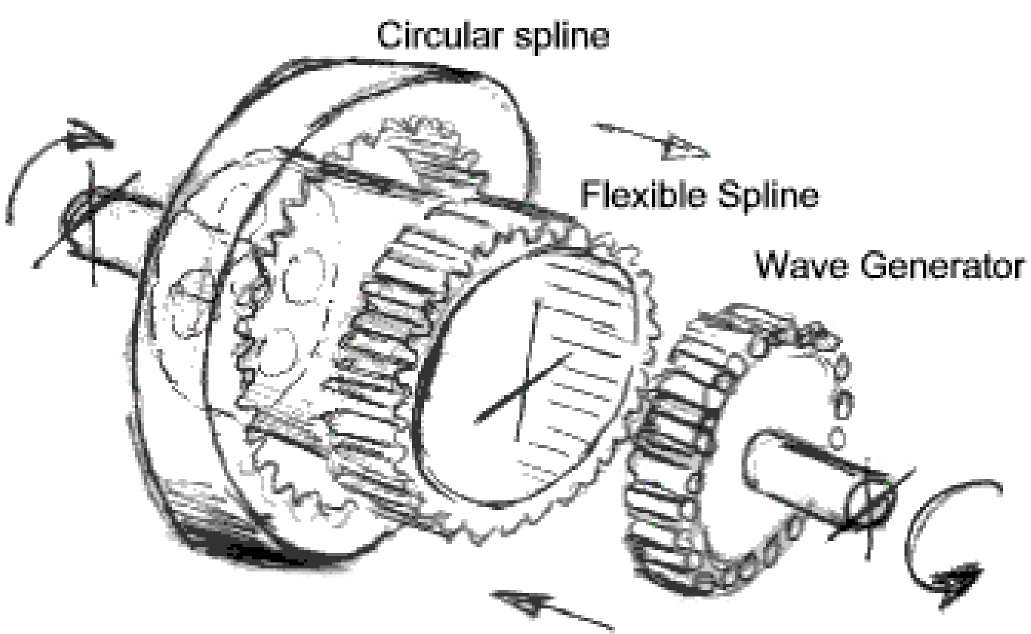
$$M = 3(4-1) - 2(3) - 1(2) = 1$$

Harmonic Gearing

"The harmonic gear allows high reduction ratios with concentric shafts and with very low backlash and vibration. It is based on a very simple construction utilizing metal's elasto-mechanical property."

"Harmonic drive transmissions are noted for their ability to reduce backlash in a motion control system. How they work is through the use of a thin-walled flexible cup with external splines on its lip, placed inside a circular thick-walled rigid ring machined with internal splines. The external flexible spline has two fewer teeth than the internal circular spline. An elliptical cam enclosed in an antifriction ball bearing assembly is mounted inside the flexible cup and forces the flexible cup splines to push deeply into the rigid ring at two opposite points while rotating. The two contact points rotate at a speed governed by the difference in the number of teeth on the two splines. This method basically preloads the teeth, which reduces backlash."

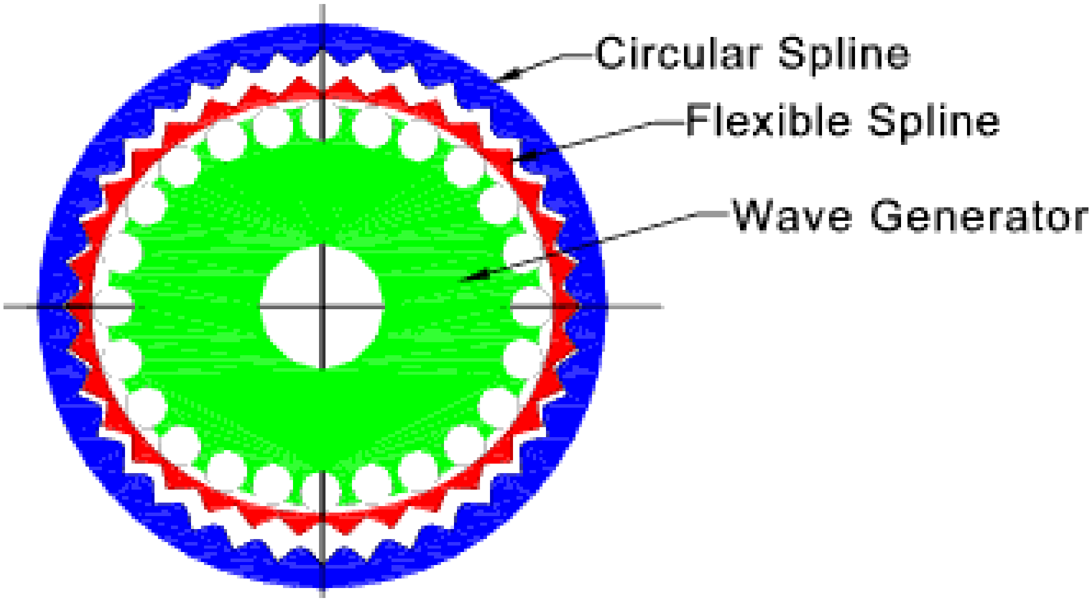
roymech.co.uk



Harmonic Gear Sketch

roymech.co.uk

The wave generator is attached to the input shaft, the flexible spline is attached to the output shaft, and the circular spline is fixed.



Harmonic Gear Diagram

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For harmonic gearing, the gear ratio is also calculated from the numbers of teeth in each gear.

$$n = \frac{\omega_{\text{IN}}}{\omega_{\text{OUT}}} = \frac{\omega_{\text{WG}}}{\omega_{\text{FS}}} = \frac{N_{\text{FS}}}{N_{\text{FS}} - N_{\text{CS}}}$$

where WG stands for wave generator, FS stands for flexible spline, and CS stands for circular spline. For example, if $N_{\text{FS}} = 200$ and $N_{\text{CS}} = 202$, the gear ratio is

