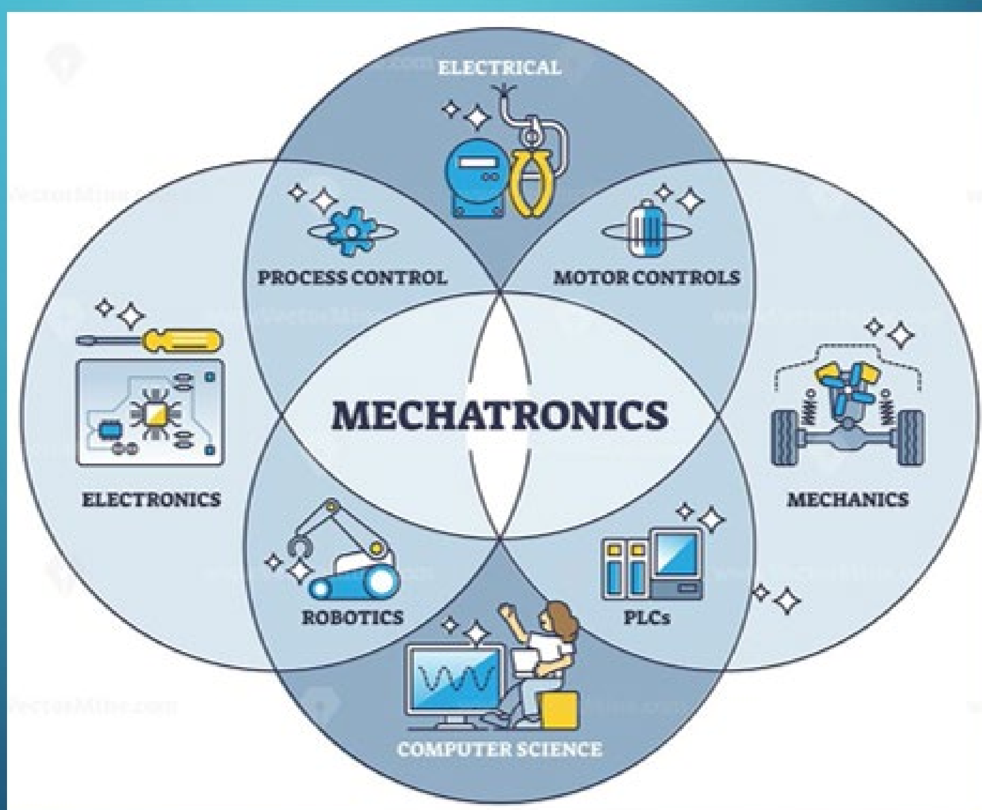
BOB WILLIAMS

MECHATRONIC COMPONENTS



Mechatronic Components

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NotesBook self-published for
ME 3550 Mechatronic Components

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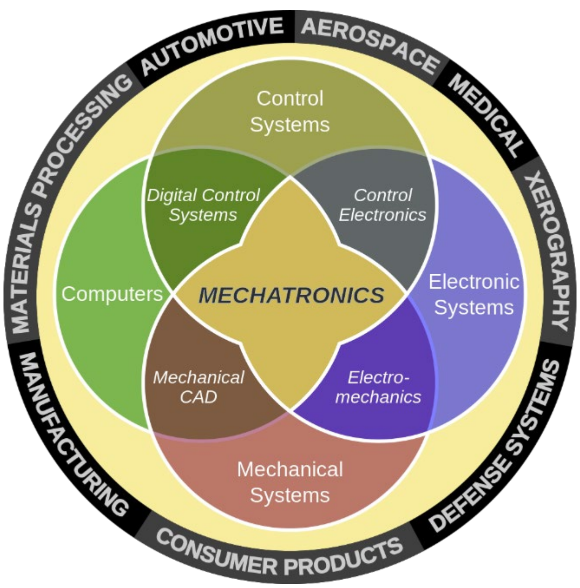
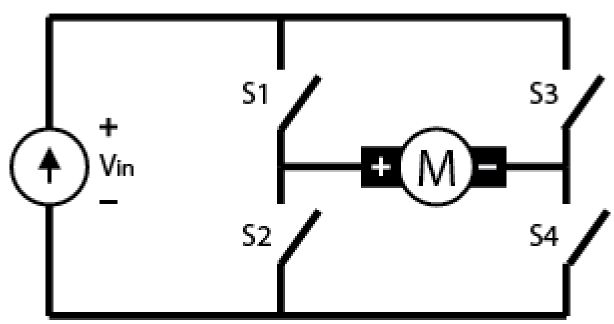


Mechatronics Venn Diagram
Mecha workaround - Mechatronics - Wikipedia



H-Bridge
https://digilent.com/blog/what-is-an-h-bridge/

Resisting the unreasonable cost of textbooks since 2008

Mechatronic Components

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No bits, nibbles, nor bytes were harmed in the making of this mechatronic NotesBook.

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Cover Venn diagram: Mechatronics engineering as electronics and mechanics mix outline diagram

The body text is set in 12-pt Times New Roman, and the headings, sub-headings, and sub-sub-headings are set in 16-pt, 14-pt, and 12-pt Arial, respectively.

This NotesBook is intended for ME 3550 Mechatronic Components, a required one-semester course in Mechanical Engineering at Ohio University, pre-requisite to ME 4550 Mechatronics. Covered are history and background, circuit components, dynamic circuit modeling with electrical/mechanical analogies, mechatronic devices, electric motors, and microcontrollers.

Keywords: mechatronics, mechatronic components, circuit modeling, mechanical/circuit analogies, resistor, capacitor, inductor, impedance, semiconductor, transistor, diode, filter, Wheatstone Bridge, H-Bridge, electric motor, PWM

"I love the smell of Bakelite in the morning!"

–Dr. Bob, NASA Langley Research Center, early 1990s

After someone accidentally fries a circuit component in the Robotics Lab

1984 Motorola 68020 Microprocessor (Dr. Bob, Ohio University)

- 16 MHz
- 200,000 transistors
- 10 MIPS
- 256 byte memory cache
- $3,000.00

2024 Raspberry Pi Pico (Dr. Bob, Ohio University)

- 133 MHz
- 1,000,000,000 transistors
- 133156 FLOPS (not a direct comparison to MIPS)
- 520kB SRAM, 4 MB on-board flash memory
- $7.50

ME 3550 Mechatronic Components Table of Contents

1. Introduction

This NotesBook is intended for ME 3550 Mechatronic Components in Mechanical Engineering at Ohio University. This course is pre-requisite to ME 4550 Mechatronics in Mechanical Engineering at Ohio University.

The subject of Mechatronics is a marriage of the best parts of Electrical and Mechanical Engineering. Such skills are in high demand in the job market and will be into the future. ME 4550 Mechatronics generally involves building a robot and controlling it in real-time with sensor feedback. ME 3550 is designed to introduce the knowledge and laboratory skills necessary in ME 4550.

This ME 3550 Mechatronic Components NotesBook covers the following topics:

- History and Background
- Circuit Components
- Dynamic Circuit Modeling
- Mechatronic Devices
- Electric Motors
- Microcontrollers

Circuit Components include resistors, capacitors, inductors, and the concept of circuit impedance. Most ME 3500 topics are in direct current (DC), but alternating current (AC) circuits and concepts are also covered.

Dynamic Circuit Modeling involves using Kirchhoff's Laws to derive the integro-differential equations representing the dynamics of voltage-driven RLC series circuits and current-driven RLC parallel circuits. Mechanical / Electrical analogies are also presented. Differential equations is not a pre-requisite to ME 3550, nonetheless some differential equations solutions will be presented and some plots will be discussed.

The Mechatronic Devices covered include semiconductors, p-n junctions and diodes, transistors, switches and relays, bridge circuits, logic circuits, op-amps, strain gauges, filters, and various sensors.

A general introduction to microcontrollers is presented, followed by number representation systems, the Raspberry Pi Pico microcontroller, and basic programming.

The operating principles of DC electric motors are presented, along with brushed DC motors, brushless DC motors, stepper motors, and linear motors. Pulse-width modulation is presented for speed control of DC motors.

1.1 Definitions

Below are presented some important definitions for circuit terms, used throughout this ME 3550 NotesBook. You may already be familiar with some these terms – if some are unfamiliar, don't freaque out, we will discuss them later.

mechatronics	Discipline combining mechanical motion / control with electronics. Analysis, design, synthesis, and selection of systems combining electronic and mechanical components with controls and microprocessors.
electronics	Discipline involving physics principles to design, create, and run devices that manipulate electrons and other electrically charged particles.
Electrical Engineering	The branch of engineering involved with electronics, circuits, and computers.
Mechanical Engineering	The branch of engineering involved with machines, materials, and manufacturing.
electricity	A form of energy resulting from charged particles such as electrons, either static (charge) or dynamic (current). The flow of electrons through a wire.
voltage	An electromotive force or electrical potential difference. Analogous to fluid pressure. Units: volts (V, 1 joule of work per 1 Coulomb of charge)
current	Flow of charged particles (electrons or ions) through an electrical conductor or space. Analogous to flow rate of a fluid. Units: amps (A).
direct current	DC, a one-directional flow of electric charge.
alternating current	AC, a flow of electric charge that periodically (sinusoidally) changes direction, and continually changes magnitude.
battery	A DC source of electric power, composed of one or more electrochemical cells having external connections for powering electrical devices.
electrical ground	1. Reference point in a circuit from which voltages are measured; 2. A common return path for electric; or 3. A direct physical connection to the Earth.
flux	The measure of an electric field through a surface. Units Volt-m.
charge	Property of matter that causes it to experience a force when subjected to an electromagnetic field. Units Coulombs.
resistor	A passive two-terminal electrical component that implements electrical resistance as a circuit element. Its primary function is to limit the flow of electric current in a circuit. Dissipates electrical energy. Units Ω.

resistivity ρ; material property that expresses how strongly the material resists electric current. Units Ω-m

conductivity σ; reciprocal of **resistivity**, units Siemens/m

capacitor A passive two-terminal device that stores electrical energy via an electric field by accumulating electric charges on two closely spaced surfaces that are insulated from each other. The primary function of a capacitor is to store electrical charge in a circuit. Units Farads (μF more common).

dielectric material An electrical insulator which can be polarized by an electric field.

inductor A passive two-terminal electrical component that stores energy in a magnetic field when electric current flows through it. Units Henrys (mH more common).

resistance The opposition to current flow in a circuit. R, units Ω.

conductance The potential for a material to conduct electricity current, i.e. how easily current flows through it. **Conductance** is the inverse of **resistance**. G, units Siemens (1 / Ω).

reactance The measure of the opposition that a circuit presents to an alternating current caused by capacitance and inductance. Higher **reactance** means less current, for a given voltage. X, units Ω.

impedance The measure of the opposition that a circuit presents to a current when a voltage is applied. The real part of **impedance** is **resistance** and the imaginary part of impedance is **reactance**. Unlike **resistance** (scalar), **impedance** has magnitude and phase. Z, units Ω.

admittance The measure of how easily a circuit allows a current to flow. **Admittance** is the inverse of **impedance**. Higher **admittance** means higher current, for a given voltage. Y, units Siemens (1 / Ω).

susceptance The measure of how much a circuit is susceptible to conducting a changing current. **Susceptance** is the imaginary part of **admittance**. **Susceptance** is also the inverse of **reactance**. B, units Siemens (1 / Ω).

memristor A memory resistor, a nonlinear two-terminal electronic component relating electric **charge** $q(t)$ and **magnetic flux** $\phi(t)$. Units Ω.

fuse An electric circuit safety device that melts when too much current flows through it.

conductor A material such as copper that allows the flow of electrical current since it has loosely-bound valence electrons that can readily move.

insulator	A material such as rubber that does not allow the flow of electrical current since it has tightly-bound valence electrons that cannot readily move.
semiconductor	A material with conductivity between that of a conductor and an insulator. Used in the manufacture of diodes, transistors, and integrated circuits.
p-n junction	A connection of p-type (positive-type) and n-type (negative-type) semiconductors, allowing current to pass in one direction only.
diode	A single p-n junction connected on both sides to a circuit. A two-terminal electronic semiconductor component that conducts current primarily in one direction.
rectifying diode	A semiconductor diode used to rectify AC (alternating current) to DC (direct current).
Schottky diode	A semiconductor diode formed by the junction of a semiconductor with a metal. Compared to conventional diodes, these allow faster response times and reduced power loss, due to the lack of a depletion layer.
Zener diode	A semiconductor diode that allows current to flow backwards when the reverse Zener voltage is reached.
light-emitting diode	LED, a semiconductor diode that emits light when current flows through it.
anode	The positively-charged electrode by which electrons leave a device. The opposite of **cathode**. The terminal for the p-type side of a semiconductor diode. The negative terminal of a battery.
cathode	The negatively-charged electrode by which electrons enter a device. The opposite of **anode**. The terminal for the n-type side of a semiconductor diode. The positive terminal of a battery.
flip-flop	An electronic circuit with two stable states, used to store binary data. It has one or more control inputs and one or two outputs. These are essential in computer memory for reliably storing 1s and 0s.
shift register	Sequential logic circuit capable of storage and transfer of data. It is a digital circuit using a cascade of flip-flops. The output of one flip-flop is connected to the input of the next flip-flop.
switch	A mechatronic component that controls the flow of current by opening and closing circuits.
transistor	A semiconductor device with at least three terminals used to amplify or switch electrical signals and power.

BJT Bipolar Junction Transistor. A current-driven three-terminal semiconductor device consisting of two p-n junctions, able to amplify a signal.

FET Field-Effect Transistor. A type of transistor in which most current is carried along a channel whose effective resistance can be controlled by a transverse electric field.

MOSFET Metal-Oxide-Semiconductor Field-Effect Transistor. A type of FET (made by the oxidation of Silicon) with an insulated gate whose voltage determines its conductivity (used for amplifying and/or switching electronic signals).

vacuum tube A device that controls electric current flow in a near vacuum between electrodes to which an electric potential difference has been applied. Largely supplanted by transistors in the 1960s and 1970s (exception: old-school electric guitar amplifiers).

integrated circuit A combination of diodes and other semiconductor devices on a single chip.

op-amp Operational-Amplifier. A DC-coupled voltage amplifier with a differential input, a single-ended output, and an extremely-high gain.

electron hole A quasi-particle denoting the lack of an electron where one could be in an atom.

relay An electrically-actuated mechanical switch.

solenoid A linear electromechanical switch wherein an electromagnet produces linear mechanical motion.

strain gauge A device for measuring strain (change of length divided by original length) of an object. Important for assessing the stress of materials.

filter A circuit designed to modify, reshape, or reject undesired frequencies of an electrical signal.

Butterworth Filter An analog signal-processing filter designed to have a flat frequency response in the passband.

thermocouple A thermoelectric device for measuring temperature, consisting of two wires of different metals connected at two points. A voltage is developed between the two junctions in proportion to the temperature difference.

accelerometer A device that measures an object’s acceleration (time rate of change of velocity, second time rate of change of position) with respect to an inertial reference frame.

frequency response The ability of a device to follow a range of input frequencies. The output of dynamic systems, including electrical circuits, have a limited ability to reproduce the input frequencies. Frequency response is graphically shown using Bode plots, amplitude in dB vs. input frequency and phase angle vs. input frequency.

decibels A unit measuring power level of an electrical signal or the intensity of a sound. One-tenth of a Bel, dB expresses the ratio of two power values, on a logarithmic scale.

photoresistor A passive component that decreases in resistance with increasing luminosity on its light-sensitive surface.

thermistor An electrical resistor whose resistance is affected by temperature.

potentiometer A variable resistor with a third adjustable terminal.

amplifier An electronic device that increase the magnitude of a time-varying voltage or current using a power supply.

transformer A passive device that transfers electrical energy from one circuit to another via electromagnetic induction. The voltage across circuits may be increased or decreased without changing the AC frequency.

Wheatstone bridge An electrical circuit used to measure an unknown electrical resistance by balancing two legs of a bridge circuit, one leg of which has a variable resistance, and the other includes the unknown component.

H-bridge A motor driver consisting of four switches arranged in an 'H' pattern that can run a motor in both directions.

pulse-width modulation A method of varying the constant input voltage from a DC power supply to achieve motor speed control. Essentially it involves switching the motor on and off at a high frequency.

microprocessor An integrated circuit providing all the functions of a computer CPU (central processing unit).

microcontroller A microprocessor used to control electromechanical devices such as motors.

bit A single digit of information, 1 or 0.

nibble Four bits.

byte Eight bits, two nibbles.

superconductor A material that offers no resistance to electric current when cooled below a critical temperature. This state is said to be of superconductivity, with zero electrical resistance and zero penetration of magnetic fields. A superconductor thus allows electrical current to flow indefinitely without energy loss. (sounds like perpetual motion?!? But it takes energy to maintain the critical temperature.) Superconducting materials include elements (mercury and lead), alloys such as niobium-titanium, ceramics such as magnesium diboride, pnictides such as fluorine-dope LaOFeAs, and organic superconductors such as carbon nanotubes.

1.2 List of Nomenclature

These lists present the names of terms used throughout this Mechatronic Components NotesBook. As much as possible, standard names from the Mechanical and Electrical fields are used.

Caution: some terms are used for two entirely different yet standard terms in the engineering world, e.g. *v* voltage and *v* velocity.

Electrical Nomenclature

B	susceptance	Siemens (1 / Ω)
C	capacitor constant	Farads
C_{EQ}	equivalent capacitor constant	Farads
E_G	band gap energy	electron-volts
f	cyclical frequency	Hz (cycles / sec)
G	conductance	Siemens (1 / Ω)
i	current	Ampere (Coulomb / sec)
j	imaginary operator	$j=\sqrt{-1}$
k	Boltzmann's constant	8.62×10^{-5} eV / K
K_B	motor back emf constant	Volt-sec / rad
K_T	motor torque constant	N-m / amp
L	inductor constant	Henry (mH, μH)
P	power	Watts (Joule / sec)
q	charge	Coulombs
R	resistor constant	Ω
R_{EQ}	equivalent resistor constant	Ω
T	temperature	Kelvin, Celsius
V	voltage	Volts
X	reactance	Ω
Y	admittance	Siemens (1 / Ω)
Z	impedance	Ω
ε	permittivity	Farads / m
ϕ	flux	Volt-m
ρ	resistivity	Ω-cm
σ	conductivity	Siemens / m
τ	time constant	sec
ω	circular frequency	rad / sec

Mechanical Nomenclature

a	acceleration	m / sec^2
A	cross-sectional area	m^2
c	viscous damping coefficient	N-sec / m
c_R	motor shaft viscous damping	N-m-sec / rad
E	Young's Modulus	Pa (N/m^2)
f	force	N
J	motor shaft inertia	kg-m^2
k	spring stiffness	N / m
k_R	torsional spring stiffness	Nm-sec / rad
m	mass	kg
v	velocity	m / sec
x, y	position	m
α	angular acceleration	rad / sec^2
ε	strain	unitless (m / m)
υ	Poisson's ratio	unitless
θ	motor shaft angle	rad
σ	stress	Pa (N / m^2)
τ	torque	N-m
ω	motor shaft velocity	rad / sec

1.3 List of Acronyms

This section presents the acronyms of terms used throughout this Mechatronic Components NotesBook. As much as possible, standard acronyms from the Mechanical and Electrical fields are used.

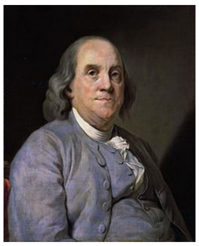
AC	**A**lternating **C**urrent (time-varying current / voltage, sinusoidal)
ADC	**A**nalog-to-**D**igital **C**onverter
ALU	**A**rithmetic **L**ogic **U**nit
BIOS	**B**asic **I**nput/**O**utput **S**ystem
BJT	**B**ipolar **J**unction **T**ransistor
CRT	**C**athode-**R**ay **T**ube
CMOS	**C**omplementary **M**etal-**O**xide **S**emiconductor
CPU	**C**entral **P**rocessing **U**nit
CW	**C**lock**W**ise
CCW	**C**ounter-**C**lock**W**ise
dB	**d**eci**B**els
DAC	**D**igital-to-**A**nalog **C**onverter
DC	**D**irect **C**urrent (constant current / voltage)
DMM	**D**igital **M**ulti**M**eters
DPDT	**D**ouble-**P**ole **D**ouble **T**hrow switch/relay
DPST	**D**ouble-**P**ole **S**ingle **T**hrow switch/relay
DRAM	**D**ynamic **R**andom-**A**ccess **M**emory
EEPROM	**E**lectronically-**E**rasable **P**rogrammable **R**ead-**O**nly **M**emory
emf	**e**lectro**m**otive **f**orce
EPROM	**E**rasable **P**rogrammable **R**ead-**O**nly **M**emory
eV	**e**lectron-**V**olts
FET	**F**ield-**E**ffect **T**ransistor
FFT	**F**ast-**F**ourier **T**ransform
FIFO	**F**irst-**I**n-**F**irst-**O**ut
GF	**G**auge **F**actor (of a strain gauge)
GPIO	**G**eneral-**P**urpose **I**nput / **O**utput
I2C	**I**nter-**I**ntegrated **C**ircuit
IDE	**I**ntegrated **D**evelopment **E**nvironment
IMU	**I**nertial **M**easurement **U**nit
IR	**I**nfra**R**ed
I/O	**I**nput / **O**utput
ISR	**I**nterrupt **S**ervice **R**outine
IVP	**I**nitial **V**alue **P**roblem (ODE)
KCL	**K**irchhoff's **C**urrent **L**aw
KVL	**K**irchhoff's **V**oltage **L**aw
LC	**I**nductor-**C**apacitor circuit
LCD	**L**iquid-**C**rystal **D**isplay
LED	**L**ight-**E**mitting **D**iode
LDR	**L**ight-**D**ependent **R**esistor
MEMS	**M**icro-**E**lectro**M**echanical **S**ystems
MOSFET	**M**etal-**O**xide-**S**emiconductor **F**ield-**E**ffect **T**ransistor
NMOS	**N**-type **M**etal-**O**xide-**S**emiconductor

List of Acronyms (continued)

NO	**N**ormally-**O**pen switch
NPN	**N**egative-**P**ositive-**N**egative
NTC	**N**egative **T**emperature **C**oefficient
ODE	**O**rdinary **D**ifferential **E**quation
Op-Amp	**O**perational **A**mplifier
PLC	**P**rogrammable **L**ogic **C**ontroller
PMOS	**P**-type **M**etal-**O**xide-**S**emiconductor
p-n	**p**ositive-**n**egative
PNP	**P**ositive-**N**egative-**P**ositive
PTC	**P**ositive **T**emperature **C**oefficient
PWM	**P**ulse-**W**idth **M**odulation
RAM	**R**andom-**A**ccess **M**emory
ROM	**R**ead-**O**nly **M**emory
RC	**R**esistor-**C**apacitor circuit
RL	**R**esistor-**I**nductor circuit
RLC	**R**esistor-**I**nductor-**C**apacitor circuit
RMS	**R**oot-**M**ean-**S**quare
rpm	**r**evolutions **p**er **m**inute
RTD	**R**esistance **T**emperature **D**etector
SCR	**S**ilicon-**C**ontrolled **R**ectifier
SPDT	**S**ingle-**P**ole **D**ouble **T**hrow switch/relay
SPI	**S**erial **P**eripheral **I**nterface
SPST	**S**ingle-**P**ole **S**ingle **T**hrow switch/relay
SRAM	**S**tatic **R**andom-**A**ccess **M**emory
TCR	**T**emperature **C**oefficient of **R**esistance
TTL	**T**ime-**T**o-**L**ive
UART	**U**niversal **A**synchronous **R**eceiver-**T**ransmitter
USB	**U**niversal **S**erial **B**us
VCC	**V**oltage at the **C**ommon **C**ollector

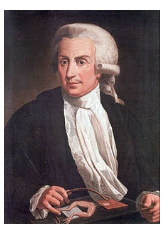
1.4 Famous Names from Electrical History

Benjamin Franklin (1706 – 1790) was a U.S. founding father, author, printer, politician, postmaster, statesman, and diplomat. He is in this list due to his fame as a scientist (discoveries and theories on electricity, experiments with early electric motors) and an inventor (lightning rod, bifocals, Franklin stove, glass armonica (not misspelled), odometer, Daylight Savings Time, fire departments, political cartoons, and mapping of the Gulf Stream).



Charles-Augustin de Coulomb (1736 – 1806) was a French engineer and physicist. He was the first to publish electrostatic force of attraction and repulsion, now called Coulomb's Law. He also worked with mechanical sliding friction. The SI unit of charge was named in his honor (Coulomb).

Luigi Galvani (1737 – 1798) was an Italian physician, physicist, biologist, and philosopher. He was a pioneer in bioelectromagnetics (muscles of a dead frog's legs twitch when excited by current). He inspired Mary Shelley, author of Frankenstein. His name lives on: galvanize, Galvanic cell, Galvani potential, galvanic corrosion, galvanometer, galvanization, Galvanic skin response.

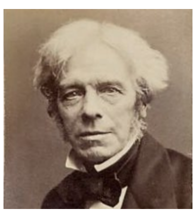


Alessandro Volta (1745 – 1827) was an Italian physicist, chemist, and electricity pioneer, inventor of the electric battery and discoverer of methane. The SI unit of electric potential was named after him (Volts), and he was a confidante of Napoleon Bonaparte.

Andre-Marie Ampere (1775 – 1836), French physicist and mathematician, a founder of classical electromagnetism. The SI unit of electric current was named after him (Ampere), and his name appears on the Eiffel Tower.

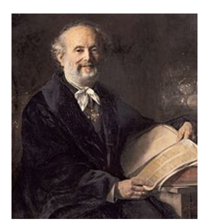
Georg Simon Ohm (1789 – 1854) developed the well-known Ohm's Law for resistors in electrical circuits, $V = i\,R$ (i.e. the voltage V across a resistor is proportional to the current i and the proportionality constant is the resistance R). The SI unit of electrical resistance was named after him (Ohms).

Michael Faraday (1791 – 1867) discovered induction (electricity can be created by moving a magnet through a wire coil). He developed the first electric motor, built the first generator and transformer, and introduced the terms ion, electrode, cathode, anode. The SI unit of capacitance was named for him (Farad). He was the first person to come up with theories that reflect the modern theory of the electromagnetic field. Faraday developed the laws of electrolysis and the relationships between light and magnetism.



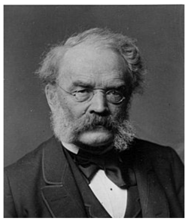
Joseph Henry (1797 – 1878), American scientist and first secretary of the Smithsonian Institute. He invented the electromagnetic relay, which enabled the telegraph. The SI unit of inductance was named after him (Henry). Discovered electromagnetic induction independently of Faraday.