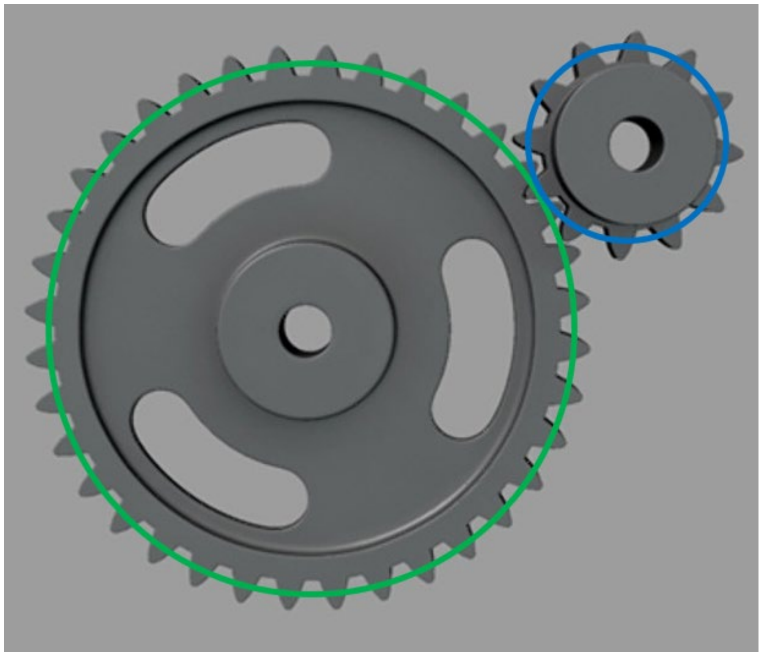
$$n = \frac{\omega_{\text{IN}}}{\omega_{\text{OUT}}} = \frac{N_{\text{FS}}}{N_{\text{FS}} - N_{\text{CS}}} = \frac{200}{200 - 202} = -100$$

which means that the output shaft rotates 100 times slower than the input shaft, but the output shaft carries 100 times more torque than the input shaft. Therefore, this example would be good for the robot joint case, i.e. reducing speed and increasing torque, with $n >> 1$. The negative sign indicates the angular velocity and torque of the output shaft are in the opposite direction of the angular velocity and torque of the input shaft.

7.1.2 Gear Ratio

Common electric motors have high speed but low torque. A robot joint needs lower rotation speed but high torque. A gear train can accomplish both objectives – reduce speed and increase torque. The gear ratio is a measure of the constant, linear degree of speed reduction and torque increase.

Here is an externally-meshing spur gear pair.



The **pitch circles** are two virtual circles (cylinders) that roll on each other without slip during the mesh.

<u>**Degrees of Freedom (mobility)**</u>

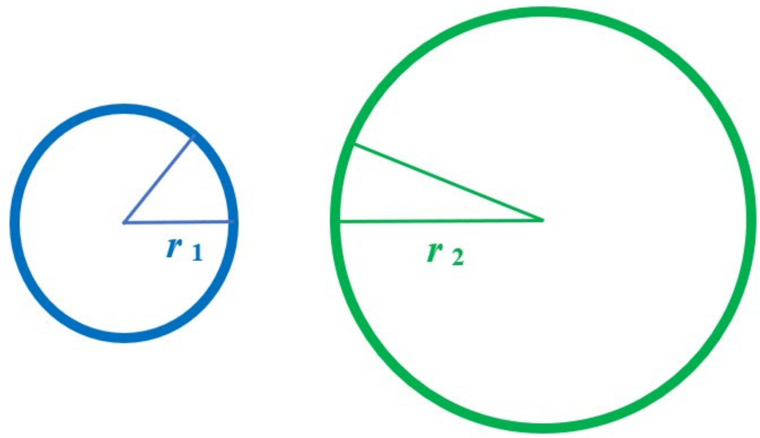
A gear joint connecting two teeth in contact allows both relative rolling and sliding. A gear joint allows 2-dof. i.e. it is a J_2 joint.

By convention, gear 1 is the input and gear 2 is the output. The pitch circles of two mating gears are like two cylinders rolling without sliding.

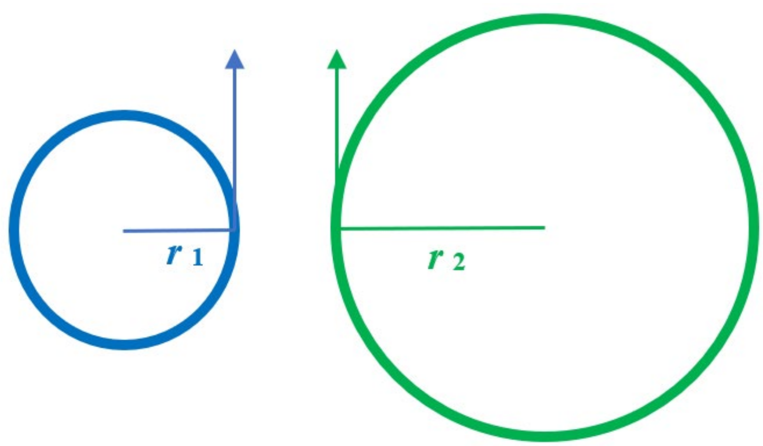
Define gear ratio n as the ratio of the output gear pitch circle radius r_2 to the input gear pitch circle radius r_1. Obviously, this ratio is also the ratio of pitch circle diameters. By **gear standardization**, n is also the direct-proportion ratio of the numbers of teeth N_i.

$$n = \frac{r_2}{r_1} = \frac{d_2}{d_1} = \frac{N_2}{N_1}$$

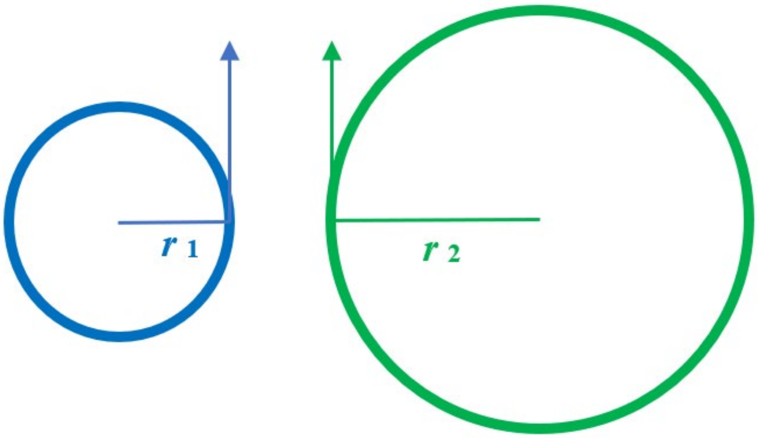
To relate angular displacements of the two gears, the contact arc lengths along the gears' pitch circles are equal.



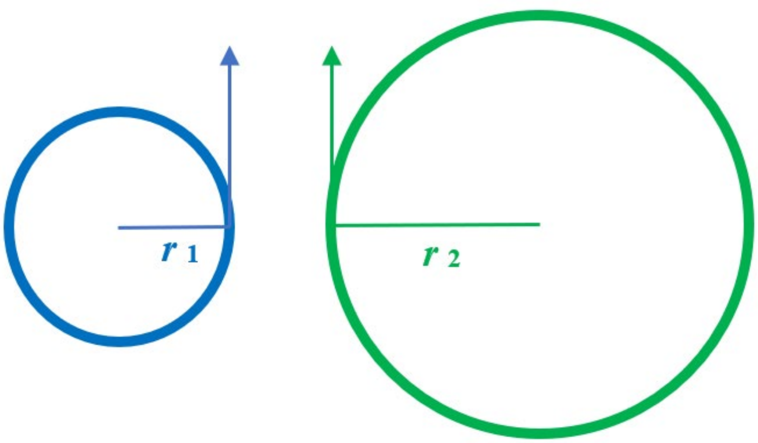
To relate angular velocities, the gears' tangential velocities are equal.



Most gear applications have constant angular velocities. For accelerating up to (or down from) constant angular velocities, the gears' tangential accelerations are equal.



To relate shaft torques, the gears' tangential forces are equal.



Gear Ratio Summary

$$n = \frac{r_2}{r_1} = \frac{d_2}{d_1} = \frac{N_2}{N_1} =$$

The gear ratio (ratio of the pitch circle radii) is **directly proportional** to the numbers of teeth, pitch diameters, and shaft torques. The gear ratio is **inversely proportional** to the shaft angles, angular velocities, and angular accelerations (when the acceleration equation applies, i.e., when both accelerations are non-zero).

Classification of gear ratios

If $n > 1$

$$\omega_2 < \omega_1$$
$$\tau_2 > \tau_1$$

The output has reduced speed and increased torque.

This is the electric motor / robot joint case, where $n >> 1$.

If $n < 1$

$$\omega_2 > \omega_1$$
$$\tau_2 < \tau_1$$

The output has increased speed and reduced torque.

This is the bicycle transmission case, except for some granny gears where n can be as high as 1.5.

If $n = 1$

$$\omega_2 = \omega_1$$
$$\tau_2 = \tau_1$$

This case is called an idler, where the output speed and torque are unchanged, but the direction reverses (for external spur gears)

Gear ratio examples – Bicycle Transmissions

Gear Ratio: $$n = \frac{N_{OUT}}{N_{IN}} = \frac{N_R}{N_F} = \frac{\omega_F}{\omega_R} = \frac{\tau_R}{\tau_F}$$

Cannondale M400 Mountain Bike

	front teeth		
N_i	44	32	22
11	0.25	0.34	0.50
12	0.27	0.38	0.55
14	0.32	0.44	0.64
16	0.36	0.50	0.73
rear teeth 18	0.41	0.56	0.82
21	0.48	0.66	0.95
24	0.55	0.75	1.09
28	0.64	0.88	1.27
32	0.73	1.00	1.45

Trek Roscoe Mountain Bike

N_i	28
11	0.39
13	0.46
15	0.54
18	0.64
21	0.75
24	0.86
28	1.00
32	1.14
36	1.29
42	1.50

The Cannondale mountain bike has a traditional front/rear derailleur transmission with three gears in the front and nine gears in the rear, for a total combination of 27 gears. The Trek Gary Fisher Roscoe mountain bike has a single front gear (28 teeth) and 10 gears in the rear; the granny gear is lower but the fast gear is not nearly so fast (0.39 vs. 0.25). The standard BikeE recumbent bike has a traditional derailleur transmission with seven gears only in the rear. Instead of a traditional derailleur transmission in the front with three gears, the BikeE has an internal hub planetary gear arrangement with three selectable ratios of 1.2913 : 1 (high), 1 : 1 (medium), and 0.7 : 1 (low). The BikeE then has a single chain ring in front to drive the rear chain rings.

The standard BikeE original front single chain ring had 46 teeth – I changed this to a smaller front chain ring of 34 teeth for more granny gear in order to climb Mulligan Hill; the cost of this is loss of high gear for the bike path. I designed the lowest gear to mimic the lowest Cannondale gear ratio since I knew that granny gear climbed well. The Cannondale mountain bike has a standard wheel size of 26" diameter and the BikeE has a rear wheel size of 20” diameter. We must consider this difference in wheel sizes to compute the overall effective BikeE gear ratios. The table below reflects this calculation.

BikeE standard recumbent bike

		front teeth		
		34	34	34
	N_i	1.2913 : 1	1 : 1	0.7 : 1
	11	0.33	0.42	0.60
	13	0.38	0.50	0.71
	15	0.44	0.57	0.82
rear teeth	18	0.53	0.69	0.98
	21	0.62	0.80	1.15
	24	0.71	0.92	1.31
	28	0.83	1.07	1.53

I was able to obtain a used deluxe BikeE recumbent bike from noted luthier Dan Erlewine. I decided to keep the front chain ring of 46 teeth (numbers of teeth in the rear chain ring and the planetary gear ratios in the rear hub are identical between the standard and deluxe BikeE models). This means my new deluxe BikeE doesn’t climb as well as my modified standard BikeE, but it flies much faster on the bike path in high gear than the standard BikeE! Again, the difference in wheel diameter is taken into account in the table below.

BikeE deluxe recumbent bike

		front teeth		
		46	46	46
	N_i	1.2913	1	0.7
	11	0.24	0.31	0.44
	13	0.28	0.37	0.52
	15	0.33	0.42	0.61
rear teeth	18	0.39	0.51	0.73
	21	0.46	0.59	0.85
	24	0.53	0.68	0.97
	28	0.61	0.79	1.13

We see that the standard BikeE that was designed to equal the granny gear of the Cannondale (it was exceeded, 1.53 vs. 1.45). However, the mountain bike still climbs better in granny gear, since your legs are positioned above the pedals in the mountain bike case, and your legs are positioned straight out in front of you in the recumbent bike case.

BikeE Standard Recumbent Bike

Unlike the robot joint example, bicycle gearing generally has $n < 1$ and so the transmission