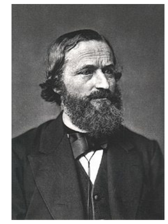
Charles Wheatstone (1802 – 1875) was an English scientist and inventor. He was important in the development of telegraphy. He is best known for improvements to the Wheatstone Bridge (originally invented by Samuel Hunter Christie).



Wilhelm Eduard Weber (1804 – 1891) was a German physicist. He invented the first electromagnetic telegraph (along with Carl Friedrich Gauss). The SI unit of magnetic flux is named for him (the Weber, Wb).

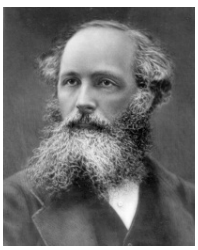
Werner von Siemens (1816 – 1892) was a German electrical engineer who invented the electric tram, trolley bus, locomotive, and elevator. He founded the Siemens corporation and his name is the SI unit for electrical conductance (Siemens).

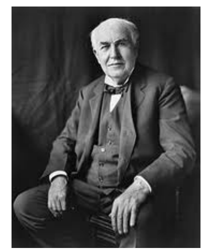




Gustav Kirchhoff (1824 – 1887) was a German physicist and mathematician who develop laws for current and voltage analysis in electric circuit, and contributed to spectroscopy, and black-body radiation.

James Clerk Maxwell (1831 – 1879) is the father of modern physics. In **1865** he published the first book about electromagnetism, showing the relation and movement between electric and magnetic charges in nature.

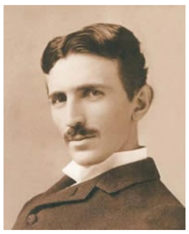




Thomas Edison (1847 – 1931) patented 1,093 inventions in his lifetime. Besides his famous light bulb innovation in **1879**, Edison developed the phonograph and the kinetoscope, a small box for viewing moving films. He also improved upon the original design of the stock tickertape, the telegraph, and Alexander Graham Bell's telephone.

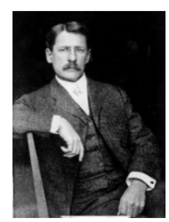
Hendrik Lorentz (1853 – 1928) was a Dutch theoretical physicist who shared the **1902** Nobel Prize in Physics with Pieter Zeeman for discovery of the Zeeman Effect (the splitting of a spectral line into several components in a magnetic field). Lorentz also discovered the Lorentz force, acting on a charged particle in a magnetic field. This is the basis for electric motors.

Nikola Tesla (1856 – 1943) emigrated to the U.S. from present-day Croatia to work briefly for Thomas Edison. Arguably the greatest electrical engineer in history, his inventions include fluorescent lighting in the **1890s**, the Tesla coil (used in the first Niagara Falls hydroelectric plant), the induction motor in **1888**, and 3-phase electricity. His patents and theoretical work paved the way for the development of radio. He developed the AC-current generation system comprised of a motor and a transformer. Tesla accidentally captured one of the earliest known X-rays in **1894**. Despite making fortunes from his patents, he died penniless due to the expense of various experiments. Tesla fell into obscurity (thanks in large part to Thomas Edison's envy of his reputation) until the **1990s** when his reputation was resurrected.



Joseph Thompson (1856 – 1940) was a British physicist and Nobel Laureate. He found the electron, the first subatomic particle to be discovered. He proved that the existing conventional current assumption (flowing from positive to negative) is in reality reversed (current really flows from negative to positive). However, by that point in history, the conventional current assumption was too deeply ingrained to be changed, and this error persists to present day!

Heinrich Hertz (1857 – 1894) confirmed in **1887** Maxwell's theories about the existence of electromagnetic radiation, proving that electricity can be transmitted in electromagnetic waves. Hertz accidentally discovered the photoelectric effect in which light falling on special surfaces can generate electricity. The SI unit of frequency is named after him (Hz, cycles per second).



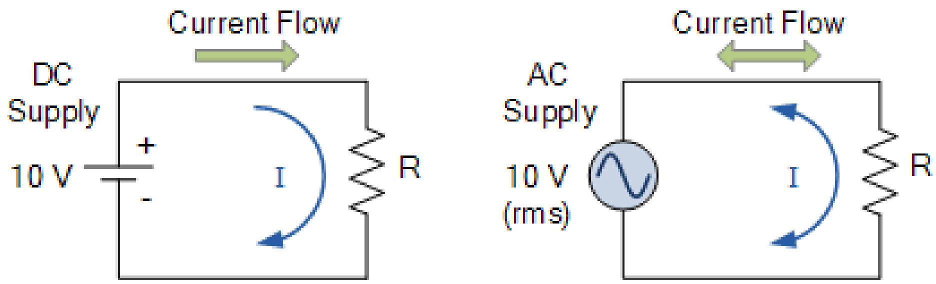
Frank Sprague (1857 – 1934) was an American naval officer and inventor, "The Father of Electric Traction". His electric motor patent led to electrically-powered underground subways, and more concentration of business in city centers due to elevators. Hired briefly by Thomas Edison, who stole his idea and put his own name on Sprague's innovative electric motor.

Charles Proteus Steinmetz (1865 – 1923) was a German-born American mathematician and electrical engineer whose mathematical theories led to alternating current. At 4 feet tall, he was afflicted with dwarfism, hunchback, and hip dysplasia.

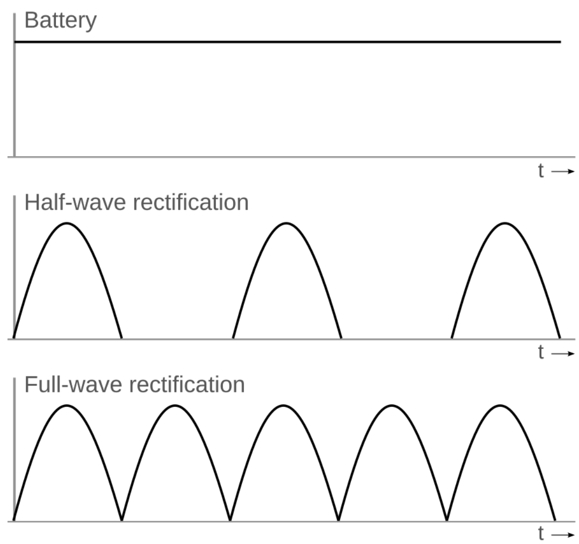
Edwin Howard Armstrong (1890 – 1954) was an American electrical engineer who invented wide-band frequency modulation, i.e. FM radio, in **1933**.

1.5 DC vs. AC

Direct Current (DC) is a one-directional flow of electric charge. An electrochemical battery cell provides DC power to devices. DC flows through wires, semiconductors, insulators, and even in a vacuum.



Simple DC vs. AC Circuits



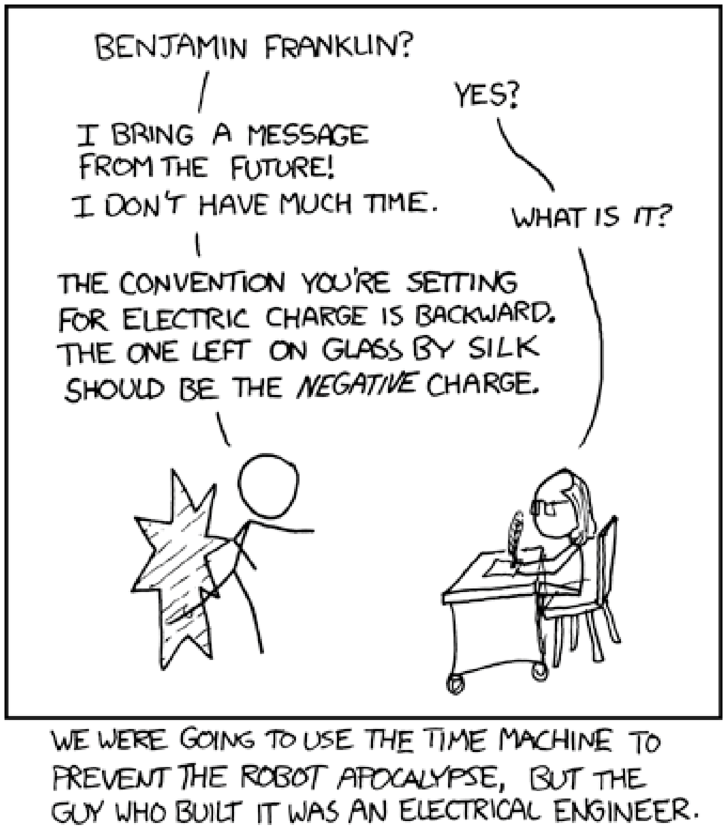
Types of DC

[Direct current - Wikipedia]

DC was first produced in 1800 by Italian physicist Alessandro Volta, using his Voltaic Pile. French physicist Andre-Marie Ampere hypothesized that current flows in one direction, from positive to negative (he got this backwards! See next page).

<u>Convention for Electron Flow</u>

When early electricity experimenters and early electrical engineer pioneers established the convention for the flow of electrons composing current, they got it backwards. In Electrical Engineering, **conventional current** assumes the current flows from positive to negative. In reality the flow of current is opposite, i.e. from negative to positive!

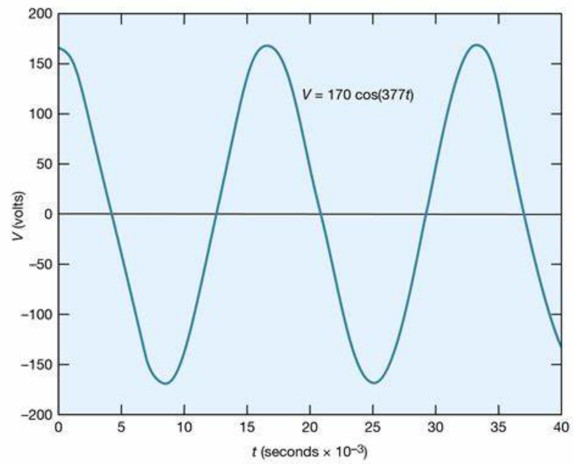


xkcd.com

When Joseph Thompson did experiments to discover the electron, he found electrons flow from negative to positive to create current. This error (conventional current assumption) persists to this day, because it really doesn't matter in the equations.

Thomas Edison championed DC electrical generation for lighting households. A huge disadvantage of DC generation is that the transmission distance is severely limited, to about 1 mile, before the voltage drops off significantly due to resistance in the wires. Because of this power loss, DC power grids used low voltage and high current, limiting the possible transmission distance, and posing a safety threat.

Alternating Current (AC) is a flow of electric charge that periodically (sinusoidally) changes direction, and continually changes magnitude.



AC Voltage Plot

[definition of alternating current - Search]

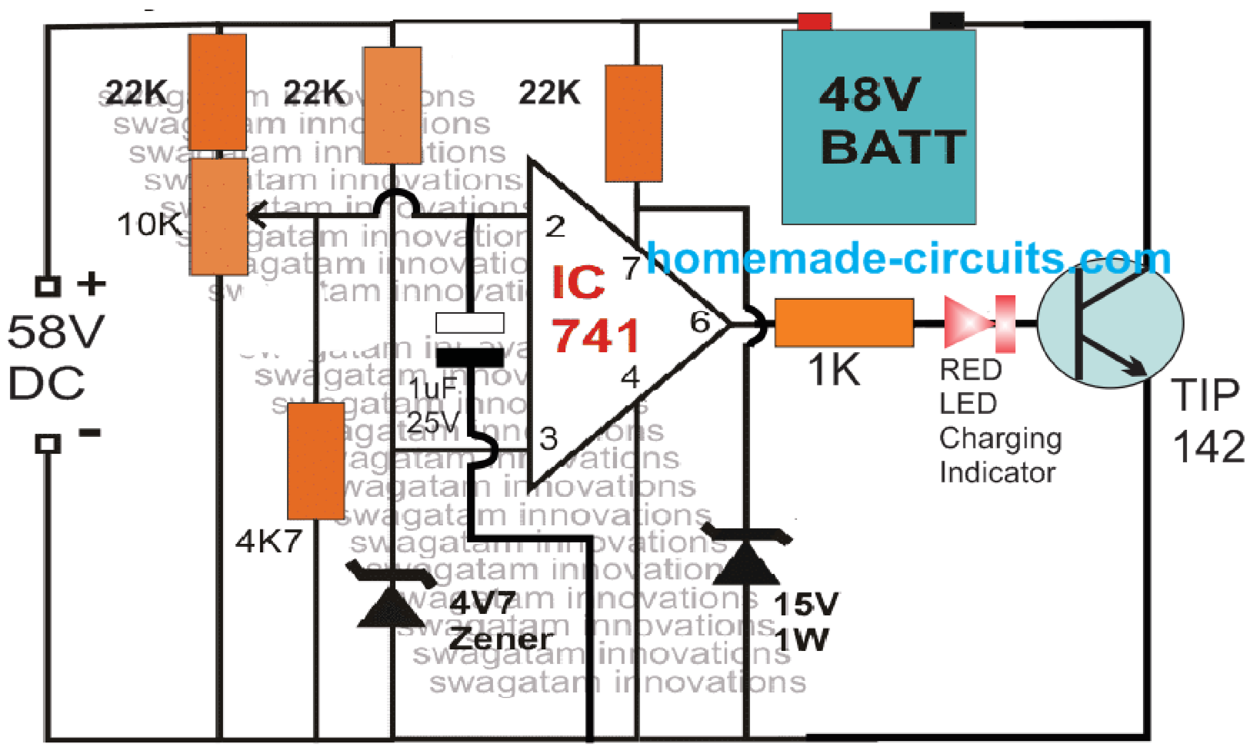
Originally Edison's employee, Nikolai Tesla championed AC generation and transmission. This interest was enabled by his invention of the AC induction motor. AC transmission involves high voltage (hundreds of thousands of volts) and relatively low current, leading to much less power loss and hence much longer transmission distances when compared to DC. Transformers are employed to step this voltage down to household levels (120 – 240 V). AC transmission is also far safer than DC transmission.

Three-Phase Electricity is a common type of AC for electricity generation, transmission, and distribution. It uses three wires (four for an optional neutral return wire), wherein the voltages on each wire are 120° phase shifted relative to each other. This allows for the voltages to be stepped up via transformers to high voltage for transmission and stepped back down for distribution. This results in high efficiency. Three-phase power is used for powering large AC induction motors and other heavy loads. Polyphase power systems were independently developed by many inventors in the late 1880s, including Nikolai Tesla.

[Three-phase electric power - Wikipedia]

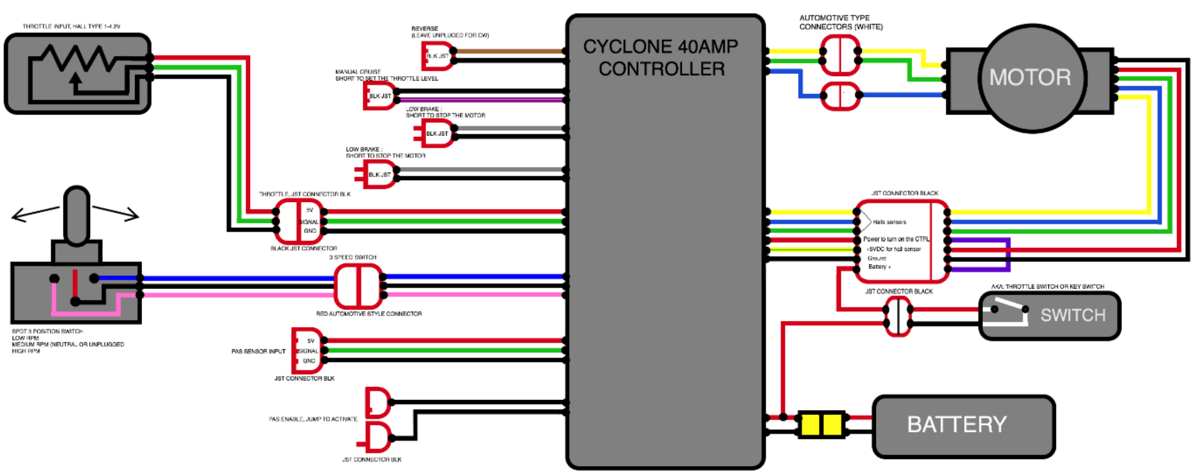
1.6 e-Bike Mechatronics

Dr. Bob’s Giant Talon e-Bike 2023



Typical e-Bike 48 V Charging Circuit

[E Bike Charger Circuit Diagram » Diagram Board]



Typical e-Bike 40 A Motor Control Diagram

https://diagramdbbecker.z21.web.core.windows.net/ebike-motor-controller-schematic.html

Typical e-Bike batteries are rated at 48 Volts and 30 Amp-hours, for 500 – 1000 Watts. They require approximately 400 – 800 Watt-hours for charging. My e-Bike charges fully in a couple hours, plugged into the AC wall source.

1.7 Electrical Circuit Computer Programs

There is an excellent, free CAD program for drawing circuits with mechatronic components. It is called DigiKey Scheme-It and is available for free on-line, allowing you to save your drawings.

https://www.digikey.com/en/schemeit

Use of this software is optional in ME 3550, but is highly encouraged.

There is another circuit design, modeling, and simulation computer program called NI MultiSim. Ohio University students have access to this for coursework via the computer labs and the remote desktop.

https://www.multisim.com

Use of this software is optional in ME 3550, but is highly encouraged.

An alternative, free, online circuit design, modeling, and simulation computer program is available at:

https://www.falstad.com/circuit/

Use of this software is optional in ME 3550, but is highly encouraged.

2. Building Blocks

This chapter presents electrical circuit building blocks. These circuit components include resistors *R*, capacitors *C*, and inductors *L*.

Following the coverage of the three classical passive circuit elements (*R*, *C*, *L*) we present Kirchhoff's DC Circuit analysis laws (Kirchhoff's Voltage Law and Kirchhoff's Current Law). Then first-order DC circuits are covered, including differential equations solutions to the applicable ordinary differential equations (ODEs).

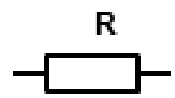
Most ME 3500 circuits are DC (direct-current) circuits. AC (alternating-current) circuits and concepts are also covered, including the concept of complex circuit impedance.

2.1 Resistors

A resistor is a passive two-terminal electrical component that implements electrical resistance as a circuit element. Its primary function is to limit the flow of current in DC and AC circuits. Resistors dissipate electrical energy through heat. Resistance is the opposition to current flow in a circuit. The units of resistance *R* is Ohms (Ω).

- Resistors restrict the flow of current in circuits.
- Resistors provide 'friction' in circuits by heating up to remove some electrical energy.
- **Resistance** is measured in units of Ohms (Ω).
- **Resistivity** has units of Ω-m (ρ).
- The Reciprocal of **Resistivity** is **Conductivity** with units of Siemens/m (σ).

US Resistor Schematic Symbol

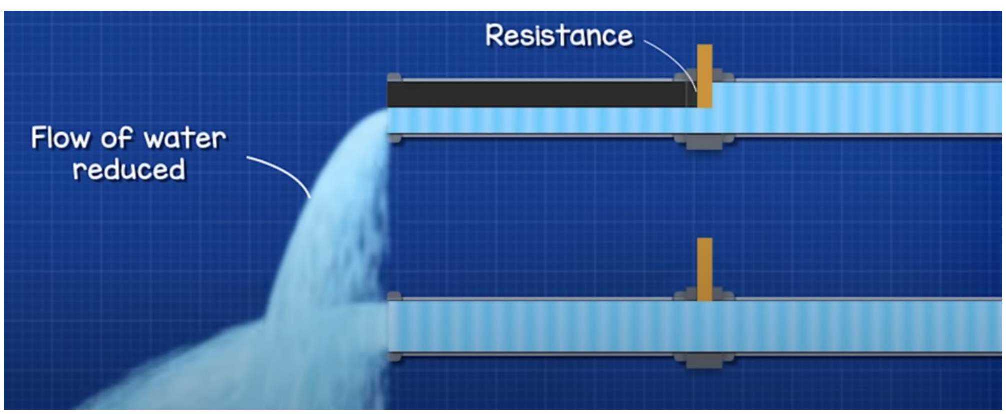


IEC Resistor Schematic Symbol

diagrams from Digikey Scheme-It

Hydraulic Analogy of Electrical Resistance

The fluid flow of water in a pipe is similar to the flow of electrical current along a wire. The flow pressure is analogous to the electrical voltage, and the fluid flow rate is analogous to the electrical current. If there is a constriction in the pipe (a length of reduced diameter), the fluid flow rate will be reduced. An electrical resistor acts much in the same way as a constricted pipe, limiting the flow of electrical current.



Water Pipes With and Without a Constriction

How Resistor Work - Unravel the Mysteries of How Resistors Work

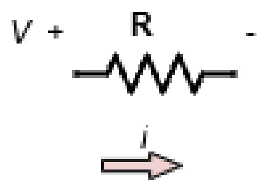
In the fluid pipe example, there is a pressure drop across the pipe constriction, with higher pressure on the entrance, and lower pressure on the exit of the pipe constriction. There is a similar voltage drop across an electrical resistor, with higher voltage 'upstream' of the resistor than 'downstream'.

Some may believe a resistor is like a speed bump for electrons. However, in a traffic analogy, resistors are more like traffic jams, impeding the flow of cars (electrical current).

<u>Ohm's Law (1827)</u>

Ohm's Law states that the voltage drop V across a resistor R is proportional to the current i flowing through the resistor:

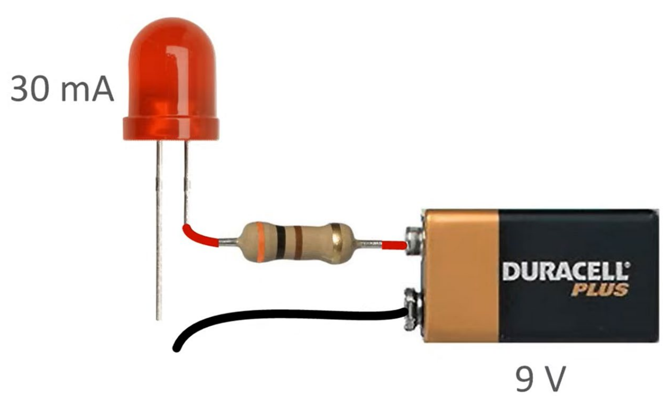
$$V = iR \qquad\qquad V(t) = i(t)R$$



We will often use Ohm's Law in a statics sense, but the equation on the right indicates that the voltage and current can vary with time, even for DC circuits.

Ohm's Law Example

It is required to make a simple DC circuit to light a light-emitting diode (LED). The LED requires 30 mA of current and has negligible resistance. Without the added resistor pictured below, the current would grow too large and quickly burn out the LED, with a possible smell of Bakelite in the morning.



DC LED Circuit

What is a resistor?

The proper sizing of the resistor for this example is performed using Ohm's Law:

$$V = iR \qquad\qquad R = \frac{V}{i} = \frac{9}{0.03} = 300 \ \Omega$$

Using this 300 Ω resistor, when the circuit is closed (it is shown as an open circuit above), the LED should operate satisfactorily and safely.

Example concludes.

Here are eleven ways to express the resistance *R* unit Ohms (Ω) in SI units:

$$\Omega = \frac{V}{A} = \frac{1}{S} = \frac{W}{A^2} = \frac{V^2}{W} = \frac{sec}{F} = \frac{H}{sec} = \frac{Wb}{C} = \frac{Jsec}{C^2} = \frac{kgm^2}{secC^2} = \frac{kgm^2}{sec^3 A^2} = \frac{J}{secA^2}$$

Resistor Types

There are many types of resistors beyond the fixed resistor pictured in the LED circuit above.

- Fixed – constant resistance, most common

- Variable – adjustable resistance, most often by mechanical movement
    - Potentiometers – variable voltage dividers
    - Rheostats – variable resistances to control current in a circuit
    - Trimpots – small potentiometers used for adjustment, tuning, and calibration in circuits.

- Dependent on Physical Quantity
    - Thermistors – resistance affected by temperature (directly and inversely both possible)
    - Photoresistors – resistance inversely related to level of light
    - Varistors – semiconductor diodes with resistance related to the applied voltage
    - Magnetos – electrical generators using permanent magnets to create periodic pulses of AC
    - Strain Gauges – measures strain via Wheatstone Bridge by balancing resistors

Resistor Construction

- Wire-wound
- Carbon Composition
- Carbon Film
- Metal Film
- Metal Oxide Film
- Foil

Resistivity

Wires that compose circuits also have inherent resistance (often ignored in circuit analysis). The resistance for a given length L and cross-sectional area A of a wire is:

$$R = \frac{\rho L}{A}$$

where **resistivity ρ** varies wildly amongst different materials. Conductors have $\rho < 10^{-3}$ Ω-m, while Carbon has $\rho = 5000$ Ω-m, and Polyethylene has $\rho = 1\text{x}10^{14}$ Ω-m! (see the table below). Conductivity σ is the reciprocal of resistivity ρ:

$$\sigma = \frac{1}{\rho}$$

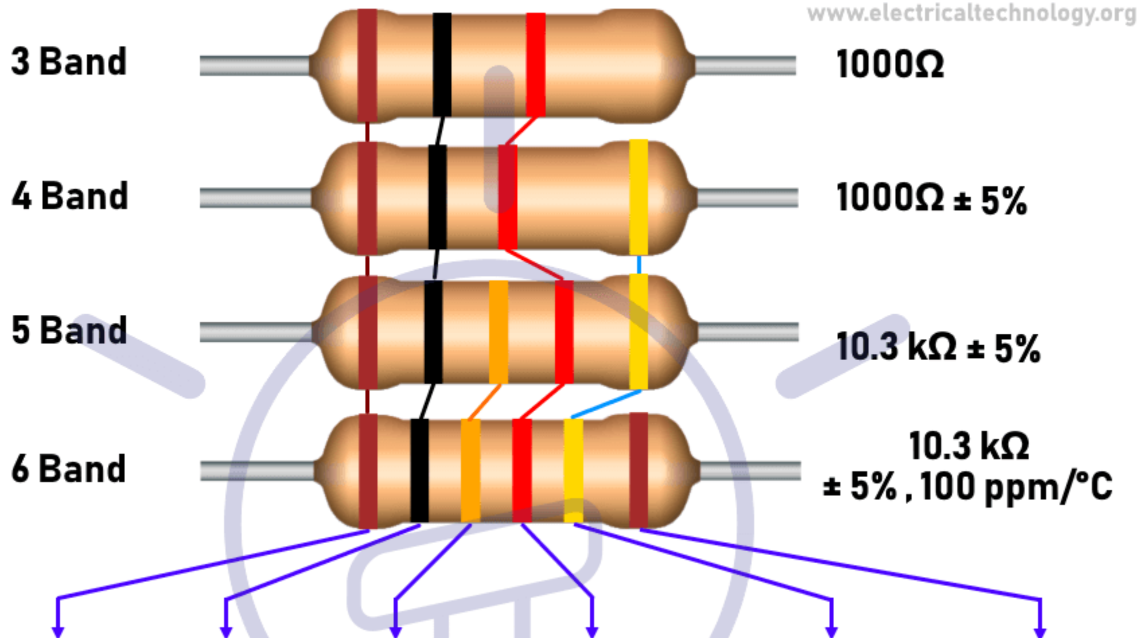
Material	Resistivity (Ω-m)
Superconductors	$\rho \approx 0$
Conductors	$\rho < 10^{-3}$
Silver	1.6
Copper	1.7
Gold	2.4
Aluminum	2.8
Aluminum alloys	4.0
Magnesium	4.5
Zinc	6.0
Nickel	6.8
Iron	9.5
Tin	11.5
Low carbon steel	17.0
Lead	20.6
Cast iron	65.0
Stainless steel	70.0
Carbon	5000
Semiconductors	$10^{-3} < \rho < 10^{5}$
Silicon	$1\text{x}10^{4}$
Insulators	$\rho > 10^{5}$
Rubber	$1\text{x}10^{12}$
Polyethylene	$1\text{x}10^{14}$

Resistivity ρ of Various Materials

table adapted from Onwubolu (2005)

Standard Resistor Colour Codes

Resistor resistance values are commonly denoted on the component using the standard resistor colour code.