- increases angular velocity
- decreases torque

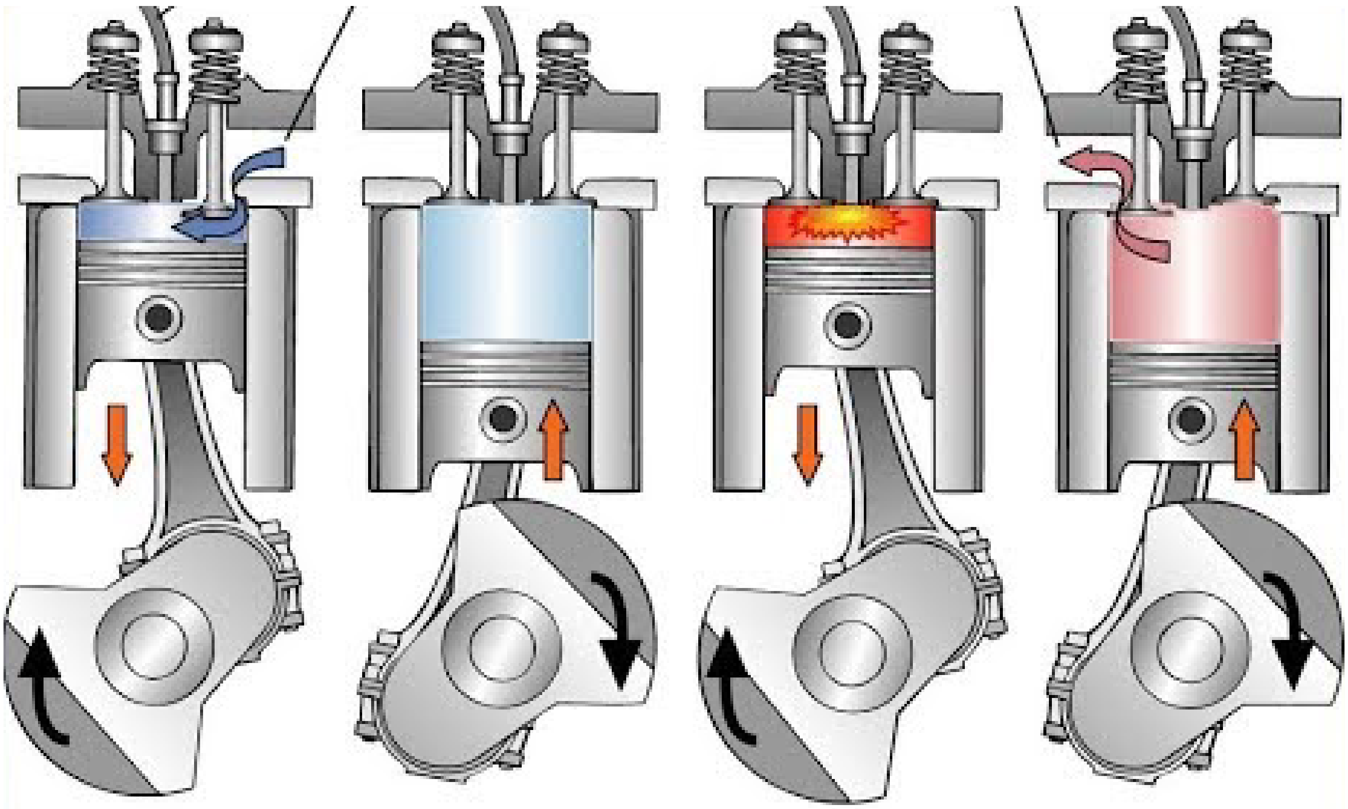
by the gear ratio n. The exception is the granny gears with $n > 1$.

7.2 Cams

7.2.1 Cam Introduction

Applications

Compared to linkages, it is easier to design desired motion with cams, but it is more expensive and difficult to produce. Also, the cam contact and wear properties are worse than for linkages.

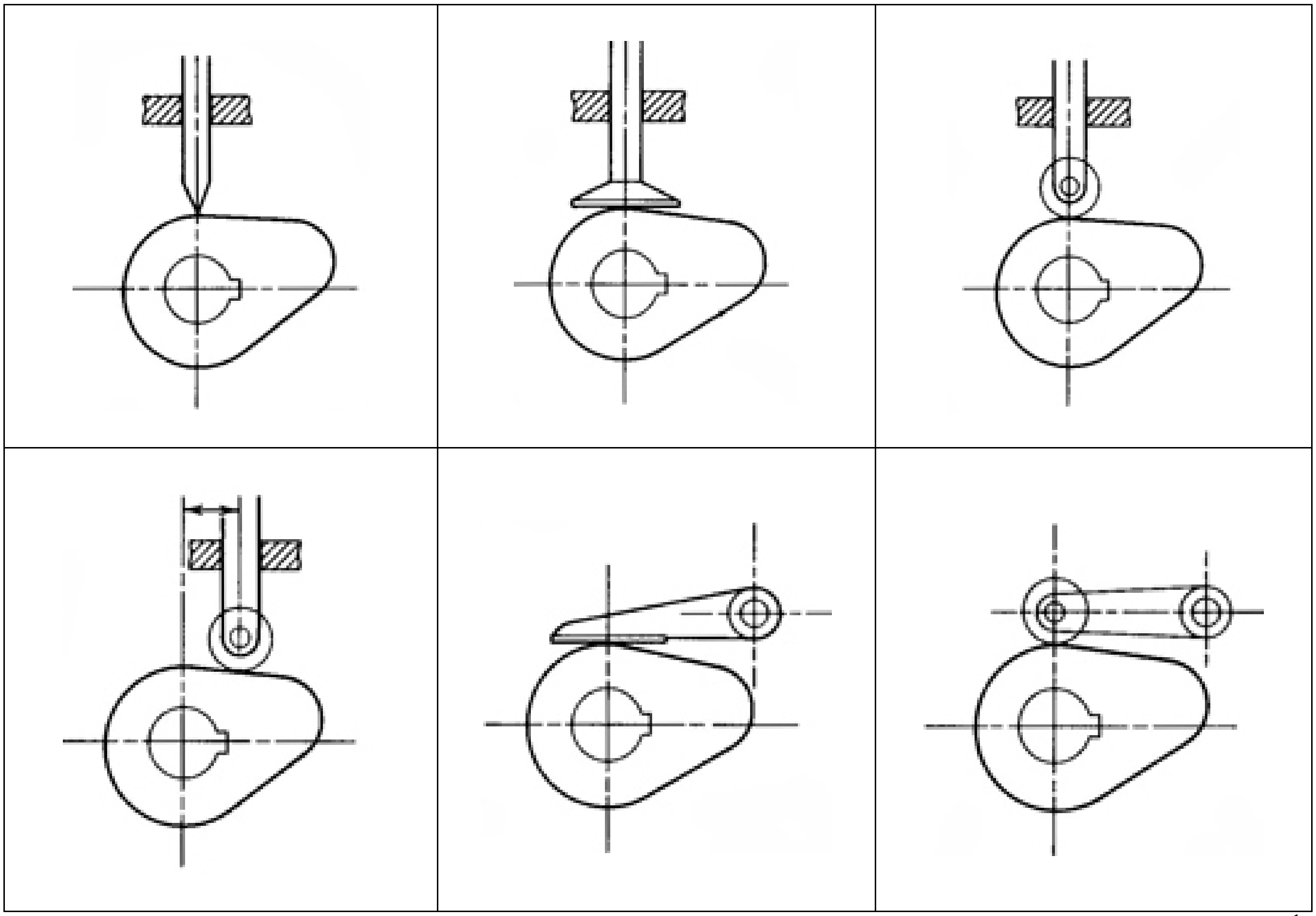


Intake **Compression** **Power** **Exhaust**

The Four Motion Phases of a Four-Stroke Engine

hqdefault.jpg (480×360) (ytimg.com)