COLOR	1st DIGIT	2nd DIGIT	3rd DIGIT	MULTIPLIER	TOLERANCE	TEMP .Co
BLACK	0	0	0	1Ω		
BROWN	1	1	1	10Ω	± 1% (F)	100
RED	2	2	2	100Ω	± 2% (G)	50
ORANGE	3	3	3	1kΩ	± 3%	15
YELLOW	4	4	4	10kΩ	± 4%	25
GREEN	5	5	5	100kΩ	± 0.5% (D)	
BLUE	6	6	6	1MΩ	± 0.25% (C)	10
VIOLET	7	7	7	10MΩ	± 0.10% (B)	5
GREY	8	8	8	100MΩ	± 0.05% (A)	
WHITE	9	9	9	1GΩ		
GOLD				0.1Ω	± 5% (J)	
SILVER				0.01Ω	± 10% (K)	

6dd635591c1f2d706304ad1ff26cd04a.png (999×907) (pinimg.com)

3-Band: bands 1 & 2 give the resistance digits, band 3 is the multiplier (all in decimal)

4-Band: bands 1 & 2 give the resistance digits, band 3 is the multiplier, band 4 is the tolerance

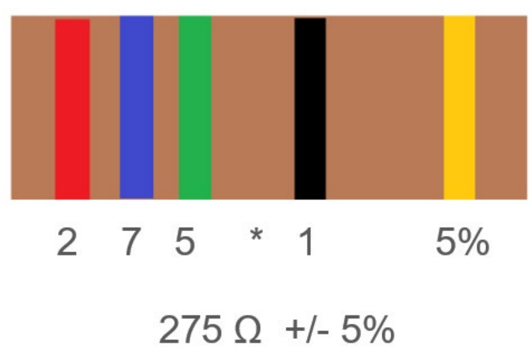
5-Band: bands 1 – 3 give the resistance digits, band 4 is the multiplier, band 5 is the tolerance

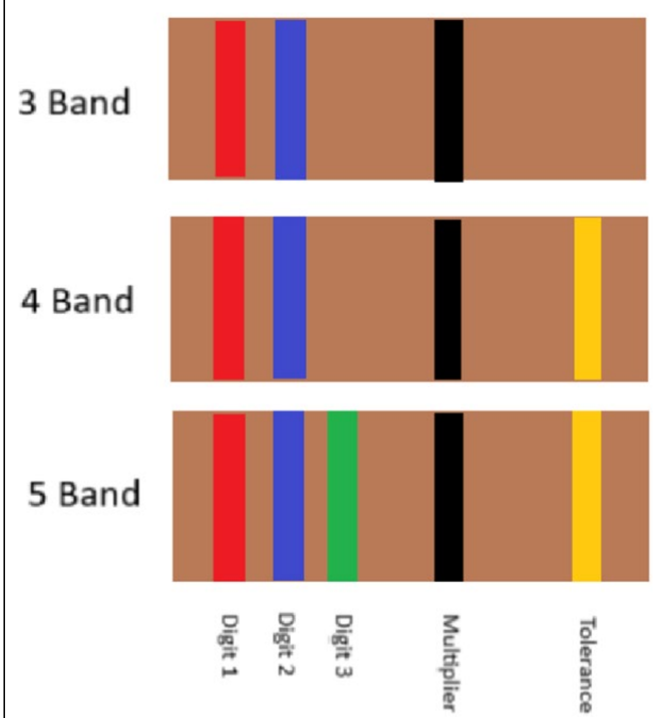
6-Band: bands 1 – 3 give the resistance digits, band 4 is the multiplier, band 5 is the tolerance, band 6 is the temperature coefficient of resistance (TCR, in 10^{-6}/K)

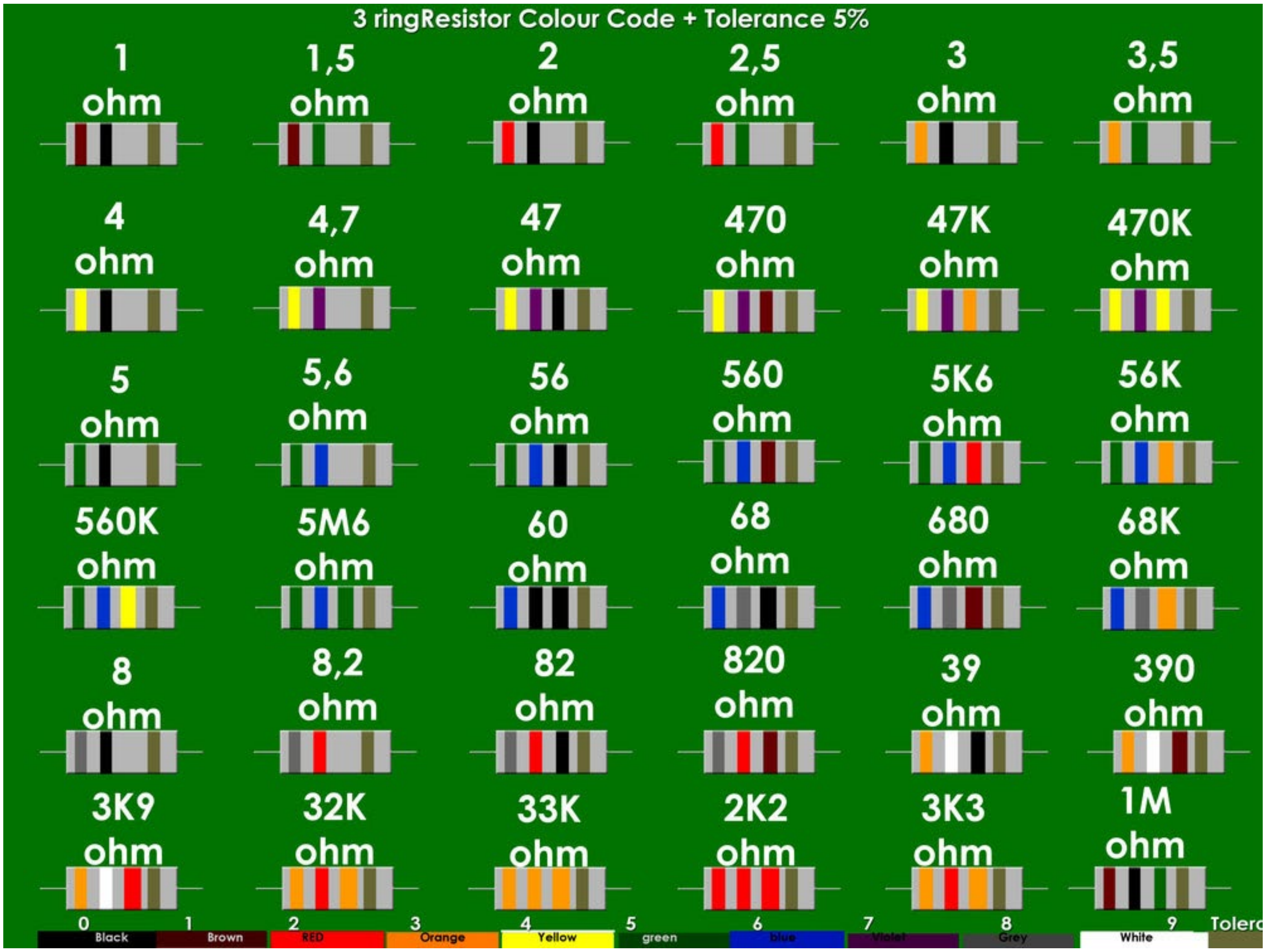
Resistor Colour Codes

Colour	Value	Colour	Value
black	0	gold	±5%
brown	1	silver	±10%
red	2	nothing	±20%
orange	3		
yellow	4		
green	5		
blue	6		
violet	7		
grey	8		
white	9		

Resistor Colour Codes Examples







Various Practical Resistance Values

Resistor Values: How to Calculate and Understand It

Here is a handy online calculator for implementing resistor colour codes:

www.allaboutcircuits.com/tools/resistor-color-code-calculator

Resistor Power Dissipation

Earlier it was stated "Resistors provide 'friction' in circuits by heating up to remove some electrical energy". The power dissipation by a resistor is:

$$P = iV = i^2 R = \frac{V^2}{R}$$

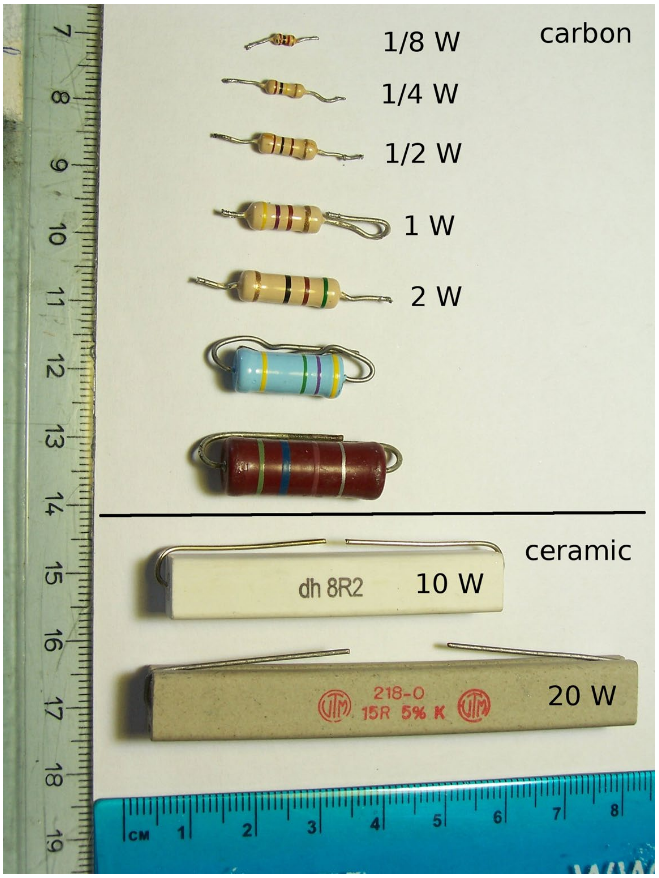
and is typically expressed in Watts. Our laboratory resistors will generally dissipate a maximum of ¼ Watts.

Example. Given: $V = 25$ V $R = 200\ \Omega$ Find the resistive power dissipation.

First calculate $i = \frac{V}{R} = \frac{25}{200} = 0.125$ A

Then $P = iV = i^2 R = \frac{V^2}{R} = 0.125(25) = 0.125^2(200) = \frac{25^2}{200} = 3.125$ W

Example concludes.



Various Resistor Power Dissipation Levels

Resistors and Power

As an example it was decided to recreate the power levels *P* of the resistors shown in the figure above. It was difficult to read the colour coding to determine the resistances. Instead here a current of 500 mA, $i = 0.5$ A, was assumed for all cases. Then the associated voltages *V* for each case was found from:

$$V = \frac{P}{i}$$

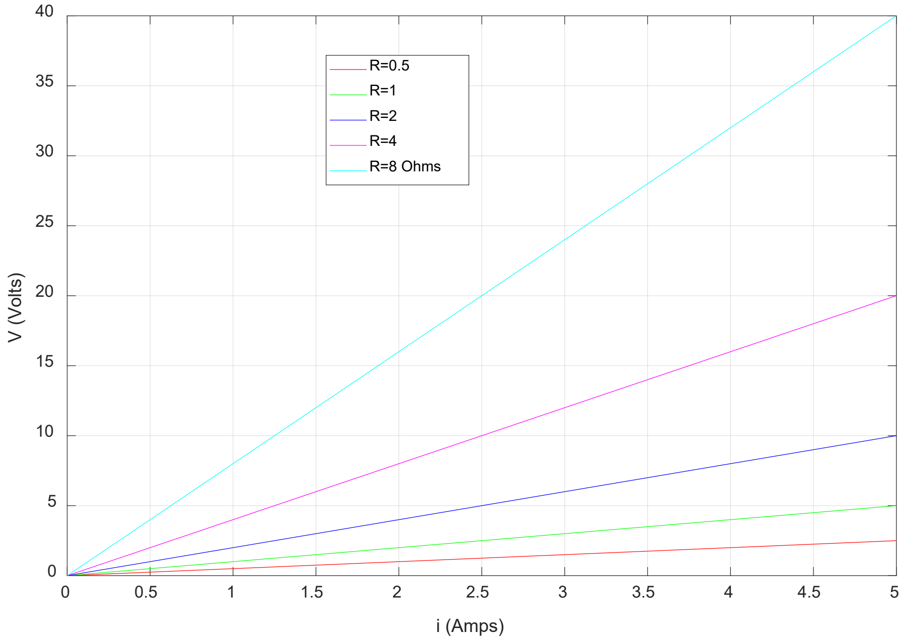
and then the resistance *R* values for this example were found from Ohm's Law:

$$R = \frac{V}{i}$$

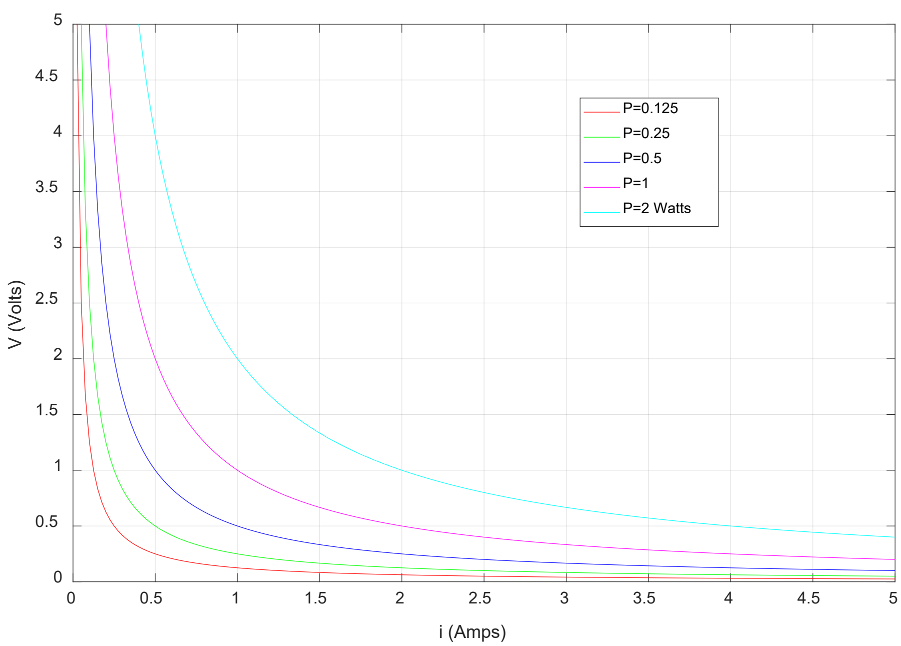
The results of this simulation are presented in the table below.

Resistors and Power Dissipation

Voltage *V* (volts)	Resistance *R* (Ω)	Current *i* (A)	Power *P* (Watts)
0.25	0.5	0.5	0.125
0.5	1	0.5	0.25
1	2	0.5	0.5
2	4	0.5	1
4	8	0.5	2
20	40	0.5	10
40	80	0.5	20

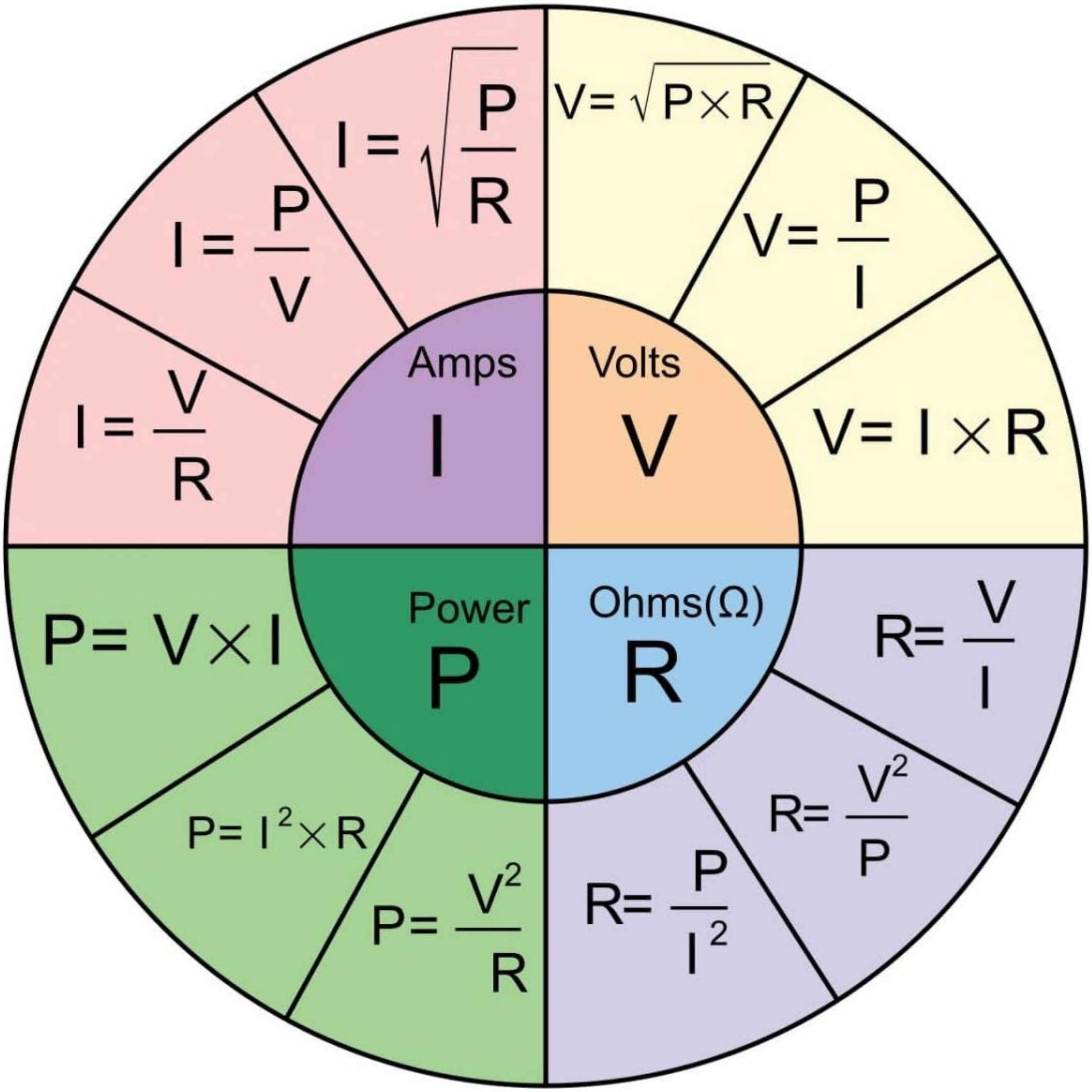


V vs. i for Different Resistances (Ohm's Law Plotted)

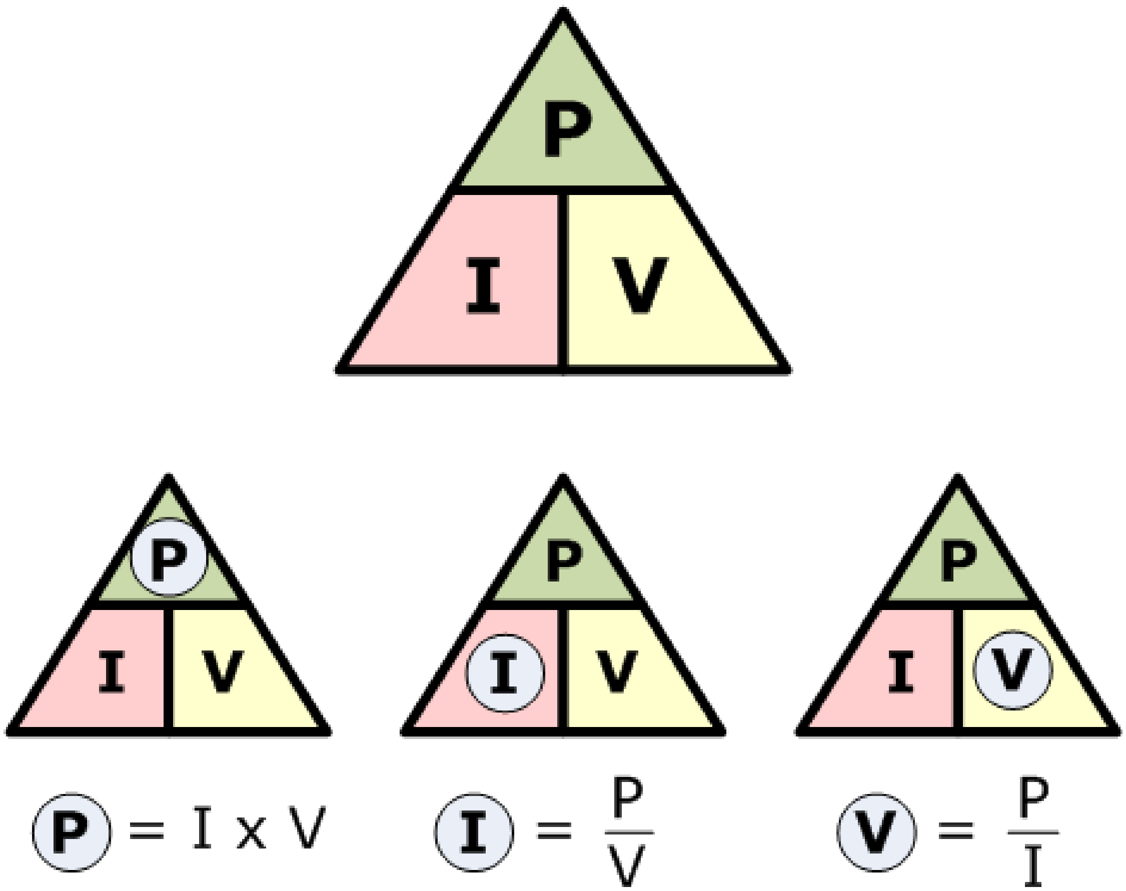


V vs. i for Different Powers

Looking at these two sets of graphs, the following conclusions can be drawn (borne out by the applicable equations). For a constant current, power increases linearly with increasing resistance ($P = i^2 R$). For a constant voltage, power decreases inversely proportional to increasing resistance ($P = \dfrac{V^2}{R}$).



Electrical Circuit Equations Summary

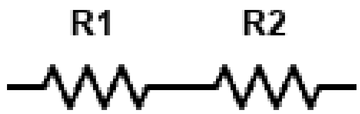


Resistor Power Triangle

Resistor Power Rating and the Power of Resistors

Resistors in Series and in Parallel (DC and AC same)

Let us consider two **resistors** in **series** as shown in the figure below.



Two Resistors in Series

To replace this arrangement with a single equivalent resistance constant R_{EQ}, note that the current flowing through both resistors is identical, but the voltages are different.

We want $V = iR_{EQ}$ so that $R_{EQ} = \frac{V}{i}$.

$$V = V_1 + V_2 = iR_1 + iR_2$$

$$R_{EQ} = \frac{iR_1 + iR_2}{i} = R_1 + R_2$$

Therefore, to find the overall equivalent resistance R_{EQ} for any number of **resistors** in pure **series**, simply add up the individual resistances. Pure series indicates there are no nodes between resistors leading to other branches of a circuit. For two and three **resistors** in **series**:

$$R_{EQ} = R_1 + R_2 \qquad\qquad R_{EQ} = R_1 + R_2 + R_3$$

In general the formula for *n* **resistors** in pure **series** is:

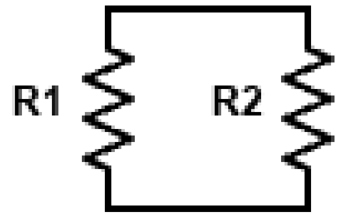
$$R_{EQ} = \sum_{i=1}^{n} R_i$$

For **resistors** in **series**, R_{EQ} is always greater than any individual R_i.

Example: 3 resistors in series

	$R_1 = 200$ $\quad R_2 = 300$ $\quad R_3 = 400\ \Omega$ $R_{EQ} = 200 + 300 + 400 = 900\ \Omega$

Now let us consider two **resistors** in **parallel** as shown in the figure below.



Two Resistors in Parallel

To replace this arrangement with a single equivalent resistance constant R_{EQ}, note that the currents flowing through each resistors are different, but the voltage across both resistors is the same.

We want $V = iR_{EQ}$ so that $R_{EQ} = \frac{V}{i}$. By KCL,

$$i = i_1 + i_2 = \frac{V}{R_1} + \frac{V}{R_2}$$

$$R_{EQ} = \frac{V}{i} = \frac{V}{\frac{V}{R_1} + \frac{V}{R_2}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

To find the overall equivalent resistance R_{EQ} for any number of **resistors** in pure **parallel** (again, not interrupted by nodes), first find the sum of the reciprocal of each resistance; then R_{EQ} is the reciprocal of that sum. For two and three **resistors** in **parallel**:

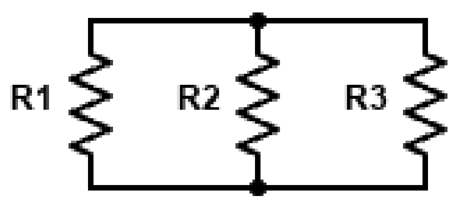
$$R_{EQ} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2} \qquad R_{EQ} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \frac{R_1 R_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

In general the formula for *m* **resistors** in pure **parallel** is:

$$R_{EQ} = \frac{1}{\sum_{i=1}^{m} \frac{1}{R_i}}$$

For resistors in parallel, R_{EQ} is always less than the least value of R_i.

Example: 3 resistors in parallel



$R_1 = 200 \qquad R_2 = 300 \qquad R_3 = 400\ \Omega$

$$R_{EQ} = \frac{1}{\frac{1}{200}+\frac{1}{300}+\frac{1}{400}}$$

$$= \frac{200(300)(400)}{200(300)+300(400)+200(400)}$$

$$R_{EQ} = 92.3\ \Omega$$

Caution:

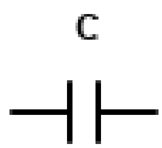
In mechanical / vibrational modeling springs in series and parallel combine exactly the opposite of resistors in series and parallel. This may be further confusing since the diagrams for springs and resistors look the same. However, they play different roles in their dynamic systems.

2.2 Capacitors

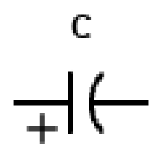
A capacitor is a passive two-terminal device that stores electrical energy between two plates via an electric field. The two plates are closely spaced surfaces and insulated from each other. The primary function of a capacitor is to store electrical charge in a circuit. The capacitance constant is C, with units of Farads (mF and μF more commonly). A Farad unit is a Coulomb per Volt (a one Farad capacitor stores a charge of one Coulomb across a potential difference of 1 V). Comparing two different capacitors, the one with a higher value of capacitance C stores more charge than the one with a lower value of capacitance C.

Since it stores charge, a capacitor is somewhat like a battery, with the following differences.

	capacitor	**battery**
energy storage	electric field	chemical
storage capacity	less	more
charge / discharge	fast	slow



Non-Polarized Capacitor
ceramic, Teflon, glass



Polarized Capacitor (electrolytic)
Aluminum, Tantalum, Niobium

Capacitor Schematic Symbols

The voltage V across a capacitor is the charge q divided by the capacitance constant C.

$$V = \frac{q}{C}$$

where the charge $q(t)$ (in Coulombs) is time integral of the current $i(t)$:

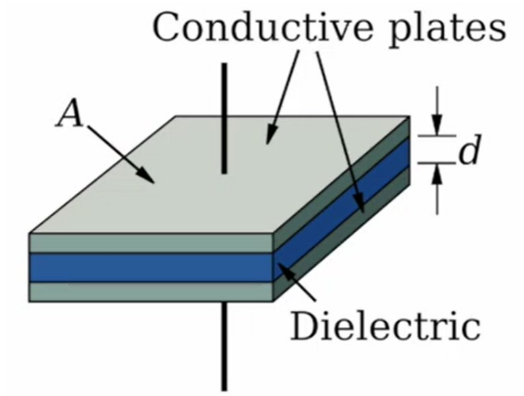
$$q(t) = \int i(t)dt$$

The capacitance C of a capacitor is constant; as the charge increases, the voltage does as well, in a constant linear proportion.

$$C = \frac{q(t)}{V(t)}$$

The capacitive energy stored E (Joules) is calculated as follows:

$$E = \frac{1}{2}CV^2$$



Physical Capacitor Construction

A **dielectric material** is an electrical insulator which can be polarized by an electric field. When placed in an electric field, electric charge cannot flow through the dielectric material as it would through an electrical conductor. In a dielectric material, free electrons only shift slightly, causing dielectric polarization. Hence, positive charges shift in the direction of the electric field and negative charges shift in the opposite direction.

Safety: never touch a charged capacitor. Even when powered off, a capacitor will still hold charge for some time, discharging according to a first-order system as shown on the following page.

The Capacitance constant C is calculated as follows (A is the plate area and d is the dielectric material gap distance):

$$C = \frac{\varepsilon A}{d} \qquad \varepsilon = \varepsilon_R \varepsilon_0$$

where ε is the absolute permittivity, ε_R is the relative permittivity, and ε_0 is the vacuum permittivity.

$$\varepsilon_0 = 8.85 \times 10^{-12} \quad \text{(Farads / m)}$$

and the relative permittivity for common dielectric materials is given in the table below.

Relative Permittivity of Dielectric Materials

Dielectric Material	Relative Permittivity ε_R (Farads / m)
air	1
Teflon	2
glass	5
Aluminum oxide	9
Tantalum Pentoxide	25
Niobium Pentoxide	40

Here are thirteen ways to express the capacitance *C* unit Farads (F) in SI units:

$$F = \frac{\sec^4 A^2}{kgm^2} = \frac{\sec^2 C^2}{kgm^2} = \frac{C}{V} = \frac{A\sec}{V} = \frac{W\sec}{V^2} = \frac{J}{V^2} = \frac{Nm}{V^2} = \frac{C^2}{J} = \frac{C^2}{Nm} = \frac{\sec}{\Omega} = \frac{1}{\Omega Hz} = \frac{S}{Hz} = \frac{\sec^2}{H}$$

Example Capacitor Applications

Power factor correction in large buildings with large capacitor banks.

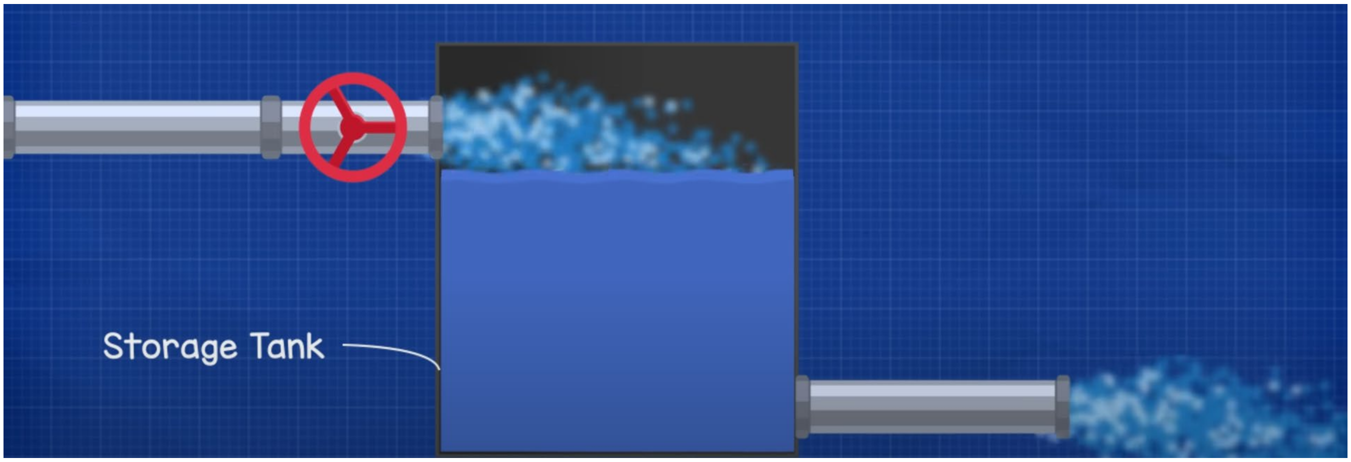
When too many inductive loads are placed into a circuit, the AC current and voltage will no longer be in phase, with the current lagging behind the voltage. Capacitor banks can counteract this to bring the current and voltage back into alignment.

Smooth out peaks when converting AC to DC power.

Using a full bridge rectifier, the negative portion of the AC signal is flipped to be only positive. This appears to the circuit to be direct current; however, there is still a gap between the half-sinusoidal peaks. A capacitor can be applied to release electrical energy such that these gaps are somewhat smoothed (not perfectly) so the signal appears more like DC.

Hydraulic Analogy of Electrical Capacitance

Consider the fluid flow of water through a pipe with a shutoff valve. The pump pressure causes the flow; this is analogous to the electrical circuit voltage potential driving the current. The fluid flow rate is analogous to the electrical charge. Now consider that the pipe's shutoff valve is closed: the fluid flow will stop. But if a water tank is introduced before the fluid outflow, when the shutoff valve is closed, fluid continues to flow until the tank is empty. So the tank can be used to smooth out interruptions in the fluid supply. Thus a fluid flow system with storage tank is analogous to an electrical capacitor storing electrical charge for release even with no voltage supply.



Fluid Flow System with Storage Tank

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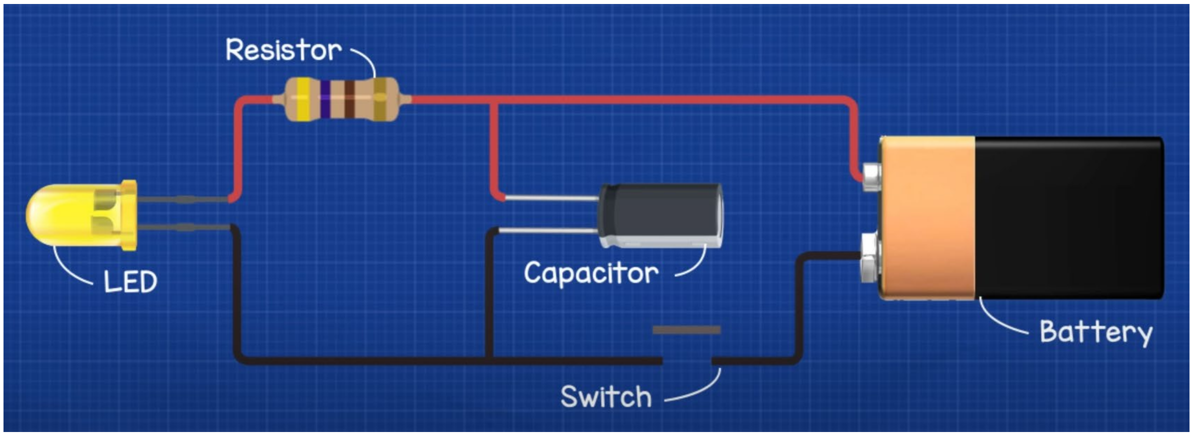
In the LED circuit example shown below, the LED lighting will be steady, even when the circuit switch is open, since the capacitor will be charged, and releasing electrical charge when the battery is disabled by the open switch.



LED Circuit with Capacitor

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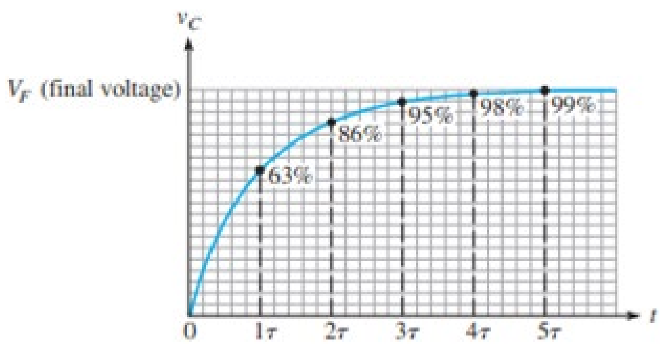
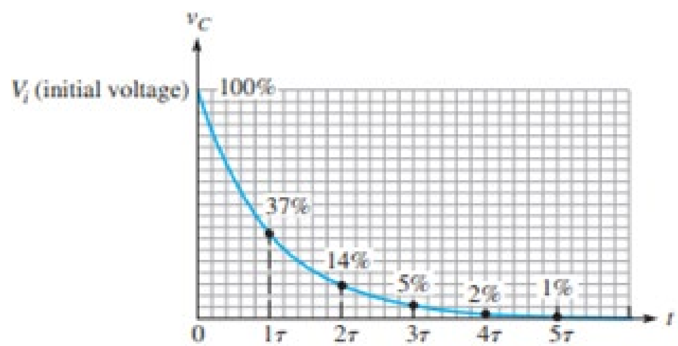
Time Constant

In a DC circuit, when not charged, capacitors act as a closed circuit (i.e. short circuit). When fully charged, capacitors act like an open circuit. The charging and discharging of capacitors follow a classic first-order ODE solution governed by the time constant $\tau = RC$ (sec).



Capacitor Charging Curve

$$v(t) = V_F(1 - e^{-\frac{t}{RC}})$$



Capacitor Discharging Curve

$$v(t) = V_i e^{-\frac{t}{RC}}$$

The classic first-order ODE solution is:

$$y(t) = y_F(1 - e^{-\frac{t}{\tau}})$$

assuming zero initial condition $y_0 = 0$. Comparing with the above specific capacitor ODE solutions, we see how the time constant $\tau = RC$ was obtained.

See Section 2.6 for first-order RC circuit ODEs, solutions, and examples.

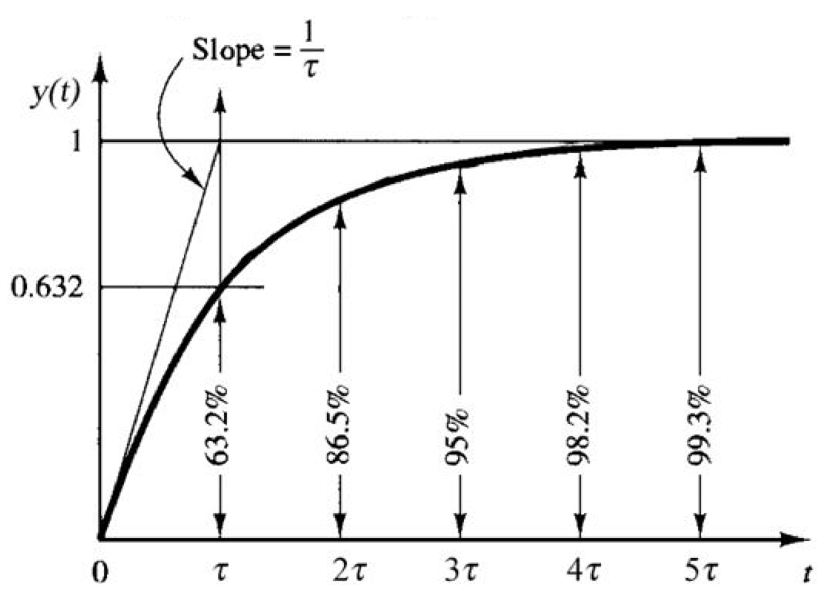
Time Constant (cont.)

Theoretically the capacitor charging response $v(t)$ never reaches the final voltage of V_F until time reaches infinity. The time constant τ gives a convenient measure of how fast this transient response occurs, starting from time $t = 0$.

elapsed time constants τ	0τ	1τ	2τ	3τ	4τ	5τ	6τ	7τ
time (sec)	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14
percentage of displacement	0	63.2	86.5	95.0	98.2	99.3	99.8	99.9
$v(t)$ value (for V_F = 0.100)	0.000	0.063	0.087	0.095	0.098	0.099	0.100	0.100

In this table the time constant is τ = RC = 0.02 sec, and so the elapsed times for 1, 2, and 3 time constants are 0.02, 0.04, and 0.06 sec, respectively, and so on. These percentages of the capacitor voltage from 0 to 0.100 can be seen in the left capacitor charging curve in the previous page. That is, after one time constant the first-order system transient response has covered 63.2% of the total displacement (value of 0.0632), after two time constants the first-order system transient response has covered 86.5% of the total displacement (value of 0.0865), after three time constants the first-order system transient response has covered 95.0% of the total displacement (value of 0.095), and so on. For some reason biologists prefer the 63.2% one-time-constant rule of thumb, while engineers favor the 95.0% three-time-constants rule of thumb.

In the figure below we see the initial slope of the first-order response is $1 / \tau$. CAUTION: this is only for the final value of $y(t)$ being 1! If the final value of $y(t)$ is K, then the initial slope of the first-order response is K / τ.



First-Order System Response: Initial Slope and First Five Time Constants

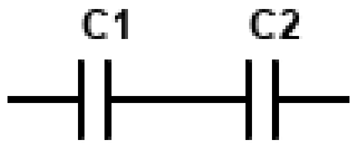
https://slideplayer.com/slide/8906092/27/

Capacitors in Series and in Parallel (DC and AC are the same)

The equivalent capacitance for multiple capacitors in series and parallel are opposite to the equations for resistors in series and parallel, presented earlier.

Let us consider two **capacitors** in **series** as shown in the figure below. The basic principle is not Ohm's Law, but the voltage V across a capacitor is the charge q divided by the capacitance constant C.

$$V = \frac{q}{C}$$



Two Capacitors in Series

To replace this arrangement with a single equivalent capacitance constant C_{EQ}, note that the voltages V across each capacitor are different, but the charge q across both capacitors is the same.

We want $V = \frac{q}{C_{EQ}}$ so that $C_{EQ} = \frac{q}{V}$.

$$V = V_1 + V_2 = \frac{q}{C_1} + \frac{q}{C_2}$$

$$C_{EQ} = \frac{q}{V} = \frac{q}{\frac{q}{C_1} + \frac{q}{C_2}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

To find the overall equivalent capacitance C_{EQ} for any number of **capacitors** in pure **series** (pure series indicates there are no nodes between capacitors leading to other branches of a circuit), first find the sum of the reciprocal of each individual capacitance; then C_{EQ} is the reciprocal of that sum. For two and three capacitors in **series**:

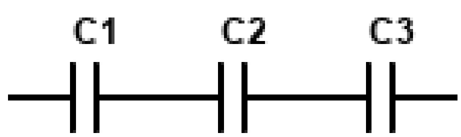
$$C_{EQ} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2} \qquad C_{EQ} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} = \frac{C_1 C_2 C_3}{C_1 C_2 + C_2 C_3 + C_1 C_3}$$

In general the formula for m **capacitors** in pure **series** is:

$$C_{EQ} = \frac{1}{\sum_{i=1}^{m} \frac{1}{C_i}}$$

For **capacitors** in **series**, C_{EQ} is always less than the least value of C_i.

<u>Example: 3 capacitors in series</u>



$C_1 = 600 \qquad C_2 = 500 \qquad C_3 = 400 \;\; \mu\text{F}$

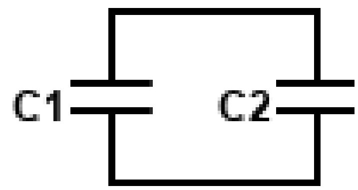
$$C_{EQ} = \frac{1}{\frac{1}{600} + \frac{1}{500} + \frac{1}{400}}$$

$$= \frac{600(500)(400)}{600(500) + 500(400) + 600(400)}$$

$C_{EQ} = 162.2 \;\; \mu\text{F}$

Now let us consider two **capacitors** in **parallel** as shown in the figure below. The basic principle is again:

$$V = \frac{q}{C}$$



Two Capacitors in Parallel

To replace this arrangement with a single equivalent capacitance constant C_{EQ}, note that the voltage across both capacitors is the same, but the charge across each individual capacitor is different.

We want $V = \frac{q}{C_{EQ}}$ so that $C_{EQ} = \frac{q}{V}$.

$$q = q_1 + q_2 = C_1 V + C_2 V$$

$$C_{EQ} = \frac{C_1 V + C_2 V}{V} = C_1 + C_2$$

To find the overall equivalent capacitance C_{EQ} for any number of **capacitors** in pure **parallel** (again, not interrupted by nodes), simply add up the individual capacitances. For two and three capacitors in pure **parallel**:

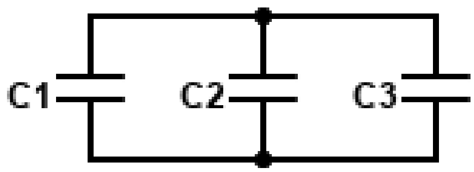
$$C_{EQ} = C_1 + C_2 \qquad\qquad C_{EQ} = C_1 + C_2 + C_3$$

In general the formula for *n* capacitors in pure parallel is:

$$C_{EQ} = \sum_{i=1}^{n} C_i$$

For capacitors in **parallel**, C_{EQ} is always greater than any individual C_i.

Example: 3 capacitors in parallel



$C_1 = 600 \qquad C_2 = 500 \qquad C_3 = 400 \ \mu F$

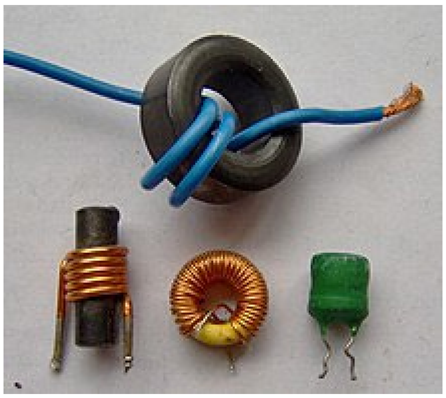
$C_{EQ} = 600 + 500 + 400 = 1{,}500 \ \mu F$

Note:

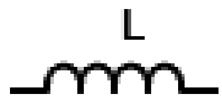
Since resistors and capacitors in series and parallel combine mathematically opposite of each other, and since resistors and springs in series and parallel combine mathematically opposite of each other, therefore, logically, springs and capacitors in series and parallel combine mathematically the same as each other.

2.3 Inductors

An **inductor** is a passive two-terminal electrical component that stores electrical energy in a magnetic field when an electric current flows through it. This stored electrical energy can be released very quickly by the inductor. Physically inductors are constructed of an insulated copper wire wound into a coil, often around an iron coil to increase the inductance (by increasing the magnetic field). Everything with a coiled wire will act like an inductor, including electric motors, transformers, and relays. The inductance constant *L* has units of Henrys (mH more commonly). 1 Henry is equivalent to 1 Volt of electromotive force (emf) across the inductor with a changing current of 1 A / sec.



Example Inductors



Inductor Schematic Symbol

In DC vs. AC circuits, we have the following duality between inductors and capacitors regarding their steady-state behaviour of open- vs. short-circuits.

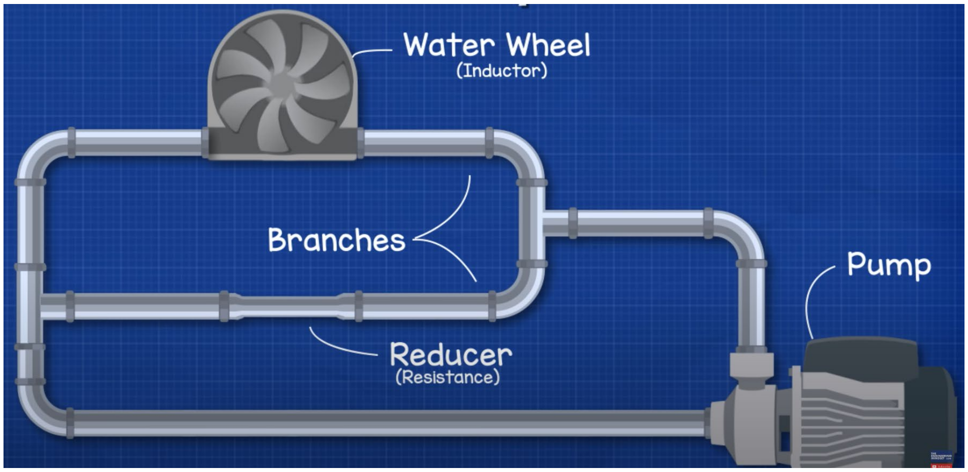
	DC	**AC high frequency**
Capacitor	open circuit	short circuit
Inductor	short circuit	open circuit

With an inductor, the current lags the voltage in AC circuits.

Hydraulic Analogy of Electrical Inductance

Consider the fluid flow of water through pipes. The pump pressure causes the flow – this is analogous to the electrical circuit voltage potential driving the current. The fluid flow rate is analogous to the electrical current. Now consider that the pipe splits into two branches. If there is a pipe constriction (a length of reduced diameter) in one of the pipe branches, the fluid flow rate will be reduced in that branch. So the constricted pipe in fluid flow is analogous to an electrical resistor limiting the flow of electrical current.

Now, consider that the second pipe branch has a water wheel built into it, analogous to the electrical circuit inductor. Starting from rest it takes some time for the fluid flow (electrical current) to increase due to the inertia of the water wheel. As the water reaches the pipe branching, it is forced through the constricted pipe (electrical resistance) because there is initially a blocking of flow in the other pipe branch due to the water wheel inertia. As time increases, the water wheel will start to turn, faster and faster, until it reaches its maximum rotational speed. Now the water wheel presents very little resistance to flow, hence the fluid flow is now easier through that branch than through the constricted pipe branch. In this condition, the water will exclusively flow through the water wheel branch, ceasing to flow through the constricted pipe branch.



Fluid Flow / Electrical Inductor Analogy

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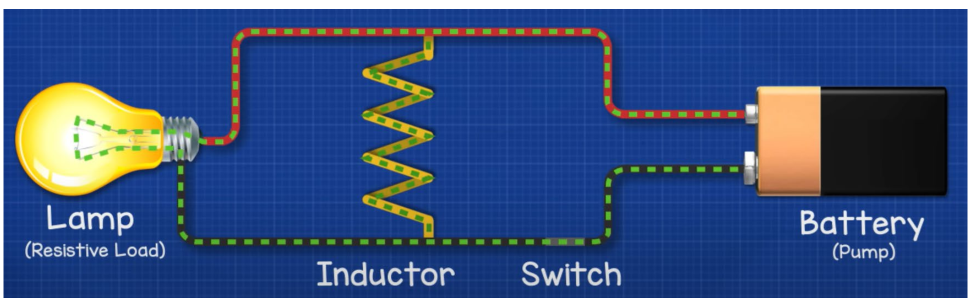
If we turn off the pump, water flows in a closed loop through the constricted pipe and water wheel. The water wheel inertia in rotational motion means it cannot stop instantaneously. Here the water wheel acts as a pump. The water will continue to flow until the resistance of the pipes stops the flow.



Electrical DC Inductive Circuit

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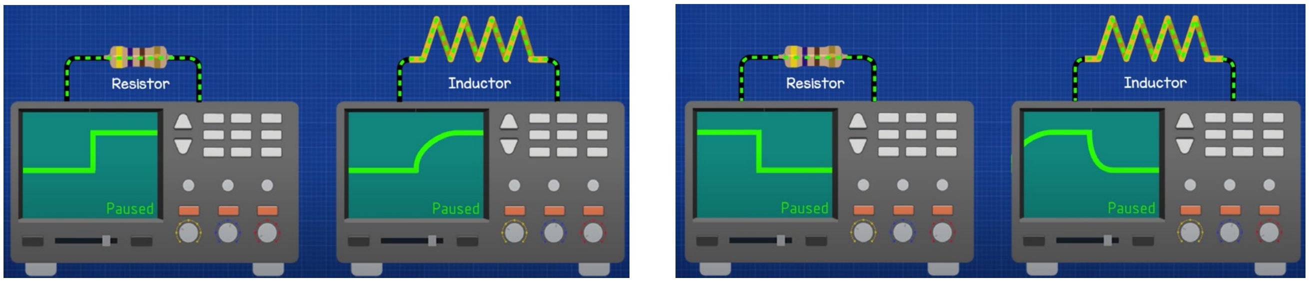
In the electrical DC inductive circuit shown above, when the switch is closed, current will flow through the resistive load (lighting the lamp) since the inductor initially presents high resistance to current flow. As the current flow continues, the inductor resistance to flow decreases, to nearly zero. Hence all the current will now flow through the inductor, switching off the lamp.

Now when the switch is opened, the inductor will continue to force electrons through the circuit, again turning on the lamp, until the resistance of the circuit dissipates the energy.