Planar Disk Cam and Follower Classification

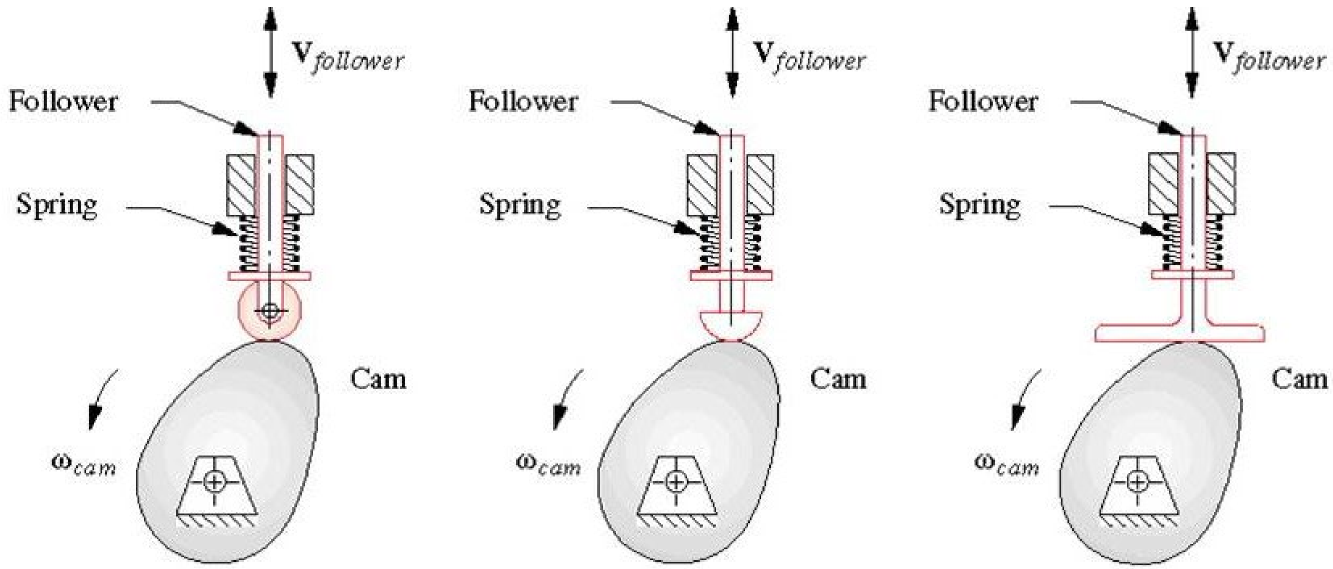


Mabie & Reinholtz (1987[1])

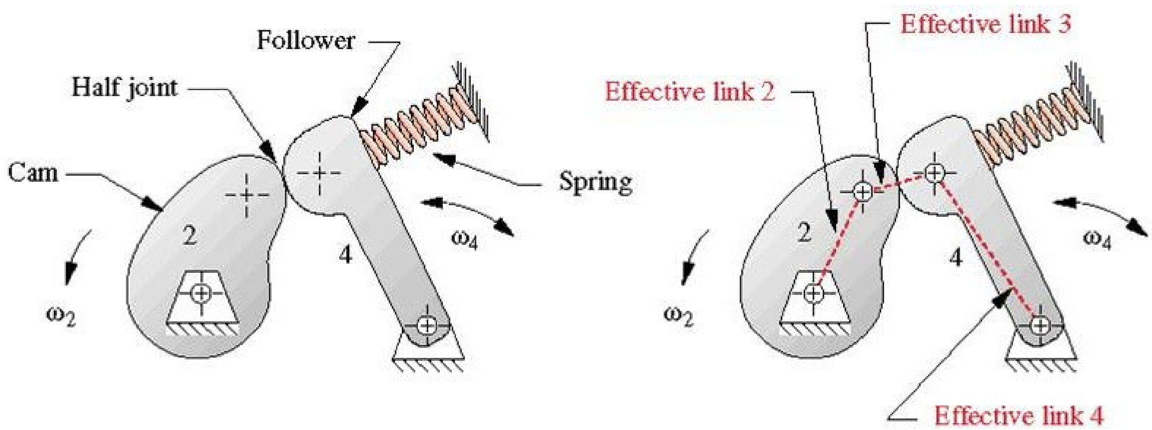
disk cam with translating knife-edge follower	disk cam with translating flat-faced follower	disk cam with translating roller follower
disk cam with offset translating roller follower	disk cam with rotating flat-faced follower	disk cam with rotating roller follower

[1] H.H. Mabie and C.F. Reinholtz, 1987, Mechanisms and Dynamics of Machinery, Wiley.

Disk cams with followers

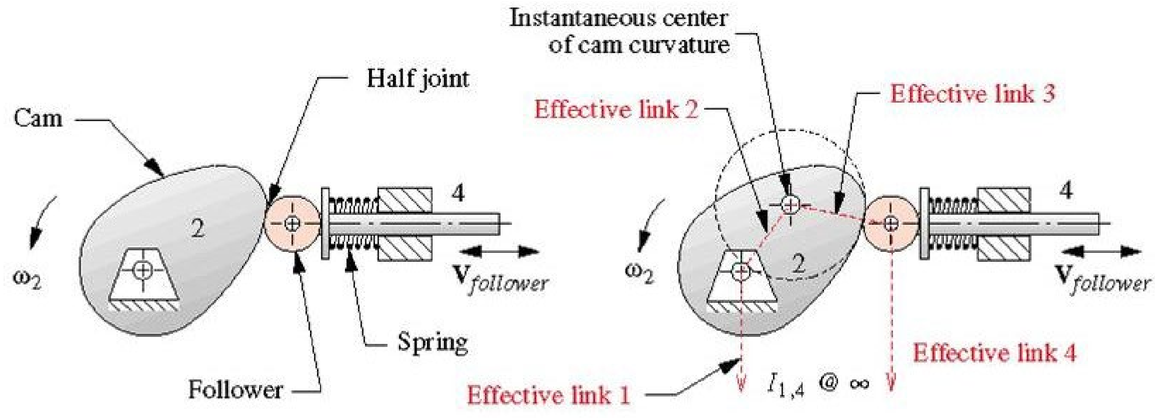


Norton (2008)



(a) An oscillating cam-follower has an effective pin-jointed fourbar equivalent

Norton (2008)



(b) A translating cam-follower has an effective fourbar slider-crank equivalent

Norton (2008)

Two cam/follower systems are shown above with equivalent four-bar and slider-crank mechanism models.

Caution

These are instantaneous equivalents only, i.e. the virtual link lengths for both the four-bar and slider-crank models change with cam mechanism configuration.