Inductors do not like change in current. When current increases, inductors try to stop the increase with an opposing electromotive force (emf). When current decreases, inductors try to stop the decrease by pushing electrons out to maintain the previous current.



Resistor vs. Inductor Response Times

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According to **Faraday's Law of Induction**, the time-varying magnetic field of an inductor causes an electromotive force (emf). According to **Lenz's Law**, the induced voltage direction is opposite to the current that causes it. From physics, the unit of magnetic flux is the Weber (Wb): 1 Wb = 1 V sec. One tesla (T) is equivalent to a magnetic flux density of 1 Wb / m. Further, 1 Wb is equivalent to 10^8 Mx (Maxwells).

The inductor constant *L* is called the inductance (units of Henrys, mH more commonly). *L* is the ratio of the voltage v_L to the time rate of change current i_L:

$$L = \frac{v_L(t)}{\frac{di_L(t)}{dt}}$$

$$v_L(t) = L\frac{di_L(t)}{dt} \qquad\qquad i_L(t) = \frac{1}{L}\int v_L(t)dt$$

Here are ten ways to express the inductance *L* unit Henrys (H) in SI units:

$$\mathrm{H} = \frac{\mathrm{kgm^2}}{\mathrm{sec^2A^2}} = \frac{\mathrm{Nm}}{\mathrm{A^2}} = \frac{\mathrm{kgm^2}}{\mathrm{C^2}} = \frac{\mathrm{J}}{\mathrm{A^2}} = \frac{\mathrm{Tm^2}}{\mathrm{A}} = \frac{\mathrm{Wb}}{\mathrm{A}} = \frac{\mathrm{Vsec}}{\mathrm{A}} = \frac{\mathrm{sec^2}}{\mathrm{F}} = \frac{\Omega}{\mathrm{radHz}} = \frac{\Omega\mathrm{sec}}{\mathrm{rad}}$$

Inductors are one of the three passive linear circuit elements (with capacitors and resistors). Inductors are used in AC electronic equipment, especially radios. Inductors called chokes are used to block AC while allowing DC to pass. Inductors can be used in electronic filters to separate signals of different frequencies, and in combination with capacitors to make tuned circuits, used to tune radio and TV receivers.

Inductor - Wikipedia

The stored electrical energy *E* of an inductor is:

$$E = \frac{1}{2}i^2 L$$

When a unit step in voltage is applied to an inductor in a DC circuit:

- In the short-term, the initial current is zero and so this behaves like an open circuit.

- As time increases, the current increases according to a classic first-order system rise over time, until inductor saturation occurs. As in capacitor charging, this current increase is governed by the time constant τ. This time constant is derived in Section 2.6:

$$\tau_{RL} = \frac{L}{R}$$

- In the long-term, as the inductor transient response dies out, the magnetic flux through the inductor becomes constant, leading to a short circuit.

Inductors are used in electrical circuits in the following ways (partial list of inductor applications):

1. Used to boost converters to increase the DC output voltage.

2. Used to choke an AC supply and allow only DC to pass.

3. Used to filter and separate signals of different frequencies.

4. Used for electric motors, transformers, and relays.

Inductance L of an inductor in Henrys cannot be measured with a standard digital multimeter. Instead, one must use an *RLC* meter.

<u>Inductors in Series and in Parallel (DC and AC same)</u>

Inductors in series and parallel combine in exactly the same manner as resistors in series and parallel. The proofs are left to the interested student.

This means inductors in series and parallel combine in the opposite manner as capacitors in series and parallel. This also means inductors in series and parallel combine in the opposite manner as mechanical/vibrational springs in series and parallel.

In general the formula for *n* **inductors** in pure **series** is:

$$L_{EQ} = \sum_{i=1}^{n} L_i$$

In general the formula for *m* **inductors** in pure **parallel** is:

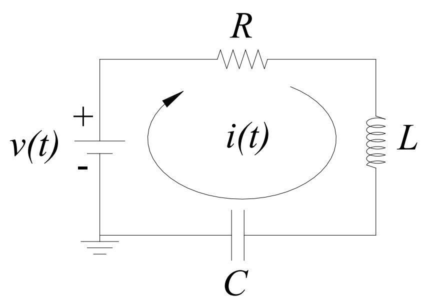
$$L_{EQ} = \frac{1}{\sum_{i=1}^{m} \frac{1}{L_i}}$$

2.4 Kirchhoff's Circuit Laws

Instead of Newton's and Euler's dynamics laws for mechanical systems, we use Kirchhoff's Circuit Laws to derive the dynamic models for electrical circuits. Kirchhoff's Circuit Laws are generally applied to DC circuits. The polarity convention is that voltage rises (positive) with a voltage source and drops (negative) across the passive elements (resistor *R*, inductor *L*, and capacitor *C*). Also, current is assumed to flow from the positive terminal of the voltage source to the negative terminal. As presented earlier, this is exactly backwards of the reality of current flow, but it makes no difference to the equations and Electrical Engineering still generally follows this fallacious convention! (This is called the conventional current-flow assumption.)

Kirchhoff's Voltage Law (KVL) states that the sum of voltages around any circuit loop is zero. Alternatively, KVL could be stated that, for a closed circuit loop, the sum of the voltage "lifts" (positive) must equal the sum of voltage "drops" (negative). A closed circuit loop is any loop of a circuit that ends where it started. Kirchhoff's Voltage Law (aka Kirchhoff's Loop Rule, KLR) is based on the principle of the Conservation of Energy.

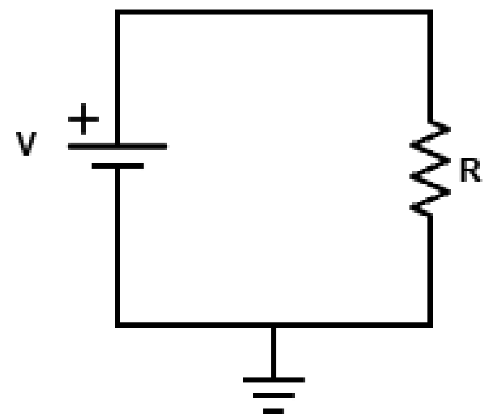
$$\sum_{k=1}^{m} v_k(t) = 0$$



KVL Diagram

KVL applied to the above single-loop circuit:

<u>Kirchhoff's Voltage Law (KVL) Example 1</u>



KVL Example 1 Circuit Diagram

$V - V_R = 0$ (KVL)

$V = iR$ (Ohm's Law)

a. Given: $V = 10$ V $R = 100\ \Omega$

Find i $i = \frac{V}{R} = \frac{10}{100} = 0.1$ A

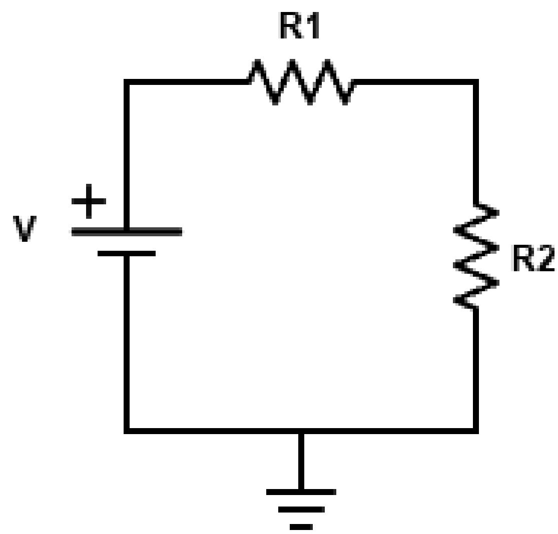
b. Given: $R = 200\ \Omega$ $i = 0.2$ A

Find V $V = iR = 0.2(200) = 40$ V

c. Given: $V = 30$ V $i = 50$ mA

Find R $R = \frac{V}{i} = \frac{30}{0.05} = 600\ \Omega$

Kirchhoff's Voltage Law (KVL) Example 2



KVL Example 2 Circuit Diagram

$V - V_1 - V_2 = 0$ (KVL) $\quad V_1 = i_1 R_1 = iR_1 \quad V_2 = i_2 R_2 = iR_2 \quad V = V_1 + V_2 = i(R_1 + R_2)$

a. Given: $V = 10$ V $\quad R_1 = 60\ \Omega \quad R_2 = 40\ \Omega$

Find i $\quad i = \dfrac{V}{R_1 + R_2} = \dfrac{10}{60 + 40} = 0.1\ \text{A}$

This current i answer is identical to that in Example 1a above since $R_1 + R_2 = R = 100$ and $V = 10$ V, same as Example 1a.

b. Given: $i = 0.2$ A $\quad R_1 = 60\ \Omega \quad R_2 = 40\ \Omega$

Find V $\quad V = i(R_1 + R_2) = 0.2(60 + 40) = 20\ \text{V}$

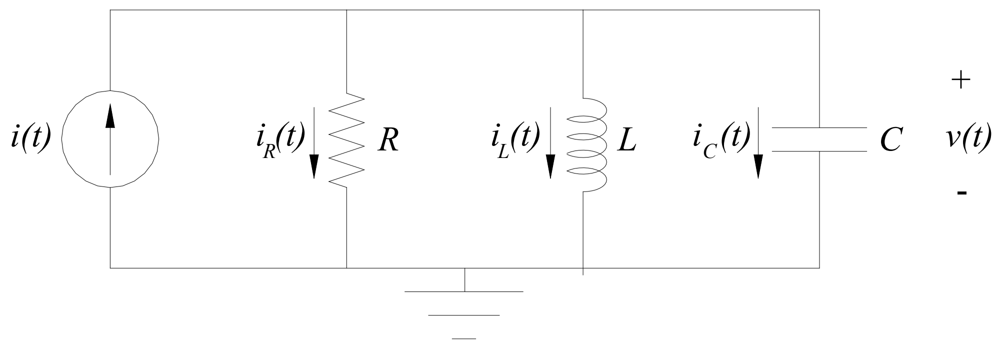
c. Given: $V = 12$ V $\quad i = 30$ mA

Find R_1, R_2 $\quad R_1 + R_2 = \dfrac{V}{i} = \dfrac{12}{0.03} = 400\ \Omega$

Here we cannot definitively solve for R_1, R_2, instead only the sum $R_1 + R_2$ (since there is one equation in the two resistance unknowns). Choosing $R_1 = 100\ \Omega$, $R_2 = R_1 + R_2 - R_1 = 400 - 100 = 300\ \Omega$.

Kirchhoff's Current Law (KCL) states that the sum of currents flowing into any circuit node is zero (in – positive, out – negative). Alternatively, KCL could be stated that the sum of currents flowing into a circuit junction must equal the sum of currents flowing out of that circuit junction. Kirchhoff's Current Law (aka Kirchhoff's Junction Rule, KJR) is based on the principle of the Conservation of Charge.

$$\sum_{j=1}^{n} i_j(t) = 0$$

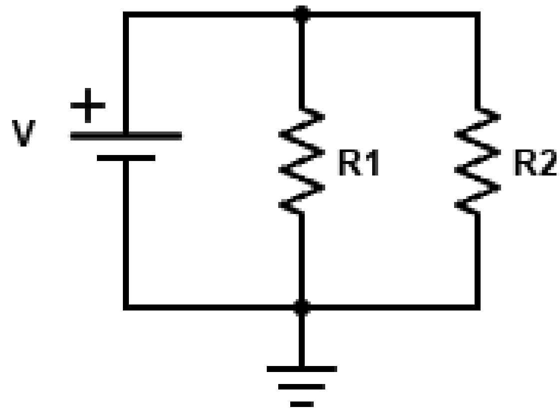


KCL Diagram

KCL applied to the above circuit:

Typically in applying Kirchhoff's Circuit Laws (KVL and KCL) to a practical DC circuit, we need one more circuit loop than the number of circuit junctions.

Kirchhoff's Current Law (KCL) Example



KCL Example Circuit Diagram

$$i - i_1 - i_2 = 0$$
$$i = i_1 + i_2 \quad \text{(KCL)} \qquad i_1 = \frac{V_1}{R_1} = \frac{V}{R_1} \qquad i_2 = \frac{V_2}{R_2} = \frac{V}{R_2} \qquad i = \frac{V}{R_1} + \frac{V}{R_2} = V\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

a. Given: $V = 10$ V $\qquad R_1 = 60\ \Omega \qquad R_2 = 40\ \Omega$

Find i $\qquad i = 10\left[\frac{1}{60} + \frac{1}{40}\right] = 0.4167\ \text{A}$

This current is significantly higher than KVL Example 2a above, with the same numbers for V, R_1, R_2. The reason for this is that the effective resistance of this parallel circuit (24 Ω) is much lower than the effective resistance of the above series circuit (100 Ω), allowing a higher current given the same 10 V input. With the relationship $i = V / R$, current is inversely related to the size of the resistance.

b. Given: $i = 0.5$ A $\qquad R_1 = 60\ \Omega \qquad R_2 = 40\ \Omega$

Find V $\qquad V = \dfrac{i}{\dfrac{1}{R_1} + \dfrac{1}{R_2}} = \dfrac{0.5}{\dfrac{1}{60} + \dfrac{1}{40}} = 12\ \text{V}$

c. Given: $V = 20$ V $\qquad i = 2$ A

Find R_1, R_2 $\qquad \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2} = \frac{i}{V} = \frac{2}{20} = 0.10\ (1/\Omega)$

Here we cannot definitively solve for R_1, R_2, since there is one equation in the two resistance unknowns. Choosing $R_1 = 60\ \Omega$, we calculate $R_2 = 12\ \Omega$.

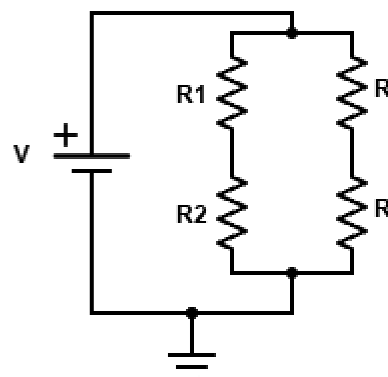
Example: Compound DC Resistive Circuit

Consider the DC resistive circuit shown below. The two resistive legs each have two resistors in series. Then those two legs are parallel to each other.

Given $R_1 = 100, R_2 = 200, R_3 = 300, R_4 = 400 \ \ \Omega$. Also given $V = 150$ V.

a. What is the current in the main circuit branch? What is the current flowing through each resistive leg?

b. What is the voltage drop across each resistor?



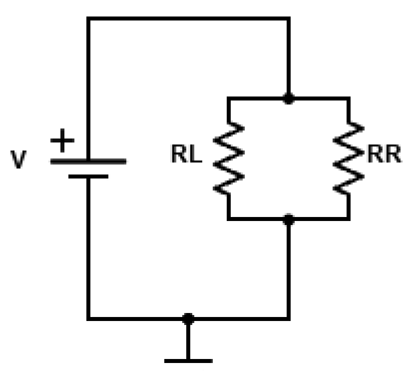
Compound DC Resistive Circuit

a. First find the equivalent series resistances R_L and R_R in each resistive leg:

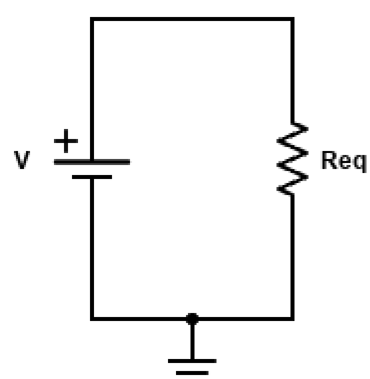
$$R_L = R_1 + R_2 = 100 + 200 = 300 \ \ \Omega \qquad\qquad R_R = R_3 + R_4 = 300 + 400 = 700 \ \ \Omega$$

Next find the overall equivalent resistance R_{EQ}, from R_L and R_R in parallel:

$$R_{EQ} = \frac{1}{\frac{1}{R_L} + \frac{1}{R_R}} = \frac{R_L R_R}{R_L + R_R} = \frac{300(700)}{300 + 700} = 210 \ \ \Omega$$



Simplified Circuit 1



Simplified Circuit 2

With these two simplifications, it is now a simple matter to find the current in the main circuit branch, using Ohm's Law:

$$V = iR_{EQ} \qquad \text{so} \qquad i = \frac{150}{210} = 0.714 \text{ A}$$

Now we use KCL to get the following expression:

$$i - i_L - i_R = 0 \qquad \text{so} \qquad i = i_L + i_R = \frac{V}{R_L} + \frac{V}{R_R} = V\left[\frac{1}{R_L} + \frac{1}{R_R}\right] = V\left[\frac{R_L + R_R}{R_L R_R}\right] = \frac{V}{R_{EQ}}$$

Above we used the fact that the voltage drop across both series resistive legs are identical, equal to the voltage supply V. Also, i_L flows through both left-leg resistors and i_R flows through both right-leg resistors. Actually we didn't need that fancy mathwork; the currents in the left and right legs are simply:

$$i_L = \frac{V_L}{R_L} = \frac{V}{R_L} = \frac{150}{300} = 0.5 \text{ A} \qquad\qquad i_R = \frac{V_R}{R_R} = \frac{V}{R_R} = \frac{150}{700} = 0.214 \text{ A}$$

Note the left resistive leg, with the lower resistance R_L, carries a higher current than the right resistive leg, with the higher resistance R_R. This is necessary since the voltage across each leg is identical, V, and thus there is an inverse relationship between current and resistance in Ohm's Law (i.e. for a constant voltage V, if R decreases, i must increase, and if R increases, i must decrease).

Current check: $\quad i = i_L + i_R = 0.5 + 0.214 = 0.714 \text{ A} \quad$ matches!

b. Now this part is easy, using Ohm's Law four times.

$$V_1 = i_L R_1 = 0.5(100) = 50 \text{ V} \qquad\qquad V_3 = i_R R_3 = 0.214(300) = 64.3 \text{ V}$$

$$V_2 = i_L R_2 = 0.5(200) = 100 \text{ V} \qquad\qquad V_4 = i_R R_4 = 0.214(400) = 85.7 \text{ V}$$

Voltage check: $\quad V = V_1 + V_2 = V_3 + V_4 = 150 \text{ V} \quad$ matches!

Example concludes.

2.5 Circuit Components Relationships

Many electrical circuits can exhibit electrical dynamics and vibrations, i.e. oscillation of the electrical output variable of interest. Such circuits lead to second-order integro-differential equations, whose output is dynamical in nature even for DC circuits. In the following Section 2.6 we only consider first-order *RC* and *RL* DC circuits, which have dynamics but no oscillation.

Electrical Circuit Elements

In order to use Kirchhoff's Laws to derive models for electrical circuits, we need the following relationships. They relate the current $i(t)$ flowing through and the voltage $v(t)$ across the three standard passive electrical elements *R*, *L*, and *C*. A capacitor *C* stores electrical charge in an electric field, a resistor *R* dissipates electrical energy, and an inductor *L* stores electrical energy in a magnetic field. Note the equations in the last two columns of each row are equivalent, just solved for the current $i(t)$ or the voltage $v(t)$, respectively.

element	notation	units	*i*(*t*)	*v*(*t*)
capacitor	*C*	Farads	$i_C(t) = C\dfrac{dv_C(t)}{dt}$	$v_C(t) = \dfrac{1}{C}\int i_C(t)dt$
resistor	*R*	Ohms	$i_R(t) = \dfrac{v_R(t)}{R}$	$v_R(t) = Ri_R(t)$
inductor	*L*	Henrys	$i_L(t) = \dfrac{1}{L}\int v_L(t)dt$	$v_L(t) = L\dfrac{di_L(t)}{dt}$

Additional Electrical Circuit Variables

In some electrical circuit modeling you may encounter two more variable types, electrical charge and electrical flux. As shown in Dr. Bob's on-line Atlas of Models and Transfer Functions:

people.ohio.edu/williams/html/PDF/ModelTFAtlas.pdf

electrical charge $q(t)$ is the time integral of current $i(t)$ and electrical flux $\phi(t)$ is the time integral of voltage $v(t)$:

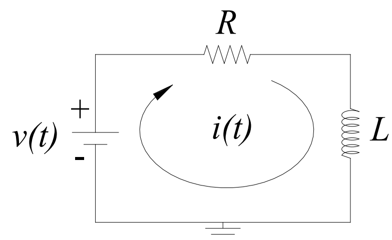
$$q(t) = \int i(t)dt \qquad\qquad \phi(t) = \int v(t)dt$$

2.6 First-Order Electrical Circuits

The two circuits presented in this section have dynamics described by first-order ordinary differential equations (ODEs). First we present the model for the voltage-driven *RL* Series Electrical Circuit, followed by the model for the voltage-driven *RC* Series Electrical Circuit. In this context, 'model' indicates the first order differential or integral equation representing the circuit dynamics of each case. Both models yield first-order initial-value problem (IVP) differential equations (ODEs) subject to one initial condition on the output and given the input voltage; hence, both may be solved in the same manner.

***RL* Series Electrical Circuit**

input: voltage $v(t)$
output: current $i(t)$



Derive the model by Kirchhoff's Voltage Law (KVL):

RL Series Electrical Circuit model:

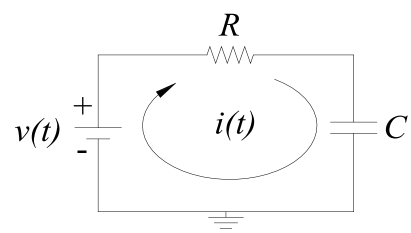
$$L\frac{di(t)}{dt} + Ri(t) = v(t)$$

The *RL* circuit time constant τ is:

$$\tau_{RL} = \frac{L}{R}$$

_RC_ Series Electrical Circuit

input: voltage $v(t)$
output: current $i(t)$ OR charge $q(t) = \int i(t)dt$



Derive the model by Kirchhoff's Voltage Law (KVL):

RC Series Electrical Circuit model with current $i(t)$ output:

$$Ri(t) + \frac{1}{C}\int i(t)dt = v(t)$$

substitute charge $q(t)$:

$$q(t) = \int i(t)dt$$
$$\dot{q}(t) = i(t)$$

RC Series Electrical Circuit model with charge $q(t)$ output:

$$R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$$

The *RC* circuit time constant τ is:

$$\tau_{RC} = RC$$

***RC* Series Electrical Circuit General First-Order ODE Solution**

Solve:

$$R\dot{q}(t)+\frac{1}{C}q(t)=v(t)$$

for the charge $q(t)$, given the resistance R and capacitance C constants, subject to the given initial condition $q(0) = q_0$ and an input step voltage $v(t)$ of magnitude v. We will first use the **Slow ME Way**[1].

Step 1: Homogeneous Solution

This is the transient response to the initial condition q_0 (and also forcing function $v(t)$). The transient response goes to zero given enough time t.

$$R\dot{q}_H(t)+\frac{1}{C}q_H(t)=0$$

Assume: $q_H(t)=Ae^{st}$

Substitute:

$$q_H(t)=Ae^{st}$$
$$\dot{q}_H(t)=sAe^{st}$$

into the homogeneous equation.

$$RsAe^{st}+\frac{1}{C}Ae^{st}=0$$

Now for some fun factoring:

$$A\left[Rs+\frac{1}{C}\right]e^{st}=0$$

The homogeneous constant A cannot possibly be zero, and e^{st} is never zero (assuming $t \ge 0$). The term in the square brackets in the above equation is called the characteristic polynomial. When set equal to zero this is called the characteristic equation.

$$Rs+\frac{1}{C}=0$$

[1] Named by Rob Belinski at OU in 1996. In proper-math-speak this is called the Variation of Parameters Method, with the Method of Undetermined Coefficients used in the particular solution.

This first-order polynomial in *s* has one root, called the characteristic value:

$$s = -\frac{1}{RC}$$

and the homogeneous solution is:

$$q_H(t) = Ae^{-\frac{t}{RC}}$$

For now the homogeneous constant *A* must be left unknown.

Step 1 concluded.

Step 2: Particular Solution

This is the steady-state response to the forcing function *v*(*t*). The steady-state response is what dominates after the transient response goes to zero.

$$R\dot{q}_P(t) + \frac{1}{C}q_P(t) = v(t) = vu(t)$$

where *u*(*t*) is the unit step function.

Assume: $q_P(t) = B$

Substitute:

$$q_P(t) = B$$
$$\dot{q}_P(t) = 0$$

into the particular equation.

$$R(0) + \frac{1}{C}B = v$$

The particular solution is entirely found in this Step 2:

$$q_P(t) = B = Cv$$

Step 2 concluded.

Step 3: Total Solution

Since the ODE to be solved is a linear ODE, linear superposition is used to find the total solution $q(t)$.

$$q(t) = q_H(t) + q_P(t) = Ae^{-\frac{t}{RC}} + Cv$$

Now in this Step 3 the initial condition is applied to find the homogeneous solution constant A:

$$q(0) = q_0 = Ae^{-\frac{(0)}{RC}} + Cv$$

$$A = q_0 - Cv$$

And so the overall general solution is:

$$q(t) = (q_0 - Cv)e^{-\frac{t}{RC}} + Cv$$

Written in standard form:

$$q(t) = q_0 e^{-\frac{t}{RC}} + Cv\left[1 - e^{-\frac{t}{RC}}\right]$$

Step 3 and solution concluded.

As presented in Section 2.2, the time constant τ in a classical first-order ODE solution is $y(t) = y_F(1 - e^{-\frac{t}{\tau}})$ (with zero initial condition). Therefore, in this problem, $\tau = RC$ (units: sec). See Section 2.2. for the important mathematical characteristics for the time constant τ in first-order ODEs.

After this solution for charge $q(t)$ in an RC circuit is obtained, the associated current $i(t)$ may easily be found. Since charge is defined as the time integral of current, it follows that current is the time derivative of charge:

$$q(t) = \int i(t)dt \qquad \text{and so} \qquad i(t) = \frac{dq(t)}{dt} = \dot{q}(t)$$

$$i(t) = -\frac{q_0}{RC}e^{-\frac{t}{RC}} + \frac{v}{R}e^{-\frac{t}{RC}} = \frac{1}{R}\left[v - \frac{q_0}{C}\right]e^{-\frac{t}{RC}}$$

Next are presented solution validation and an alternative solution method, followed by a numerical example with plots.

ODE Solution Validation

A good engineer must always be concerned with validation, beyond obtaining a solution. Here two validation methods are presented, the first weak and the second strong.

Weak Validation

Given the candidate solution:

$$q(t) = q_0 e^{-\frac{t}{RC}} + Cv\left[1 - e^{-\frac{t}{RC}}\right]$$

the initial condition can be checked:

$$q(0) = q_0 e^{-\frac{(0)}{RC}} + Cv\left[1 - e^{-\frac{(0)}{RC}}\right] = q_0(1) + Cv[1-1] = q_0$$

This result is correct, but it is a weak validation since only one special time, $t = 0$, is checked.