Degrees of Freedom (mobility)

A cam joint is a J_2, i.e. it has two-dof since it allows both rolling and sliding, like a gear joint.

Function Generation

In function generation, the output parameter is a continuous function of the input parameter in a mechanism. With linkages, we can only satisfy a function exactly at a finite number of points: 3, 4, or 5, usually. For example, for a four-bar linkage with $\theta_4 = f(\theta_2)$, this function is only exact at a few points.

With a cam and follower mechanism, however, we can satisfy function generation at infinite points. θ is the cam input angle and the output is S for a reciprocating (translating) follower and the output is ϕ for an oscillating (rotating) follower.

$$S = f(\theta) \qquad\qquad \phi = g(\theta)$$

7.2.2 Cam Motion Profiles

Up to this point, we have been mostly concerned with mechanism **analysis**: given a mechanism design and its input parameters, determine the position, velocity, acceleration, and dynamics behavior. With cams we must consider mechanism **synthesis** for the first time: given the motion requirements (follower motion and timing with the input cam angle), **design** the cam. The first step is to determine a smooth cam follower motion profile. In general a cam follower has 4 motion zones (rise, dwell, fall, dwell), as shown below.

When the motion transitions between different motion functions, we must ensure smooth motion.

Fundamental Law of Cam Design

The cam function must be continuous through the first and second derivatives of displacement across the entire motion interval.

Which means:

Position, velocity, and acceleration must be continuous for the entire 360° of cam rotation. The jerk function (the derivative of the acceleration) must be finite, but need not be continuous.

If the **Fundamental law of Cam Design** is satisfied, the resulting dynamic performance will be acceptable for high-speed cam/follower operation. If not, there will be performance degradation due to noise, vibrations, high wear, etc. There is a cyclical impulse hammering at each point in the cam cycle when acceleration is not continuous (even worse if position or velocity is not continuous).

***S V A J* Diagrams**

In cam synthesis (design), we are only given the total motion range and perhaps some timing requirements. It is the engineer's job to determine the position curves and to match the velocity and acceleration across junctions. Position is automatically matched by shifting the dependent function axes. Draw *S V A J* diagrams vs. time to graphically see if the **Fundamental Law of Cam Design** is satisfied for candidate curves. We can plot vs. time or vs. input cam angle θ (assuming constant angular velocity, $\theta = \omega t$).

The slope of a function is the value of its derivative at a point in time (or θ). Therefore, for continuous velocity and acceleration curves, the slopes of the position and velocity curves must match across all junctions. The slope of the acceleration can be discontinuous (leading to finite jumps in jerk), but the acceleration itself must be continuous.

Cam motion curves are very much like the input link motion curves discussed earlier, for input links that start and stop at zero velocity and acceleration. In fact, I adapted the input motion curves from cam motion curve design.

Generic Cam-Follower Motion Profile Figure

Define each separate function so the value is zero at the initial angle, which is zero. Then to put the whole cam motion profile together, just shift the θ and S axes.

Match *S* easy, just shift the S axis.

Match *V* slope of S must match across junctions.

$$v_i(\theta_i = \beta_i) = v_{i+1}(\theta_{i+1} = 0)$$

apply to all functions / junctions.

Match *A* slope of V must match across junctions.

$$a_i(\theta_i = \beta_i) = a_{i+1}(\theta_{i+1} = 0)$$

apply to all functions / junctions.

<u>Cam Follower Motion Profile Examples</u> **Example 1**

rise – dwell portion. Specify parabolic (constant acceleration) to straight line (constant velocity) *rise*, followed by a *dwell*.

	parabolic function	**constant velocity function**	**dwell**
S:	$f_1(\theta_1)=\frac{1}{2}A_0\theta_1^2$	$f_2(\theta_2)=V_0\theta_2$	$f_3(\theta_3)=0$
V:	$v_1(\theta_1)=A_0\theta_1$	$v_2(\theta_2)=V_0$	$v_3(\theta_3)=0$
A:	$a_1(\theta_1)=A_0$	$a_2(\theta_2)=0$	$a_3(\theta_3)=0$
J:	$j_1(\theta_1)=0$	$j_2(\theta_2)=0$	$j_3(\theta_3)=0$

Match ***S*** at junction *B* just shift the vertical axis up.

Match ***V*** at junction *B*

$$v_1(\theta_1=\beta_1)=v_2(\theta_2=0) \qquad A_0\beta_1=V_0 \qquad \text{so } V_0=A_0\beta_1$$

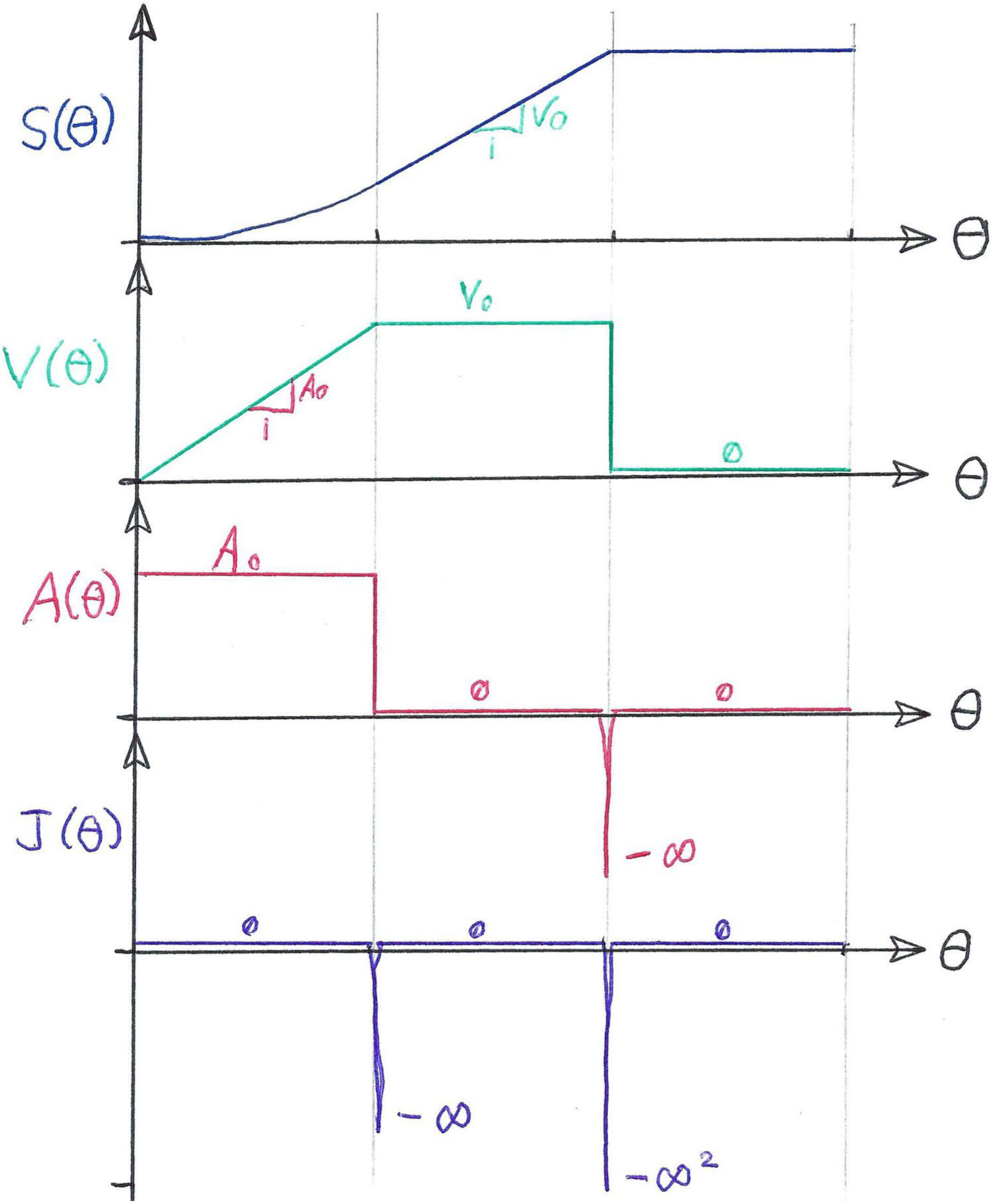
Try to match ***A*** at junction *B*:

$$a_1(\theta_1=\beta_1)=a_2(\theta_2=0) \qquad A_0=0$$

$A_0=0$ is impossible, or else the parabola is degenerate, which we cannot allow. This case violates the **Fundamental Law of Cam Design** since the acceleration function cannot be made to be continuous at junction *B*. Therefore, this cam motion profile example cannot be used for cam design.

We have a bigger problem at junction *C*, between functions 2 and 3: the velocity function cannot be made to be continuous at junction *C*. Discontinuous velocity is one level worse than discontinuous acceleration; either renders the resulting cam motion profile unacceptable.

Example 1 Plots



<u>Cam Follower Motion Profile Examples</u> **Example 2**

Let us fix the *rise* portion only, at junction *B*. Then the problem at junction *C* can be fixed using symmetry. We specify a half-cycloidal function (sinusoidal in cam angle) to a straight line (constant velocity) *rise*.

	half-cycloidal function	**constant velocity function**
S	$f_1(\theta_1) = L_1\left(\frac{\theta_1}{\beta_1} - \frac{1}{\pi}\sin\frac{\pi\theta_1}{\beta_1}\right)$	$f_2(\theta_2) = V_0\theta_2$
V	$v_1(\theta_1) = \frac{L_1}{\beta_1}\left(1 - \cos\frac{\pi\theta_1}{\beta_1}\right)$	$v_2(\theta_2) = V_0$
A	$a_1(\theta_1) = \frac{\pi L_1}{\beta_1^2}\left(\sin\frac{\pi\theta_1}{\beta_1}\right)$	$a_2(\theta_2) = 0$
J	$j_1(\theta_1) = \frac{\pi^2 L_1}{\beta_1^3}\left(\cos\frac{\pi\theta_1}{\beta_1}\right)$	$j_2(\theta_2) = 0$

Match ***S*** at junction *B* just shift the vertical axis up

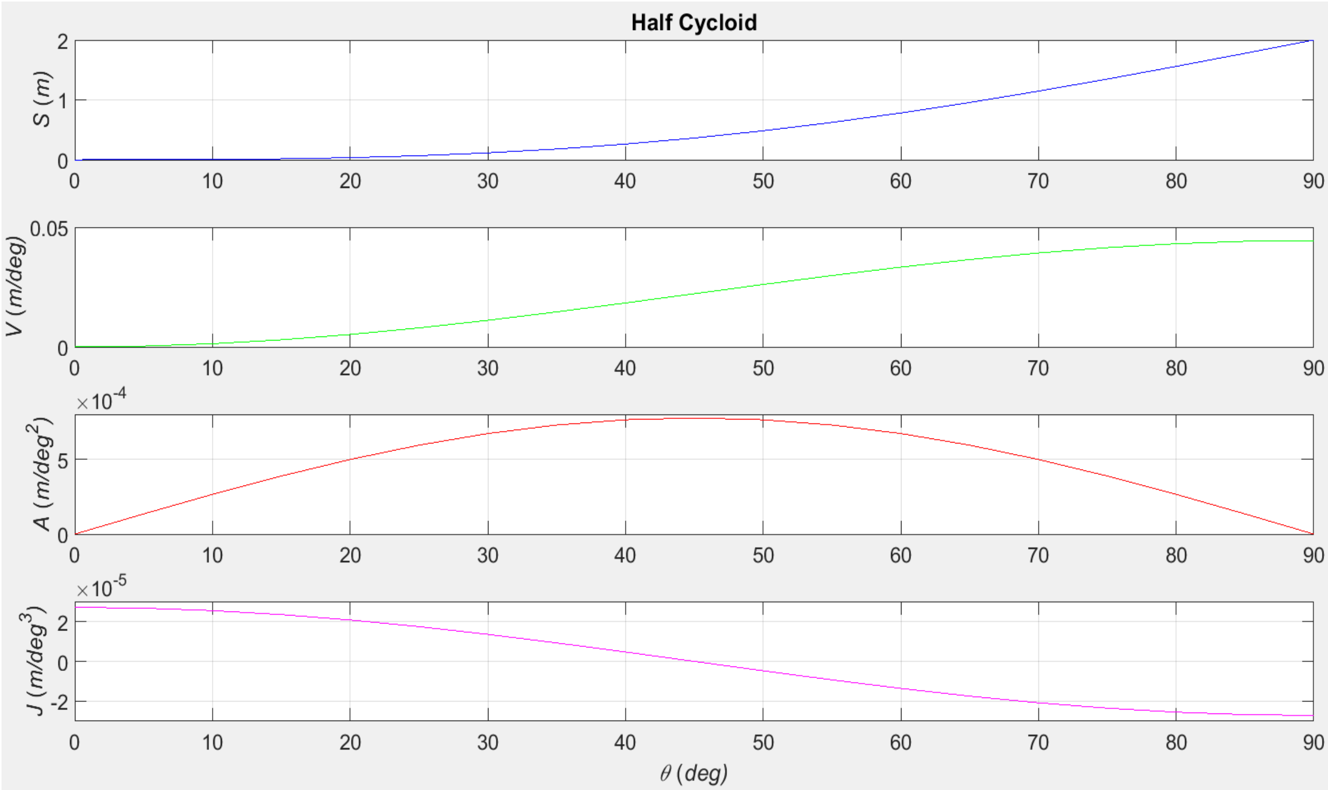
Match ***V*** at junction *B*

$$v_1(\theta_1 = \beta_1) = v_2(\theta_2 = 0) \qquad \frac{L_1}{\beta_1}\left(1 - \cos\frac{\pi\beta_1}{\beta_1}\right) = V_0 \qquad \text{so } V_0 = \frac{2L_1}{\beta_1}$$