Strong Validation

Given the candidate solution:

$$q(t) = q_0 e^{-\frac{t}{RC}} + Cv\left[1 - e^{-\frac{t}{RC}}\right]$$

the entire solution for all time t can be checked by substituting $q(t)$ and its first time derivative into the original ODE:

$$R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$$

$$\dot{q}(t) = -\frac{q_0}{RC}e^{-\frac{t}{RC}} + \frac{v}{R}e^{-\frac{t}{RC}}$$

$$R\left[-\frac{q_0}{RC}e^{-\frac{t}{RC}} + \frac{v}{R}e^{-\frac{t}{RC}}\right] + \frac{1}{C}\left[q_0 e^{-\frac{t}{RC}} + Cv\left[1 - e^{-\frac{t}{RC}}\right]\right] ?=? v$$

$$\left[-\frac{q_0}{RC}e^{-\frac{t}{RC}}+\frac{v}{R}e^{-\frac{t}{RC}}\right]+\frac{1}{RC}\left[q_0e^{-\frac{t}{RC}}+Cv\left[1-e^{-\frac{t}{RC}}\right]\right]?=?\frac{v}{R}$$

$$-\frac{q_0}{RC}e^{-\frac{t}{RC}}+\frac{v}{R}e^{-\frac{t}{RC}}+\left[\frac{q_0}{RC}e^{-\frac{t}{RC}}+\frac{Cv}{RC}\left[1-e^{-\frac{t}{RC}}\right]\right]?=?\frac{v}{R}$$

$$-\frac{q_0}{RC}e^{-\frac{t}{RC}}+\frac{v}{R}e^{-\frac{t}{RC}}+\frac{q_0}{RC}e^{-\frac{t}{RC}}+\frac{v}{R}\left[1-e^{-\frac{t}{RC}}\right]?=?\frac{v}{R}$$

$$\frac{q_0}{RC}e^{-\frac{t}{RC}}-\frac{q_0}{RC}e^{-\frac{t}{RC}}+\frac{v}{R}e^{-\frac{t}{RC}}+\frac{v}{R}-\frac{v}{R}e^{-\frac{t}{RC}}?=?\frac{v}{R}$$

$$0+\frac{v}{R}e^{-\frac{t}{RC}}-\frac{v}{R}e^{-\frac{t}{RC}}+\frac{v}{R}?=?\frac{v}{R}$$

$$0+0+\frac{v}{R}?=?\frac{v}{R}$$

$$\frac{v}{R}=\frac{v}{R}$$

Q.E.D.

Since an equality results, this proves the entire ODE solution is correct for all time *t*, so it is a strong validation.

Linear ODEs have a unique solution.

_RC_ Series Electrical Circuit First-Order ODE: Alternate Solution using Laplace Transforms

Solve:

$$R\dot{q}(t)+\frac{1}{C}q(t)=v(t)$$

for the charge $q(t)$, given the resistance R and capacitance C, subject to the initial condition $q(0) = q_0$ and an input step voltage $v(t)$ of magnitude v. Here the **Laplace Transform Method** is used, which solves the converted ODE (to frequency domain s) using algebra. Also, this a one-step process, including initial condition and forcing function $v(t)$ simultaneously.

Here is a tiny table of Laplace Transforms, only the rows required for this solution. For a more complete Laplace Transform Table, please see Dr. Bob's Controls NotesBook[2]. Or Google.

Partial Laplace Transform Table

	$f(t)$	$F(s)$
	$\dot{x}(t)$	$sX(s)-x(0)$
2	unit step $u(t)$	$\frac{1}{s}$
5	e^{-at}	$\frac{1}{(s+a)}$
6	$1-e^{-at}$	$\frac{a}{s(s+a)}$

Take the Laplace transform of both sides (don't forget the initial condition on charge $q(t)$ which enters via $\dot{q}(t)$).

$$\mathcal{L}\left\{R\dot{q}(t)+\frac{1}{C}q(t)=vu(t)\right\}$$

$$R\left[sQ(s)-q(0)\right]+\frac{1}{C}Q(s)=\frac{v}{s}$$

[2] R.L. Williams II, 2024, Linear Systems Control for Mechanical Engineers, Dr. Bob Productions.

Use algebra to solve for the variable of interest, $Q(s)$, which is the Laplace transform of the answer $q(t)$.

$$\left[Rs+\frac{1}{C}\right]Q(s)-Rq_0=\frac{v}{s}$$

$$\left[Rs+\frac{1}{C}\right]Q(s)=\frac{v+Rsq_0}{s}$$

$$Q(s)=\frac{v+Rsq_0}{s\left[Rs+\frac{1}{C}\right]}$$

$$Q(s)=\frac{v}{s\left[Rs+\frac{1}{C}\right]}+\frac{Rq_0}{\left[Rs+\frac{1}{C}\right]}$$

Now the inverse Laplace Transform of $Q(s)$ will yield the desired result $q(t)$.

$$q(t)=\mathcal{L}^{-1}\left\{Q(s)\right\}$$

$$q(t)=\mathcal{L}^{-1}\left\{\frac{v}{s\left[Rs+\frac{1}{C}\right]}+\frac{Rq_0}{\left[Rs+\frac{1}{C}\right]}\right\}$$

The inverse Laplace Transform distributes over addition, so there are two components to take the inverse Laplace Transform of:

$$q(t)=\mathcal{L}^{-1}\left\{\frac{v}{s\left[Rs+\frac{1}{C}\right]}\right\}+\mathcal{L}^{-1}\left\{\frac{Rq_0}{\left[Rs+\frac{1}{C}\right]}\right\}$$

Use the tiny Laplace Transform table entry line 6 for the left term and tiny Laplace Transform table entry line 5 for the right term.

We must algebraically modify the left term of $Q(s)$ so that the leading coefficient of s in the square brackets is 1, by dividing numerator and denominator by R:

$$\mathcal{L}^{-1}\left\{\frac{\frac{v}{R}}{s\left[s+\frac{1}{RC}\right]}\right\}$$

Then we must algebraically modify this term so that the same constant a appears in the numerator and denominator, by multiplying by "1":

$$1=\frac{\frac{1}{Cv}}{\frac{1}{Cv}} \qquad \text{then} \qquad \mathcal{L}^{-1}\left\{Cv\frac{\frac{1}{RC}}{s\left[s+\frac{1}{RC}\right]}\right\}$$

Taking the inverse Laplace transform of this portion of $Q(s)$ yields:

$$Cv\left[1-e^{-\frac{t}{RC}}\right]$$

We must algebraically modify the right term of $Q(s)$ so that the leading coefficient of s in the square brackets is 1, by dividing numerator and denominator by R:

$$\mathcal{L}^{-1}\left\{\frac{q_0}{\left[s+\frac{1}{RC}\right]}\right\}$$

Taking the inverse Laplace transform of the right portion of $Q(s)$, using tiny Laplace Transform table entry line 5 now yields:

$$q_0e^{-\frac{t}{RC}}$$

And the total solution is:

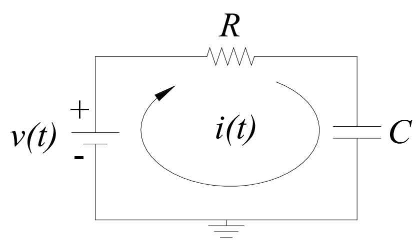
$$q(t)=q_0e^{-\frac{t}{RC}}+Cv\left[1-e^{-\frac{t}{RC}}\right]$$

This is the same solution obtained previously via the Slow ME Way, so the validation is already accomplished.

<u>*RC* Series Electrical Circuit First-Order ODE Example</u>

RC series electrical circuit

input:	voltage $v(t)$		
output:	current $i(t)$	OR	charge $q(t) = \int i(t)dt$



Summary of earlier work:
Model (from KVL and electrical circuit element table):

$$Ri(t) + \frac{1}{C}\int i(t)dt = v(t)$$

substitute charge $q(t)$:

$$\int i(t)dt = q(t)$$

$$i(t) = \dot{q}(t)$$

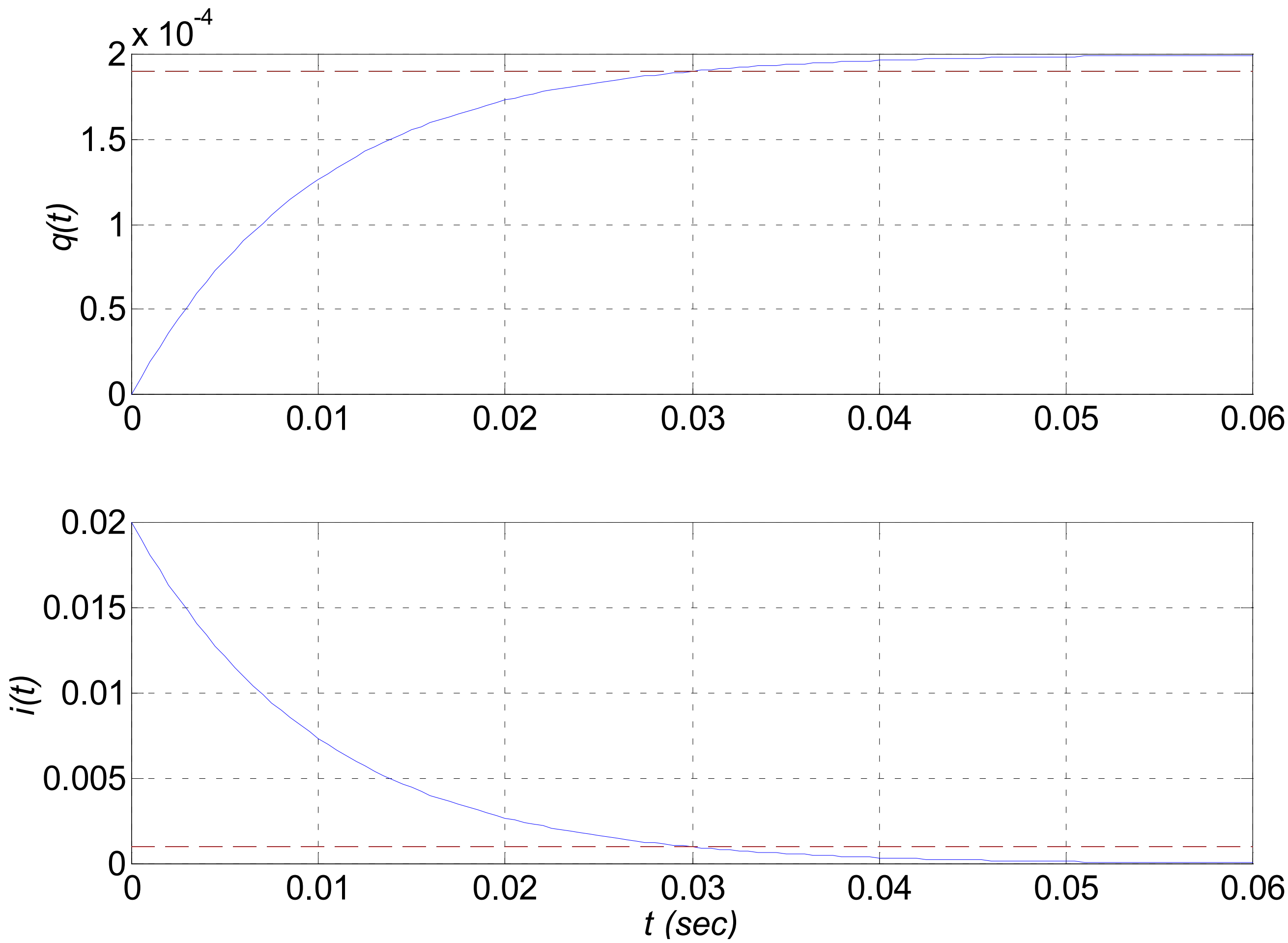
$$R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$$

Given R = 50 Ω and C = 0.2 mF, solve $50\dot{q}(t) + 5000q(t) = 1$ subject to $q(0) = 0$ and a unit step voltage input $v(t)$ (i.e. $v = 1$). From the previous general solution, with the given zero initial condition, $q(0) = q_0 = 0$:

$$q(t) = \frac{1}{5000}(1 - e^{-100t})$$

$$i(t) = \dot{q}(t) = \frac{1}{50}e^{-100t}$$

First-Order *RC* Series Electrical Circuit ODE Example Plots



- As expected from the circuit dynamics, the charge $q(t)$ in the capacitor builds up to a constant given a constant voltage input.
- Also as expected, the capacitor current $i(t)$ goes to zero at steady-state, acting as an open circuit.
- The steady state charge value is $q_{SS} = 1/5000$.
- The time constant is $\tau = RC = 0.01$, so at 3 time constants ($\tau = 0.03$ *sec*) both the $q(t)$ and $i(t)$ values have reached 95% of their respective final values.
- As required (given), the initial charge is $q(0) = q_0 = 0$.
- The initial current is 0.02 (1 / 50).
- As explained in Section 2.2, the initial slope of the charge $q(t)$ is then the initial slope of the first-order response, $K / \tau = 0.0002 / 0.01 = 0.02$, which is precisely the initial current value.

Numerical ODE Solution Validation

As a final ODE solution validation, here we will use the MATLAB function **`dsolve`** to solve the same numerical ODE problem from above. The problem statement is the same.

Solve:

$$50\dot{q}(t) + 5000q(t) = 1$$

subject to $q(0) = q_0 = 0$ and a unit step voltage input $v(t)$.

MATLAB `dsolve` Code

```
clear; clc;
q = dsolve('50*Dq+5000*q=1','q(0)=0');
pretty(q);
```

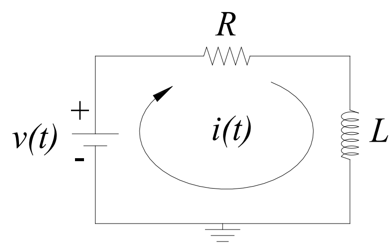
MATLAB `dsolve` Output

```
  1     exp(-100 t)
---- - -----------
5000       5000
```

Though we asked nicely for MATLAB to output the result in a pretty manner, it still does not meet the human expectation (the AI singularity has not yet arrived! Dr. Bob, December 2024). However, if we stare at this result for a while, and factor, we will see MATLAB obtained the same result as the solution by hand:

$$q(t) = \frac{1}{5000}(1 - e^{-100t})$$

_RL_ Series Electrical Circuit General First-Order ODE Solution



The *RL* circuit model (first-order ODE representing its dynamics) was already derived at the start of this section. Solve:

$$L\frac{di(t)}{dt}+Ri(t)=v(t)$$

for the output current $i(t)$, given the resistance R and inductance L, subject to initial condition $i(0) = i_0$ and an input step voltage $v(t)$ of magnitude v.

We have already solved a similar first-order ODE for the *RC* series circuit earlier in this section. Therefore, we need not do another ODE solution; instead we can adapt that previous solution. The *RC* model and solution is summarized below.

model: $R\dot{q}(t)+\frac{1}{C}q(t)=v(t)$ solution: $q(t)=q_0e^{-\frac{t}{RC}}+Cv\left[1-e^{-\frac{t}{RC}}\right]$

In the new *RL* ODE of this subsection, L plays the role of R, R plays the role of $1 / C$, and current $i(t)$ plays the role of charge $q(t)$. Therefore the new solution is:

model: $L\frac{di(t)}{dt}+Ri(t)=v(t)$ solution: $i(t)=i_0e^{-\frac{Rt}{L}}+\frac{v}{R}\left[1-e^{-\frac{Rt}{L}}\right]$

and the *RL* circuit time constant τ is:

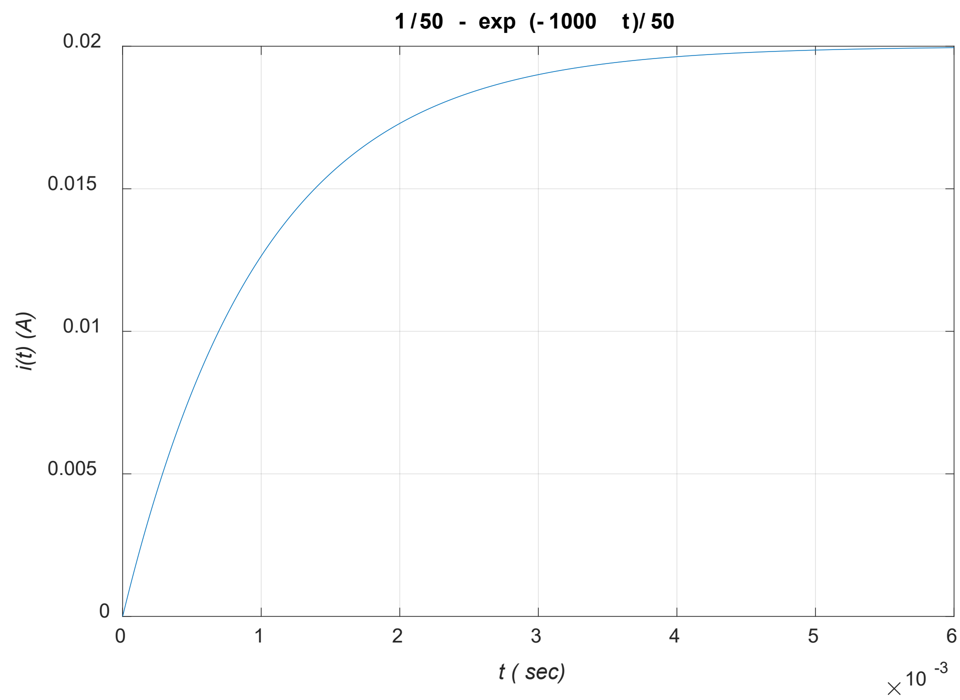
$$\tau_{RL}=\frac{L}{R}$$

Numerical *RL* Electrical Series Circuit Example

Given R = 50 Ω and L = 0.05 H, solve $0.05\dfrac{di(t)}{dt}+50i(t)=1$ subject to $i(0)$ = 0 and a unit step voltage input $v(t)$ (i.e. v = 1). From the above general solution, with the initial condition, $i(0) = i_0 = 0$:

$$i(t)=\frac{1}{50}\left[1-e^{-1000t}\right]\text{ A} \qquad \tau_{RL}=\frac{L}{R}=\frac{0.05}{50}=0.001\text{ sec}$$

and the solution plot is (verified and generated by MATLAB function **`dsolve`**):



- As expected from the circuit dynamics, the current $i(t)$ in the circuit rises to a constant in a classical first-order system response, given a constant voltage input.

- By Ohm's Law $v = iR$, the steady-state current constant is $i_{SS} = v/R = 0.02$ A.

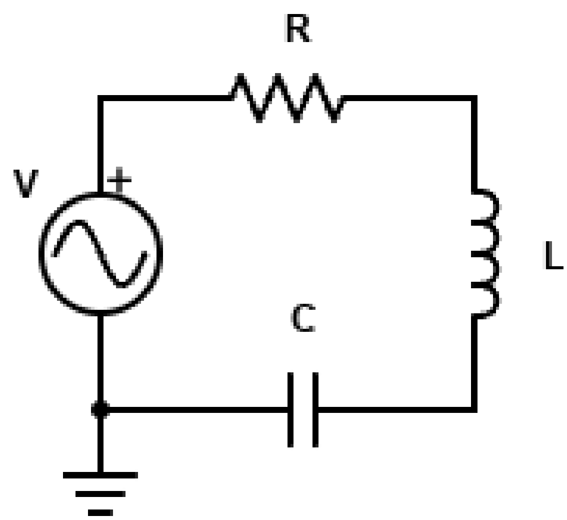
- The time constant is $\tau = 0.001$, so at 3 time constants ($t = 0.003$ *sec*) the $i(t)$ curve has reached 95% of its final value i_{SS}.

- As required (given), the initial current is $i(0) = i_0 = 0$.

2.7 Impedance

Impedance Z is the measure of opposition that an AC circuit presents to a current when a voltage is applied. The real part of impedance is resistance and the imaginary part of impedance is reactance. Unlike resistance (scalar), impedance is a vector and thus has magnitude and direction. The units of impedance are Ohms (Ω). Impedance is defined to be the ratio of voltage to current in an AC circuit, i.e. $Z = V / i$.

In AC circuits, opposition to current flow is not just resistors, but resistance R, capacitance C, and inductance L. Opposition to current flow is also dependent on the frequency of operation (assumed to be the same in the entire circuit). Impedance is used to determine a load in an AC circuit with more than just resistors. $j = \sqrt{-1}$ is the imaginary (complex) number operator; complex number maths is required. Resistance R is the real part of impedance, while reactance (C and L) are the imaginary terms in impedance. For the following derivation, consider the series RLC circuit shown:



AC Series *RLC* Circuit

Let f (Hz) be the cyclical frequency of the AC voltage source supplying the circuit. Then the circular frequency (rad / sec) is:

$$\omega = 2\pi f$$

In order to derive the impedance of each of the three passive circuit elements resistor R, inductor L, and capacitor C, we need Euler's Identity relating sinusoidal functions to the imaginary exponential:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

where $j = \sqrt{-1}$ is the imaginary number operator. $i = \sqrt{-1}$ is also widely applied in maths as the imaginary number operator, but not in Electrical Engineering, to avoid confusion with current i.

In the following impedance derivations we assume that the AC current and voltages applied to a circuit are purely sinusoidal in form, amongst the infinite possible input signal shapes. The following two mathematical identities relate the cosine and sine functions to the imaginary natural logarithm base *e*:

$$\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right] \qquad \sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$$

It is left to the interested student to prove these relationships using Euler's identity.

Now we assume a cosine form for the input current and voltage:

$$\cos(\omega t) = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$$

where we have used $\theta = \omega t$, where ω is the circular driving frequency in rad/sec. For impedance derivation purposes, we only need choose the first term of the above cosine for the input current i (and voltage V for capacitance):

$$i(t) = I_0 e^{j\omega t}$$

where I_0 is the current amplitude. For the resistor R impedance derivation we start with Ohm's Law:

$$V = i_R R = I_0 e^{j\omega t} R$$

The resistor R impedance Z_R is defined as:

$$Z_R = \frac{V}{i_R} = \frac{I_0 e^{j\omega t} R}{I_0 e^{j\omega t}} = R$$

And thus the real impedance Z_R due to resistance (in Ω) is:

$$Z_R = R$$

Big freakin' deal, we just chased our tails back to Ohm's Law! The next two derivations are more interesting.

For the inductor L impedance derivation we start with the inductor voltage / current differential relationship, again using $i(t) = I_0 e^{j\omega t}$:

$$V = L\frac{di_L}{dt} = j\omega L I_0 e^{j\omega t}$$

The inductor L impedance Z_L is defined as:

$$Z_L = \frac{V}{i_L} = \frac{j\omega L I_0 e^{j\omega t}}{I_0 e^{j\omega t}} = j\omega L$$

The imaginary inductor impedance Z_L (in Ω) is therefore:

$$Z_L = X_L = j\omega L$$

where the inductor reactance X_L is part of the imaginary impedance (the other part is the capacitor reactance X_C). We see that the inductor impedance Z_L is a function of the driving frequency ω.

For the capacitor C impedance derivation we start with the capacitor voltage / current differential relationship:

$$i_C = C\frac{dV}{dt}$$

Now we must assume a sinusoidal form for the applied voltage:

$$V(t) = V_0 e^{j\omega t}$$

where V_0 is the voltage amplitude. Thus we have:

$$i_C = C\frac{dV}{dt} = j\omega C V_0 e^{j\omega t}$$

The capacitor C impedance Z_C is defined as:

$$Z_C = \frac{V}{i_C} = \frac{V_0 e^{j\omega t}}{j\omega C V_0 e^{j\omega t}} = \frac{1}{j\omega C}$$

The imaginary inductor Z_C impedance (in Ω) is therefore:

$$Z_C = X_C = \frac{1}{j\omega C}$$

where the capacitor reactance X_C is the other part of the imaginary impedance (along with the inductor reactance). We see that the capacitor impedance Z_C is a function of the driving frequency ω, like the inductor.

It is important to note that the capacitor impedance Z_C must always be along the **negative Im axis**, while the inductor impedance Z_L must always be along the **positive Im axis**. The resistor impedance Z_R must always be along the **positive Re axis**.

Here 'imaginary' simply denotes perpendicular to the resistance. Let's make a Re-Im plot of resistance and reactance (don't forget the capacitive reactance X_C **must be negative**):

Z is the equivalent total impedance of the series RLC circuit:

$$Z = ze^{j\theta}$$

The above $ze^{j\theta}$ is a **phasor** representation for a vector. A phasor is a convenient, compact notation for a vector wherein the magnitude z is the vector length and the exponential gives the direction along θ from the positive (right) horizontal; using Euler's Identity:

$$\underline{Z} = ze^{j\theta} = z(\cos\theta + j\sin\theta)$$

where the real component $z\cos\theta$ is understood to be along the horizontal Real axis (traditional X axis) and the imaginary component $jz\sin\theta$ is understood to be along the vertical Imaginary axis (traditional Y axis). Let's make a Re-Im plot of this phasor:

Phasors are especially convenient for taking time derivatives of vectors. The phasor can also be expressed in polar notation. Formulae for the impedance magnitude z and angle θ are (leaving out the imaginary operator j in X_L and X_C):

$$z = |Z| = \sqrt{R^2 + (X_L - X_C)^2} \qquad \theta = \tan^{-1}\left[\frac{X_L - X_C}{R}\right]$$

Impedance Examples

1. Given a circuit with R = 100 Ω in series with L = 10 H, what is the impedance for f = 4 Hz?

$$\omega = 2\pi f = 25.13 \text{ rad/sec} \qquad Z_R = R = 100 \;\Omega \qquad X_L = j\omega L = 251.3j \;\Omega$$

$$\begin{aligned} Z &= Z_R + jX_L \\ Z &= 100 + 251.3j \end{aligned} \;\Omega \qquad Z = 270.5 \;@\; 68.3^\circ$$

2. Given a circuit with R = 100 Ω in series with C = 0.001 F, what is the impedance for f = 4 Hz?

$$\omega = 2\pi f = 25.13 \text{ rad/sec} \qquad Z_R = R = 100 \;\Omega \qquad X_C = \frac{1}{j\omega C} = -39.8j \;\Omega$$

$$\begin{aligned} Z &= Z_R - jX_C \\ Z &= 100 - 39.8j \end{aligned} \;\Omega \qquad Z = 107.6 \;@\; -21.7^\circ$$

3. Given a circuit with R = 100 Ω in series with L = 1 H and C = 0.01 F, what is the impedance for f = 4 Hz?

$$\omega = 2\pi f = 25.13 \text{ rad/sec} \qquad Z_R = R = 100 \;\Omega \qquad X_L = j\omega L = 25.13j \;\Omega \qquad X_C = \frac{1}{j\omega C} = -3.98j \;\Omega$$

$$\begin{aligned} Z &= Z_R + j(X_L - X_C) \\ Z &= 100 + 21.15j \end{aligned} \;\Omega \qquad Z = 102.2 \;@\; 11.9^\circ$$

Examples conclude.