Match ***A*** at junction *B*

$$a_1(\theta_1 = \beta_1) = a_2(\theta_2 = 0) \qquad \frac{\pi L_1}{\beta_1^2}\left(\sin\frac{\pi\beta_1}{\beta_1}\right) = 0 \qquad 0 = 0$$

In this case the acceleration function is continuous because the half-cycloidal function ensures that the acceleration is zero at the end of the function range. This case obeys the **Fundamental Law of Cam Design** and so this cam motion profile example portion can be used for cam design.



Example 2: Half-Cycloidal Rise (to connect with constant velocity)

Cam Follower Motion Profile Examples **Example 3**

We now specify a full-cycloidal function (sinusoidal in cam angle). This will *rise* all the way to meet a *dwell* smoothly; it satisfies the **Fundamental Law of Cam Design**. This is the same function used in term project input link motion specification earlier, when starting at stopping at rest.

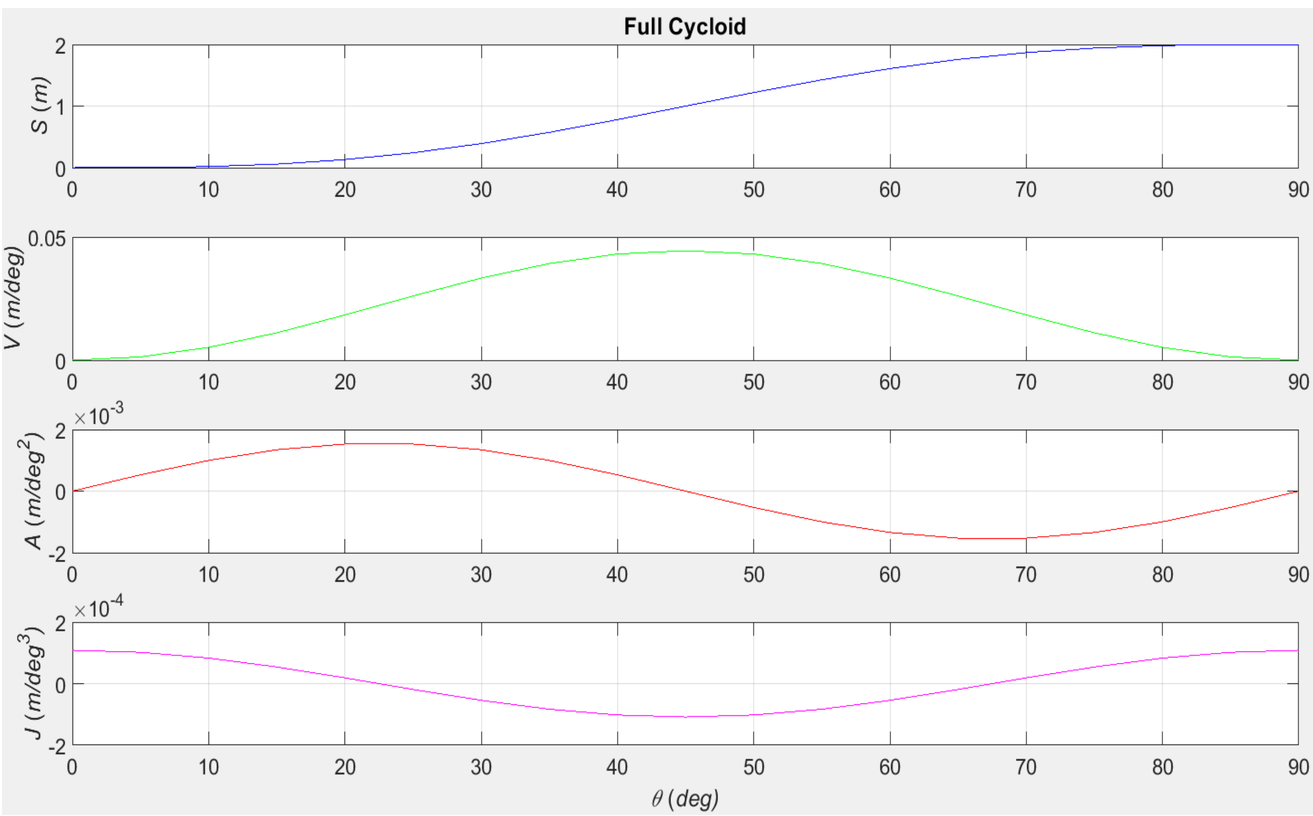
S $$f_1(\theta_1) = L_1\left(\frac{\theta_1}{\beta_1} - \frac{1}{2\pi}\sin\frac{2\pi\theta_1}{\beta_1}\right) \qquad f_2(\theta_2) = 0$$

V $$v_1(\theta_1) = \frac{L_1}{\beta_1}\left(1 - \cos\frac{2\pi\theta_1}{\beta_1}\right) \qquad v_2(\theta_2) = 0$$

A $$a_1(\theta_1) = \frac{2\pi L_1}{\beta_1^2}\left(\sin\frac{2\pi\theta_1}{\beta_1}\right) \qquad a_2(\theta_2) = 0$$

J $$j_1(\theta_1) = \frac{4\pi^2 L_1}{\beta_1^3}\left(\cos\frac{2\pi\theta_1}{\beta_1}\right) \qquad j_2(\theta_2) = 0$$

The full-cycloidal function plots are shown below, rising through the displacement to connect with a dwell.



Example 3: Full-Cycloidal Rise (to connect with a dwell)

Note that the derivatives above are with respect to θ (*deg*). To find the time derivatives, use the chain rule (e.g., for velocity, multiplying by $d\theta(t)/dt = \dot{\theta}(t) = \omega$, a constant). However, for a constant ω, it is customary to use the derivatives with respect to θ for cam design. The full-cycloidal function matches the ensuing dwell: the displacement functions are made to match, and the velocity and acceleration are zero at the end of the full-cycloidal function and the start of the ensuing dwell. The jerk does not match, but the discontinuity in jerk is finite, which satisfies the **Fundamental Law of Cam Design**.