Here is a handy on-line impedance calculator:

www.mathforengineers.com/AC-circuits-calculators/series-RLC-circuit-Impedance.html

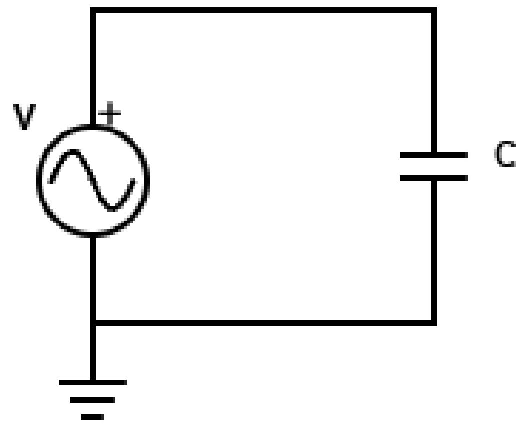
Caution: this impedance calculator is only for full *RLC* series AC circuits. That is, if one attempts to enter a zero value for one of the circuit constants, the following error message is given:

"*The input resistance* (or inductance, or capacitance) *must be a number greater than 0.*"

My spreadsheet for the same impedance calculations does not suffer from this problem. (Obviously do not divide by zero capacitance C, simply skip that part of the calculation.)

Example: Current in an AC capacitive circuit

In an AC capacitive reactance-only circuit, given input voltage $V(t) = 7\sin\omega t$, capacitance $C = 1.0 \quad \mu\text{F}$, and $\omega = 500$ rad/sec, find the current *i*.



AC Capacitive Circuit

From the AC version of Ohm's Law:

$$V = iZ$$

The impedance *Z* consists only of the capacitive reactance:

$$X_C = \frac{1}{j\omega C} = \frac{1}{j500(0.000001)} \ \Omega$$

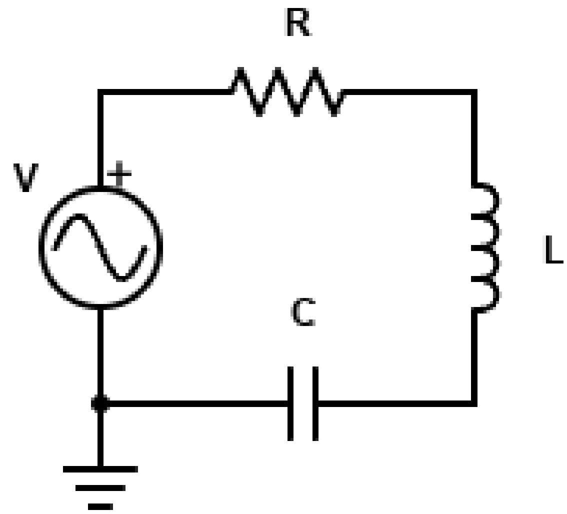
so the magnitude of the capacitive reactance is 2000 Ω, in the negative Im direction. Therefore, the current is (using the X_C magnitude):

$$i = \frac{V}{X_C} = \frac{7}{2000} = 0.0035 \text{ A} \qquad \text{(3.5 mA)}$$

Example concludes.

Example: Current with AC *RLC* circuit

In an AC *RLC* circuit, R = 200 Ω, L = 400 mH, C = 60 μF, and an AC voltage source of $V(t) = 90\cos(\omega t)$ is given. Assuming ω = 1000 rad/sec, find the total impedance Z, the current i and the voltage drop across each passive element.



AC *RLC* Circuit

From the AC version of Ohm's Law:

$$V = iZ$$

The circuit current is:

$$i = \frac{V}{Z}$$

The total impedance Z includes the resistance and the inductive and capacitive reactance:

$$\underline{Z} = \underline{R} + \underline{X}_L + \underline{X}_C$$

The resistive impedance is simply R (along the positive Real axis). The inductive and capacitive reactance magnitudes are (directions: positive and negative Imaginary axis, respectively):

$$X_L = \omega L = 1000(0.4) = 400\ \Omega \qquad X_C = \frac{1}{\omega C} = \frac{1}{1000(0.00006)} = 16.7\ \Omega$$

Therefore the total impedance Z is:

$$Z = 200 + 383.3j \qquad Z = 432.4\ @\ 62.5^\circ$$

and the current *i* is:

$$i = \frac{V}{|Z|} = \frac{90}{432.4} = 0.208 \text{ A} \qquad \text{(208 mA)}$$

The voltage drops across the resistor, inductor, and capacitor are then:

$$V_R = iZ_R = 0.208(200) = 41.6 \qquad \text{V}$$

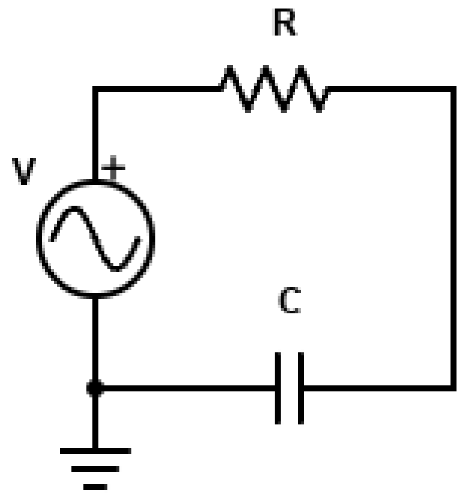
$$V_L = iX_L = 0.208(400) = 83.3 \qquad \text{V}$$

$$V_C = iX_C = 0.208(16.7) = 3.5 \qquad \text{V}$$

Example concludes.

Example: Impedance Phase Angle

In an AC capacitive-resistive circuit, given input voltage $V(t) = 100\sin\omega t$, capacitance $C = 40\ \mu\text{F}$, $R = 50\ \Omega$, and $\omega = 500$ rad/sec, find the current i and the phase angle ϕ.



AC *RC* Circuit

From the AC version of Ohm's Law:

$$V = iZ$$

The circuit current is:

$$i = \frac{V}{Z}$$

The total impedance Z includes the resistance R and the capacitive reactance:

$$Z = R + X_C$$

The resistive impedance is simply R (along the positive Real axis). The capacitive reactance magnitude is (direction: negative Imaginary axis):

$$X_C = \frac{1}{\omega C} = \frac{1}{500(0.00004)} = 50\ \Omega$$

Therefore the total impedance Z is:

$$Z = 50 - 50j \qquad Z = 70.7\ @\ -45^{\circ}$$

Calculate $i = \frac{V}{Z}$ in two steps:

1. Drop the sin(ωt) portion of the AC voltage; keep only the amplitude of V in $i = V / Z$: $1 - j$

$$i = \frac{V}{|Z|} = \frac{100}{70.7} = 1.414 \text{ A}$$

2. Factor in the phase angle ϕ

$$\phi = \tan^{-1}\left[\frac{-50}{50}\right] = -45^{\circ} \qquad (\phi = -0.785 \text{ rad})$$

Finally,

$$i(t) = |Z|\sin(500t + \phi) = 70.7\sin(500t - 0.785)$$

Example concludes.

Alternating Current (AC) Circuits

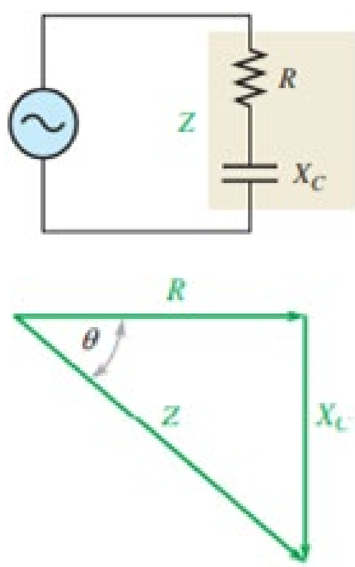
In AC circuits, capacitors act like resistors act in a DC circuit (restricting current flow). This characteristic is called **reactance**, in units of Ohms.

$$X_C = \frac{1}{j\omega C} = \frac{1}{j2\pi fC}$$

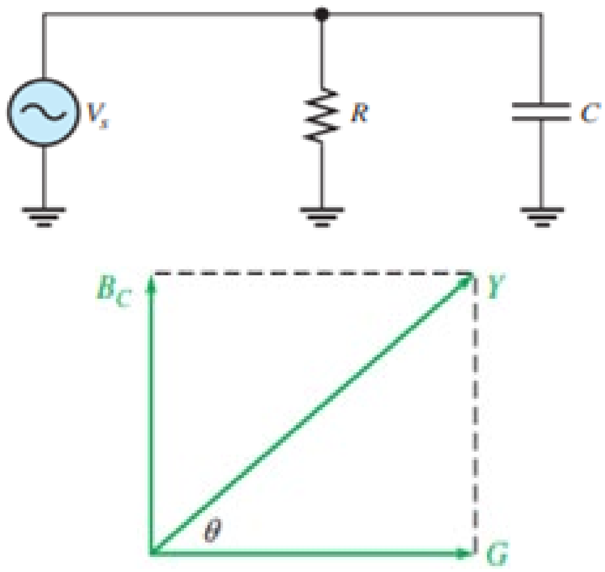
where ω is the input voltage circular frequency (rad / sec) and f is the cyclical input voltage frequency (Hz, i.e. cycles per second). The imaginary operator j doesn't mean this is unreal, it simply means the capacitive reactance along the negative Imaginary axis is perpendicular to the resistive reactance along the positive Real axis.

In AC circuits, there is a concept of imaginary power. This is not dissipative power via heat like resistance. Instead, imaginary power involves storing and releasing current in different phase via capacitance.

<u>Series RC circuit</u>

	$V = i\,Z$ (Ohm's Law) $Z = \sqrt{R^2 + X_C^2}$ $\theta = \tan^{-1}\left[\frac{-X_C}{R}\right]$

Parallel RC circuit



$$B_C = \frac{1}{X_C}$$

$$Y = \frac{1}{Z}$$

$$G = \frac{1}{R}$$

$$V = \frac{i}{Y} \text{ (Ohm's Law)}$$

$$Y_{tot} = \sqrt{G^2 + B_C^2}$$

where:

- B susceptance Siemens (1 / Ω)
- X reactance Ω
- Y admittance Siemens (1 / Ω)
- Z impedance Ω
- G conductance Siemens (1 / Ω)

When using resistors in AC circuits with negligible inductance and capacitance, the same principles and equations from DC resistive circuits apply (KVL and KCL are also applied in the same manner). That is, Ohm's Law, equations for voltage, current, and power such as presented in Chapter 2 still apply. However, the RMS (root-mean-square) values for voltage and current must be used. These values are calculated below, where V_0 is the peak voltage value and i_0 is the peak current value.

$$V_{RMS} = \frac{V_0}{\sqrt{2}} \qquad i_{RMS} = \frac{i_0}{\sqrt{2}}$$

In AC resistive-only circuits, the impedance Z is identical to the resistance R.

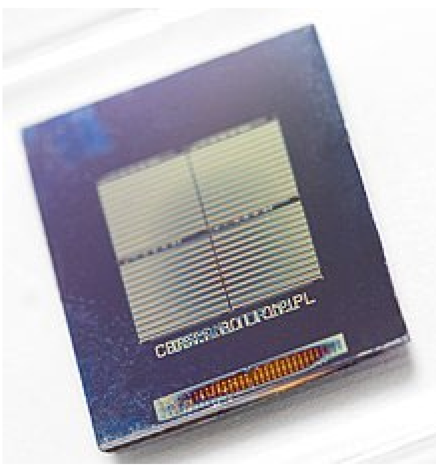
2.8 Other

This section covers other devices of interest to mechatronics. These include memristors, batteries, household electricity, and electrical fuses. Then some handy laboratory tools are covered: multimeters, power supplies, oscilloscopes, and signal generators.

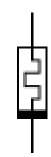
2.8.1 Memristors

A **memristor** (memory resistor) is a nonlinear two-terminal electronic component relating electric charge $q(t)$ and magnetic flux $\phi(t)$. Chua first inferred the memristor (1971), completing a theoretical quartet of basic electrical components, joining the resistor, capacitor, and inductor.

Memristors are still theoretical and controversial – an ideal memristor has yet to be developed in the lab, hence no commercial memristor products exist. Therefore, there are currently no applications. Some have posited memristors can be used as analog memory in superconducting quantum computers. In recent experiments it was found that memristor memory can be 100x faster and require 1% of the energy of existing Flash memory.



UICU Memristor



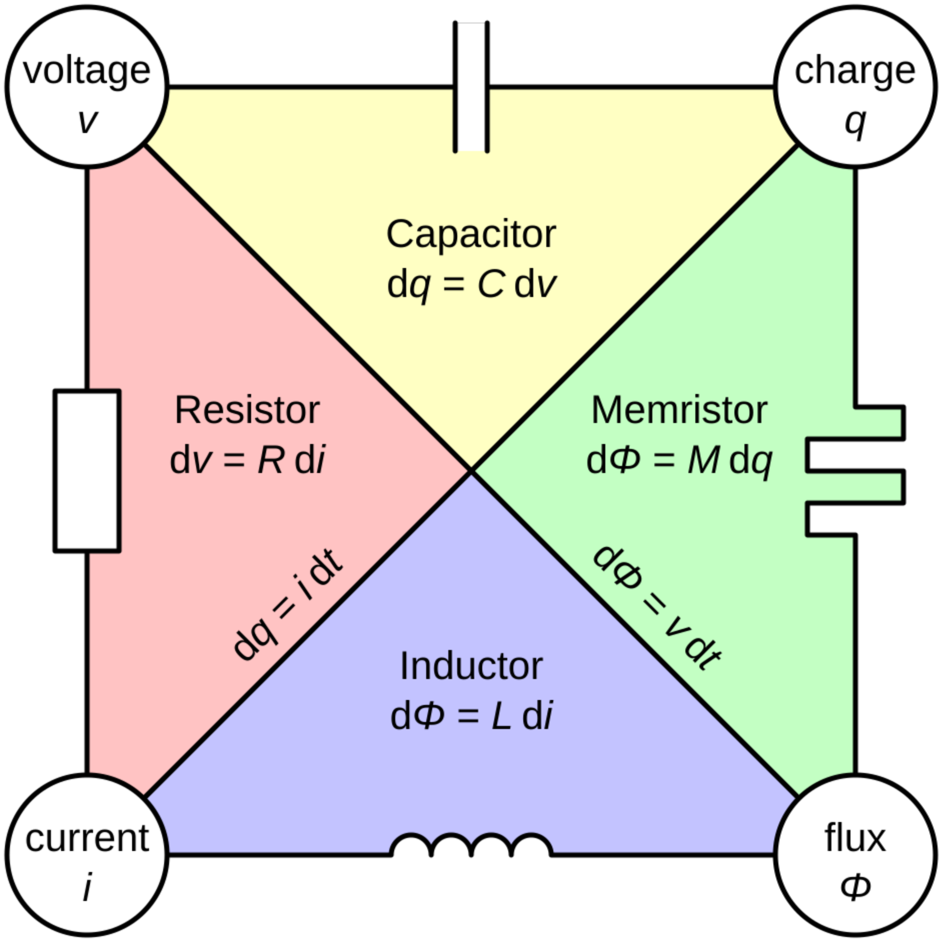
Memristor Schematic Symbol

Memristor - Wikipedia

Unlike a linear resistor, the memristor has a dynamic relationship between current and voltage, including a memory of past voltages or currents. The present resistance of a memristor depends on the amount and direction of electric charge previously flowing through. When the electric power supply is turned off, a memristor remembers its most recent resistance until it is again powered on.

The **memristance** $\boldsymbol{M}$ of a memristor has units of Ω, and is a function of charge $q(t)$:

$$M(q(t)) = \frac{\frac{d\phi(t)}{dt}}{\frac{dq(t)}{dt}} = \frac{V(t)}{i(t)}$$



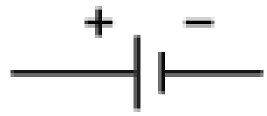
Conceptual Symmetries of the Four Components including Memristor

Memristor - Wikipedia

2.8.2 Batteries

An electric battery is a source of DC electric power, composed of one or more electrochemical cells having external connections for powering electrical devices. When a battery is providing power, the cathode is its positive terminal and the anode is its negative terminal. Practical batteries used to power portable and other devices come in a variety of sizes, shapes, and voltage. Tiny batteries power hearing aids and watches, and room-sized battery banks provide emergency power for telephone and computer data centers.

Commercial Batteries



Battery Schematic Symbol

Electric battery - Wikipedia

Benjamin Franklin coined the term ‘battery’ in 1749 in electricity experiments involving a set of linked Leyden jar capacitors. Alessandro Volta invented the first electrochemical battery in 1800.

The negative battery terminal is the source of electrons. When the battery is connected to an external electrical load, these negatively-charged electrons flow through the circuit to reach the positive terminal, causing a redox reaction by attracting cations (positively-charged ions). The free-energy difference between high-energy reactants and lower-energy products is delivered to the external circuit as electrical energy.

Single-use (disposable) batteries are discarded (recycled) after their charge is used up. Such an example is alkaline batteries in flashlights. Rechargeable batteries can be discharged and recharged many times. These are used in vehicles (lead-acid batteries) and portable electronic devices (lithium-ion batteries) like laptops and cellphones.

2.8.3 Household Electricity Consumption

In the U.S., the standard electric supply via normal wall outlets is 60 Hz, 120 V, 15 A alternating current (AC). Therefore, the maximum possible power is:

$$P = i_{\text{RMS}} V_{\text{RMS}} = \frac{i}{\sqrt{2}}\frac{V}{\sqrt{2}} = \frac{15(120)}{2} = 900 \text{ Watts} \qquad (0.9 \text{ kW})$$

and the effective household circuit resistance is:

$$R_{\text{EFF}} = \frac{V}{i} = \frac{V_{\text{RMS}}}{i_{\text{RMS}}} = \frac{120}{15} = 8 \ \Omega$$

The average kilowatt-hours (kWh) energy used per household in the United States is about 30 kWh per day. Kilowatt-hours are a non-SI unit (1 kWh = 3.6 MJ). It is a measure of electrical energy consumption equal to 1,000 Watts for one hour. There is something wrong with this analysis since there aren't enough hours in a day to agree with this energy usage (30 kWh / 0.9 kW = 33.3 hours in a day!)

Electrical Wire Sizes

Often electricity flows through copper wires (which is a good conductor), insulated by a rubber cover (which is a good insulator).

Wire Gauge Size and Current Capacity

Wire Gauge	Current (amps)	Use
3 / 0	200	From utility pole to meter
1 / 0	150	To panel box
3	100	To panel box
6	55	Feeder and large appliances
8	40	Feeder and large appliances
10	30	Dryer, air-conditioning, hot water heater
12	20	Laundry, bathroom, and kitchen
14	15	Lighting, fans, outlets

The wire gauge sizes in the above table are arranged from biggest to smallest diameter, top to bottom.

2.8.4 Electrical Fuses

An electrical fuse is an electrical circuit overcurrent protection safety device that melts when too much current flows through it (activated by the current heat). Once tripped, the circuit becomes an open circuit and the non-reusable fuse must be replaced for continued operation. Household electric fuses are of the breaker-switch type that can be reactivated after protecting the circuit from an overload of current.

Miniature Time-delay 250 V Fuse

Fuse (electrical) - Wikipedia

The example fuse shown above is 32 mm long. It will interrupt a 0.3 A current at after 100 sec, or a 15 A current in 0.1 sec.

2.8.5 Multimeters

A multimeter (aka volt-ohm-ammeter) is a measuring instrument that can measure multiple electrical properties. Typically multimeters can measure voltage (voltmeter), resistance (ohmmeter), and current (ammeter). Some also measure temperature, capacitance, conductance, decibels, duty cycle, frequency, and inductance. The first usage of 'multimeter' appeared in the Oxford English dictionary in 1907.

Analog multimeters have a moving pointer to display readings. Digital multimeters (DMMs) have numerical displays and hence are more precise than analog multimeters. Multimeters vary in size, features, and cost. Some are portable while others are benchtop units.

A multimeter will load the tested circuit; low readings may result if a high-impedance circuit is loaded by the multimeter so that the circuit behaviour is changed. Full-scale deflection current is called the sensitivity of the multimeter, and is expressed in units of Ω / V.

Multimeter - Wikipedia

Digital Multimeter in the ME 3550 Laboratory

2.8.6 DC Power Supply

A DC power supply provides electrical power to an electrical load. It can be an external unit, as pictured below, or it can be integrated into a product such as a computer. A DC power supply plugs into the wall outlet AC power source and rectifies this to a DC source for your laboratory applications. Power supplies may also provide safety limitations such as protecting against voltage surges and limiting current to safe levels.

DC Power Supply in the ME 3550 Laboratory

The HP E3620A DC power supply pictured above provides two isolated 0 – 25 V outputs rated at a current of 1 A. Each output voltage is continuously variable over 0 – 25 V. Separate current limits protect each output against overload or short-circuit damage. The AC supply is via a grounded three-pin plug cord.

In addition to the DC power jacks, the front panel also includes a line switch, output voltage controls, a digital voltmeter, a digital ammeter, and two meter select pushbutton switches.

HP E3620A DC Power Supply online manual

2.8.7 Oscilloscope

An oscilloscope is an electronic test instrument that graphically displays one or more time-varying signal voltages vs. time. The use is for capturing electrical signal information for debugging, analysis, design, and characterization. Waveforms can be analyzed for amplitude, frequency, rise time, time periods, distortion, among other properties.

Oscilloscopes are widely applied in science, engineering, biomedicine, and the automotive and telecommunications industries. They can be used for electronic equipment maintenance and laboratory experiments.

Oscilloscope in the ME 3550 Laboratory

The SIGLENT SDS 1052DL Digital Storage Oscilloscope pictured above is a two-channel, portable, benchtop instrument used to take ground-referenced readings. Its two channels have a bandwidth of 25 – 200 MHz with single real-time sampling rate of 500 MSa/sec (mega-samples per second, i.e. 10^6 samples per second). It has edge, pulse, video, and alternative trigger types.

This specific oscilloscope allows automatic measurement of 32 parameters. There are manual, track-mode, and auto-mode cursors. The waveform and its fast-Fourier transform (FFT, in the frequency domain) display simultaneously on a split screen.

SIGLENT SDS 1052DL Digital Storage Oscilloscope online manual

2.8.8 Signal Generator

A signal generator is an electronic device that creates electrical signals with user-specified amplitude, frequency, and wave shape. The output can be used in designing, testing, troubleshooting, and maintenance/repair of electronic devices. The various types of signal generators include function generators, RF and microwave signal generators, pitch generators, waveform generators, digital pattern generators, and frequency generators. Signal generators can be external benchtop units or incorporated into larger machines.

Signal Generator in the ME 3550 Laboratory

The SeeSii DDS Signal Generator/Counter pictured above is a waveform generator with two output channels. There is a power button, 2" display, function buttons, control buttons, and a main dial. It takes a DC 5 V input, transformed via the supplied AC plug. Up to 99 user-defined state parameters may be stored. Sine wave output can be up to 60 MHz frequency. The time sampling rate is 200 MSa/sec (mega-samples per second, i.e. 10^6 samples per second). This specific signal generator has a linear sweep up to 999.9 sec, and a logarithmic sweep function.

DDS Signal Generator and Counter

3. Dynamic Circuit Modeling

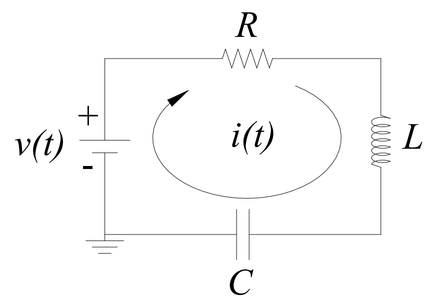
This chapter presents dynamic modeling of electrical circuits. Like mechanical translational and rotational systems, electrical circuits have dynamics described by IVP ODEs, including input and output variables that are functions of time. We will make analogies between the mechanical and electrical systems, i.e. the models they yield are identical mathematically. Hence, we can use the same techniques for solving these mechanical and electrical ODEs.

Dynamic Circuit Modeling involves using Kirchhoff's Voltage and Current Laws to derive the integro-differential equations representing the dynamics of voltage-driven *RLC* series circuits and current-driven *RLC* parallel circuits. Mechanical / Electrical analogies are also presented. Differential equations is not a pre-requisite to ME 3550, nonetheless some differential equations solutions will be presented and some plots will be discussed.

The Hydraulic Flow / Electrical Circuit analogy is also presented in this Chapter.

3.1 Series *RLC* Circuit Model

The system diagram for the voltage-driven series *RLC* DC electrical circuit is shown below. The input is voltage $v(t)$, and the output is current $i(t)$. The current $i(t)$ flowing through all elements is the same. The input voltage source $v(t)$ increases as shown negative to positive; the voltage drops across each of the passive circuit elements, positive to negative in each case. All voltage drops are different, in general.



Series *RLC* Circuit Diagram

Using **Kirchhoff's Voltage Law** (the sum of voltages around the circuit loop is zero), we have:

$$v(t) - v_R(t) - v_L(t) - v_C(t) = 0$$

$$v_L(t) + v_R(t) + v_C(t) = v(t)$$

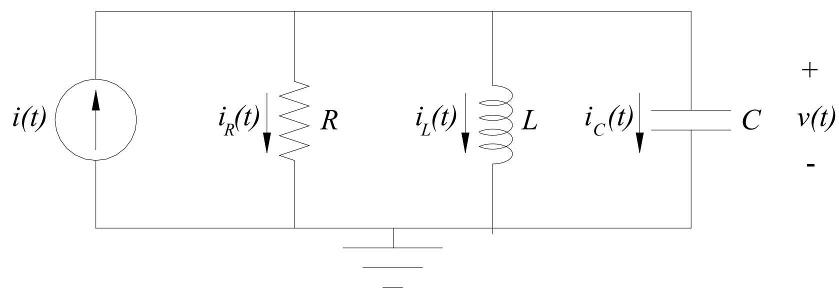
Substituting the voltage drop across each element as a function of current i(t), from the table in Section 2.5, we obtain the model for the voltage-driven series *RLC* circuit:

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt = v(t)$$

This model is a linear, lumped-parameter, constant-coefficient, integro-differential equation. It is second-order, since, although the highest derivative order is first-order, there are two time differentiation steps between the integral and the first derivative. This equation is written in standard form, where the output variable $i(t)$ appears with its derivative and integral on the left-hand-side and the input forcing function $v(t)$ appears on the right-hand-side of the equation. The external input is voltage $v(t)$ (V) and the output is current $i(t)$ (A).

3.2 Parallel *RLC* Circuit Model

The system diagram for the current-driven parallel *RLC* DC electrical circuit is shown below. The input is current $i(t)$, and the output is voltage $v(t)$. The voltage across the current source, the resistor R, the inductor L, and the capacitor C are all identical, $v(t)$. The currents in each leg are different.



Parallel *RLC* Circuit Diagram

Using **Kirchhoff's Current Law** (the sum of currents into any circuit node is zero), we have:

$$i(t) - i_R(t) - i_L(t) - i_C(t) = 0$$

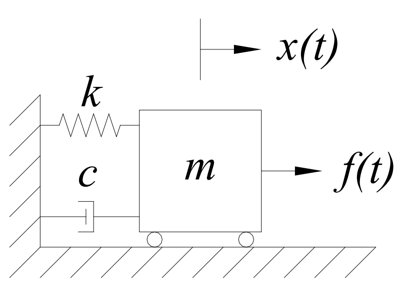
$$i_C(t) + i_R(t) + i_L(t) = i(t)$$

Substituting the current through each element as a function of voltage $v(t)$, from the table in Section 2.5, we obtain the model for the current-driven parallel *RLC* circuit:

$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) + \frac{1}{L}\int v(t)dt = i(t)$$

This model is a linear, lumped-parameter, constant-coefficient, integro-differential equation. Again, it is a second-order model, since there are two time differentiation steps between the integral and the first derivative. This equation is written in standard form, where the output variable $v(t)$ appears with its derivative and integral on the left-hand-side and the input forcing function $i(t)$ appears on the right-hand-side of the equation. The external current input is $i(t)$ (A) and the output is voltage $v(t)$ (V).

Here is the diagram for the translational mechanical vibrational system, necessary in the following section. Force $f(t)$ is the input, displacement $x(t)$ is the output, and m (kg), c (N-sec/m), and k (N/m) are the lumped point mass, viscous damping coefficient, and spring stiffness, respectively. To derive the model given on the next page, draw the FBD and apply Newton's Second Law.



3.3 Mechanical / Electrical Analogies

Force-Voltage Analogy

We can make an analogy between the mechanical and electrical modeling worlds by comparing the translational mechanical system with the voltage-driven series *RLC* DC circuit model. This is called the **force-voltage analogy**.

We use the translational mechanical system model, rewritten in terms of velocity $v(t)$ instead of displacement $x(t)$ (alternatively, one could use charge $q(t)$ in place of current to obtain the same analogy):

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=f(t)$$
$$m\dot{v}(t)+cv(t)+k\int v(t)dt=f(t)$$

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)dt=v(t)$$

Compare the mechanical system ODE to the series *RLC* circuit model.

Energy Role	**Translational Mechanical**	**Electrical**
store energy	$f_S(t)=m\frac{dv(t)}{dt}$	$v_S(t)=L\frac{di(t)}{dt}$
dissipate energy	$f_D(t)=cv(t)$	$v_D(t)=Ri(t)$
oscillate energy	$f_O(t)=k\int v(t)dt$	$v_O(t)=\frac{1}{C}\int i(t)dt$

Force-Voltage Analogy

Variable Type	**Translational Mechanical System**	***RLC* Series Circuit**
input (through)	$f(t)$	$v(t)$ (voltage)
output (across)	$v(t)$ (velocity)	$i(t)$
inertia	m	L
damping	c	R
stiffness	k	$1 / C$

Caution: the **Force-Current Analogy**, presented next, based on the current-driven parallel *RLC* DC circuit model, is different.

Force-Current Analogy

We can make a different analogy between the mechanical and electrical modeling arenas by comparing the translational mechanical system with the current-driven parallel *RLC* DC circuit model. This is called the **force-current analogy**.

We first rewrite the left-hand-side of the translational mechanical system model in terms of velocity $v(t)$ instead of displacement $x(t)$ (alternatively, one could use flux $\phi(t)$ in place of voltage to obtain the same analogy):

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=f(t)$$
$$m\dot{v}(t)+cv(t)+k\int v(t)dt=f(t)$$

$$C\frac{dv(t)}{dt}+\frac{1}{R}v(t)+\frac{1}{L}\int v(t)dt=i(t)$$

Compare the mechanical system ODE to the parallel *RLC* circuit model.

Energy Role	**Translational Mechanical**	**Electrical**
store energy	$f_S(t)=m\frac{dv(t)}{dt}$	$i_S(t)=C\frac{dv(t)}{dt}$
dissipate energy	$f_D(t)=cv(t)$	$i_D(t)=\frac{1}{R}v(t)$
oscillate energy	$f_O(t)=k\int v(t)dt$	$i_O(t)=\frac{1}{L}\int v(t)dt$

Force-Current Analogy

Variable Type	**Translational Mechanical System**	***RLC* Parallel Circuit**
input (through)	$f(t)$	$i(t)$
output (across)	$v(t)$ (velocity)	$v(t)$ (voltage)
inertia	m	C
damping	c	$1/R$
stiffness	k	$1/L$

Caution: the **Force-Voltage Analogy**, presented previously, based on the voltage-driven series *RLC* DC circuit model, is different.

Important Mechanical Vibrations Terms and Concepts

Mechanical **stiffness** is defined as the ratio of the force to the displacement. From Hooke's Law, the system or element **stiffness** is the spring constant *k*, with N/m units. The translational stiffness *k* is the slope of the force-displacement line, $f_S(t) = kx(t)$.

The mechanical **compliance** is defined as the inverse of the system stiffness, i.e. the ratio of the displacement to the force. **Compliance** is $1/k$ with $x(t) = f_S(t)/k$ and units m/N.

The mechanical **impedance** is defined as the ratio of the force to the velocity. From the Linear Dashpot Law (analogous to Hooke's Law), the system or element **impedance** is then the damping coefficient *c*, with N-sec/m units. The translational damping coefficient *c* is the slope of the force-velocity line, $f_D(t) = cv(t)$.

The mechanical **mobility** is defined as the inverse of the system damping, i.e. the ratio of the velocity to the force. **Mobility** is $1/c$ with $v(t) = f_D(t)/c$ and units m/(N-sec). Note that this definition conflicts with the mechanisms definition of **mobility**, the number of degrees of freedom in a device.

The mechanical **inertia** (or **mass**) is defined as the ratio of the force to the acceleration, with units of kg, or N-s^2/m, from Newton's Second Law. Evidently there is no technical term for the inverse of mass.

We see one of these terms is shared with AC circuits, **impedance**. From the Force-Voltage Analogy presented above, the mechanical / electrical through (input) variables are force $f(t)$ and voltage $V(t)$, while the mechanical / electrical across (output) variables are velocity $v(t)$ and current $i(t)$. From Ohm's Law applied to AC circuits with impedance, we have:

$$V(t) = i(t)Z \qquad \text{or} \qquad Z = \frac{V}{i}$$

i.e. the electric impedance is the ratio of the through variable to the across variable. From the definition of mechanical impedance above:

$$Z = \frac{f}{v}$$

i.e. the mechanical impedance is the ratio of the through variable to the across variable, so the mechanical and electrical definitions of impedance are identical.

Hydraulic Flow / Electrical Circuit Analogy

This analogy arose historically because it was difficult to envision the flow of current in a circuit. Early electrical engineers visualized electrical current as a fluid flow.

The fluid flow of water in a pipe is similar to the flow of electrical current along a wire. The flow pressure is analogous to the electrical voltage potential, and the fluid flow rate is analogous to the electrical current.

Hydraulic-Circuit Analogy

Type	Hydraulic	Electric
Potential	pressure P (N/m^2)	potential V (V)
Quantity	volume V (m^3)	charge q (Coulombs)
Quantity Flux	flow rate F (m^3/sec)	current i (A)
Flux Density	velocity v (m/sec)	current density (A/m^2)

Hydraulic analogy - Wikipedia

A constricted pipe (a length of reduced diameter) in hydraulics is analogous to an electrical resistor *R*, limiting the flow. A flexible diaphragm sealed inside a pipe in hydraulics is analogous to an electrical circuit capacitor *C*. A fluid water wheel in hydraulics is analogous to an electrical circuit inductor *L*. A one-way ball-type check valve in hydraulics is analogous to an electrical diode.

Hydraulic	Electric
pipe	wire
pipe T	circuit node
constricted pipe	resistor
flexible diaphragm	capacitor
water wheel	inductor
one-way check valve	diode

3.4 Additional Circuit Modeling Topics

Subset Electrical Circuit Models

This subsection presents subset models derived from the standard current-driven parallel *RLC* electrical circuit and the voltage-driven series *RLC* electrical circuit models. Each column below starts with the full models already derived above.

***RLC* Parallel Circuit**

$$C\frac{dv(t)}{dt}+\frac{1}{R}v(t)+\frac{1}{L}\int v(t)dt=i(t)$$

***RL* Parallel Circuit**

$$\frac{1}{R}v(t)+\frac{1}{L}\int v(t)dt=i(t)$$

***RC* Parallel Circuit**

$$C\frac{dv(t)}{dt}+\frac{1}{R}v(t)=i(t)$$

***LC* Parallel Circuit**

$$C\frac{dv(t)}{dt}+\frac{1}{L}\int v(t)dt=i(t)$$

***RLC* Series Circuit**

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)dt=v(t)$$

***RL* Series Circuit**

$$L\frac{di(t)}{dt}+Ri(t)=v(t)$$

***RC* Series Circuit**

$$Ri(t)+\frac{1}{C}\int i(t)dt=v(t)$$

***LC* Series Circuit**

$$L\frac{di(t)}{dt}+\frac{1}{C}\int i(t)dt=v(t)$$

We have already derived, solved the ODEs, and presented examples for the *RC* series and *RL* series circuits in Section 2.6. The *RC* series involved current as above, but also was modified to feature charge $q(t)$ (see next page).

RLC parallel models based on flux $\phi(t)$ and *RLC* series models based on charge $q(t)$

Recall that in an electrical circuit, electrical flux $\phi(t)$ is the time integral of voltage $v(t)$ and electrical charge $q(t)$ is the time integral of current $i(t)$. The subset models for the current-driven *RLC* parallel DC electrical circuit and voltage-driven *RLC* series DC electrical circuit can thus be rewritten as follows, using flux $\phi(t)$ as the output variable in place of voltage $v(t)$ and using charge $q(t)$ as the output variable in place of current $i(t)$, respectively. Here are the flux/voltage and charge/current substitution relationships:

$$\phi(t) = \int v(t)dt \qquad q(t) = \int i(t)dt$$

$$\dot{\phi}(t) = v(t) \qquad \dot{q}(t) = i(t)$$

$$\ddot{\phi}(t) = \frac{dv(t)}{dt} \qquad \ddot{q}(t) = \frac{di(t)}{dt}$$

Parallel *RLC* Circuit

$$C\ddot{\phi}(t) + \frac{1}{R}\dot{\phi}(t) + \frac{1}{L}\phi(t) = i(t)$$

Series *RLC* Circuit

$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$$

Parallel *RL* Circuit

$$\frac{1}{R}\dot{\phi}(t) + \frac{1}{L}\phi(t) = i(t)$$

Series *RL* Circuit

$$L\ddot{q}(t) + R\dot{q}(t) = v(t)$$

Parallel *RC* Circuit

$$C\ddot{\phi}(t) + \frac{1}{R}\dot{\phi}(t) = i(t)$$

Series *RC* Circuit

$$R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$$

Parallel *LC* Circuit

$$C\ddot{\phi}(t) + \frac{1}{L}\phi(t) = i(t)$$

Series *LC* Circuit

$$L\ddot{q}(t) + \frac{1}{C}q(t) = v(t)$$

In the Force-Current Analogy the new output variable would be flux $\phi(t)$ instead of voltage $v(t)$ and in the Force-Voltage Analogy the new output variable would be charge $q(t)$ instead of current $i(t)$. The rest of these analogies are unchanged.

4. Mechatronic Devices

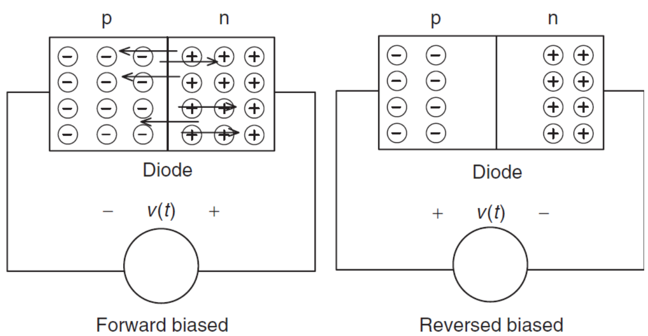
This chapter presents an overview for various mechatronic devices. The mechatronic devices covered include semiconductors, p-n junctions and diodes, transistors, switches and relays, bridge circuits, logic circuits, op-amps, strain gauges, filters, thermocouples, accelerometers, photoresistors, potentiometers, and thermistors.

These are all important system-level (i.e. not simple one-unit devices) building blocks for the mechatronics laboratory.

4.1 Semiconductors

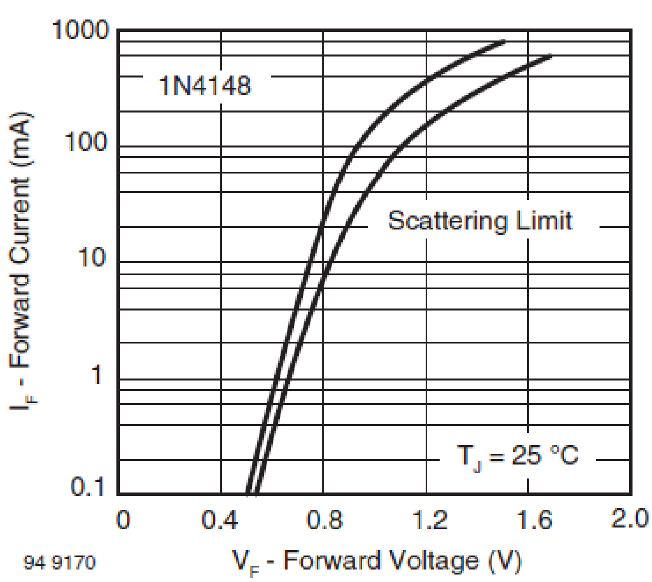
Prior to the development of semiconductors, **vacuum tubes** were the only technology available for switching and signal amplification. Vacuum tubes are bulky, require high operating voltages, and are inefficient. Vacuum tubes get hot, making for slow switching and often burning out. A **semiconductor** is a material with conductivity between a **conductor** and an **insulator**. Semiconductors are used in the manufacture of diodes, transistors, and integrated circuits. The most common semiconductor materials are germanium and silicon. Impurities (doping materials) are added to these bases, highly purified tetravalent atoms (boron, aluminum, gallium, or indium), and highly purified pentavalent atoms (phosphorus, arsenic, or antimony).

An **n-type semiconductor** has an excess of electrons. Applying a battery voltage causes the negative terminal to repel the free electrons while the positive terminal attracts the free electrons. A **p-type semiconductor** doped with tetravalent atoms has a deficit of electrons (holes). Applying a battery voltage results in electron flow that is opposite to that in the n-type semiconductors.



p-n Junction Diode

Onwubolu (2005)



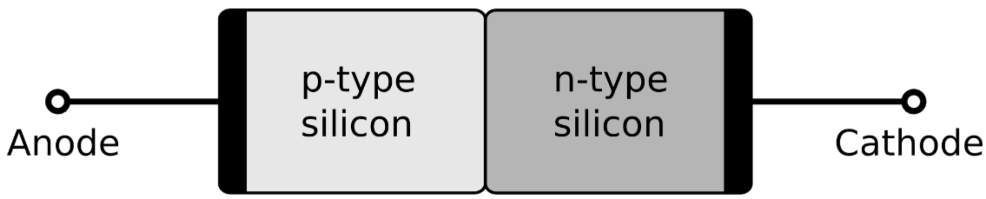
Typical Diode Current-Voltage Curve

[Diodes - SparkFun Learn]

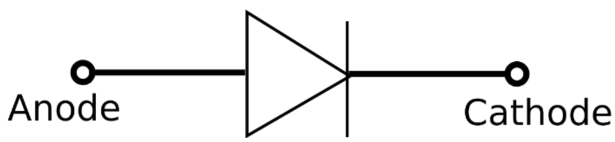
4.2 p-n Junctions and Diodes

A p-n junction is a connection of p-type (positive-type)and n-type (negative-type) semiconductors in a single crystal. The n-side has freely-moving electrons, and the p-side has freely-moving electron holes. At the connection there is a depletion region near the boundary as free electrons fill the holes; this is called diffusion. This allows current to pass through the junction in one direction only.

When a p-n junction is connected on both sides to a circuit, this is called a **diode**. A diode is a two-terminal electronic component that conducts current primarily in one direction. It has low (theoretically zero) resistance in one direction, and high (theoretically infinite) resistance in the opposite direction. This is called **asymmetric conductance**.



p-n Junction Diode



Diode Schematic Symbol

PN diode with electrical symbol - p–n junction - Wikipedia

Electronic Materials

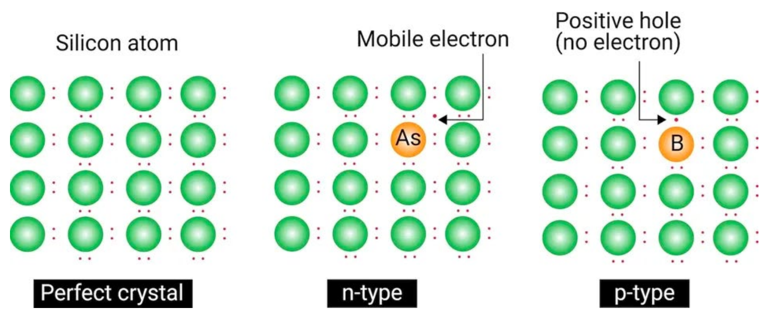
Insulators	$\rho > 10^5$	Ω-cm	recall ρ is resistivity
Semiconductors	$10^{-3} < \rho < 10^5$	Ω-cm	
Conductors	$\rho < 10^{-3}$	Ω-cm	

Conductors are materials with high electrical conductivity (low resistivity ρ) due to many free electrons that carry electric current. Conductors generally have 1, 2, or 3 valence electrons in their atoms. These valence electrons have high energy and are loosely attached to their atoms, creating an ideal situation for the flow of current.

Insulators are non-metal materials with low electrical conductivity (high resistivity ρ) due to tightly-bound electrons that cannot readily move. Insulators are used in circuits and household devices for protection against electric current. Perfect insulators are impossible since a high-enough voltage will cause current to flow (electrical breakdown voltage).

Semiconductor Doping

Silicon is by far the most common circuit material. Germanium, which has a higher energy band gap, was used historically. The Silicon atom has 4 electrons in its valence shell, and it wants 8. Through covalent bonding, Silicon atoms share electrons to satisfy the desired 8 electrons.

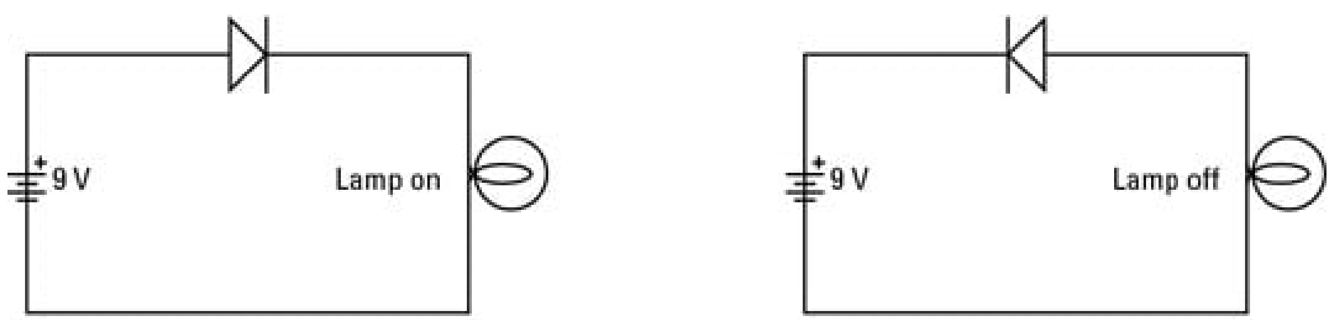


n- and p-type Semiconductor Doping **Partial Periodic Table**

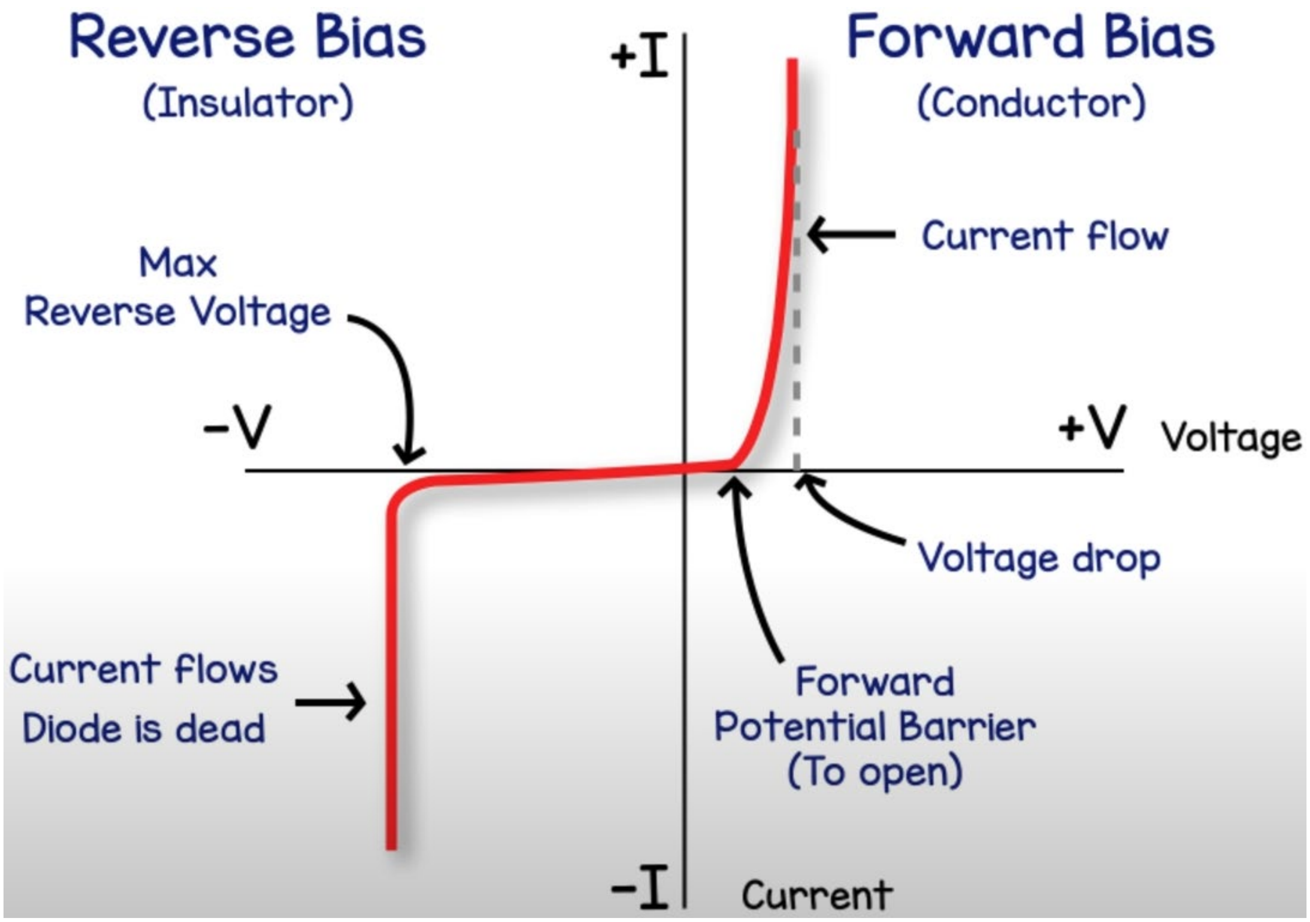
With **n-type doping**, another element (such as Phosphorus from Group 5A) is introduced into the Silicon. Phosphorus has 5 electrons in its valence shell, but the Silicon atoms crave only 4 of these, allowing a free (excess) electron to flow given each Phosphorus atom. With **p-type doping**, another element (such as Aluminum from Group 3A) is introduced into the Silicon. Aluminum has only 3 electrons in its valence shell; since the Silicon atoms crave 4 electrons, an electron hole is created.

At the p-n junction a depletion region forms with a buildup of electrons and electron holes on opposite sides. Electrons are negatively-charged, hence electron holes are considered to be positively-charged. This results in a slightly negatively-charged region abutting a slightly positively-charged region, creating an electric field that prevents more electrons from moving across the junction. The potential difference across this region is about 0.7 V.

When an external battery is supplied such that the positive terminal is connected to the p-type material via the anode, this is called **Forward Bias (diode acts as a conductor)**, with electrons flowing from negative to positive (assuming the voltage source is greater than the 0.7 V threshold). When the battery polarity is reversed such that the positive terminal is connected to the n-type material via the cathode, this is called **Reverse Bias (diode acts as an insulator)**, wherein the junction electrons will be pulled towards the p-type material, and the electron holes will be pulled towards the n-type material.



Forward Bias **Reverse Bias**

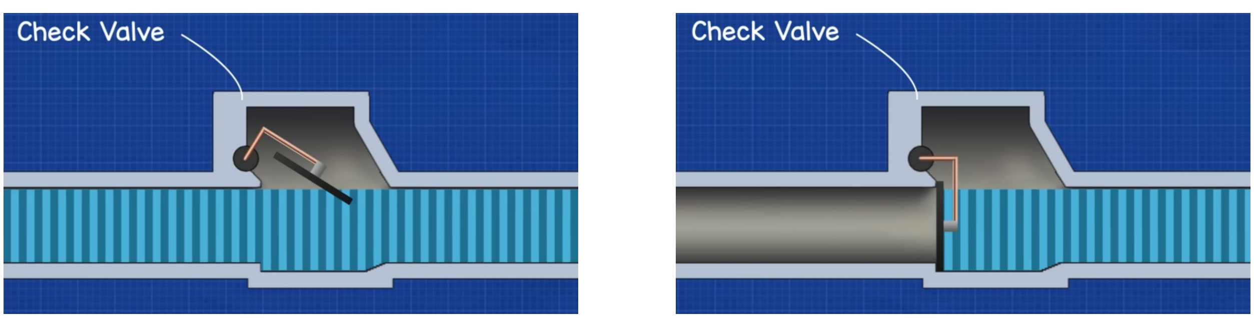


Current-Voltage Curve

Bing Videos

Hydraulic Analogy of a Semiconductor Diode

The hydraulic flow analogy for a semiconductor diode is a swing-gate (check valve) in a pipe. This allows water to flow freely in one direction. If the flow of water tries to reverse, the swing-gate will close, disallowing water to flow in the opposite direction. So it is with diodes and electrical current.



Hydraulic Flow / Diode Analogy

Bing Videos

In an NPN transistor (Section 4.3), there are two p-n junctions. The emitter (N) is heavily doped, with many excess electrons. The base (P) is lightly doped so a few electron holes exist. The collector (N) is moderately doped, with a few excess electrons.

The Intrinsic Carrier Concentration (aka Charge Carrier Density) n_i is calculated as follows:

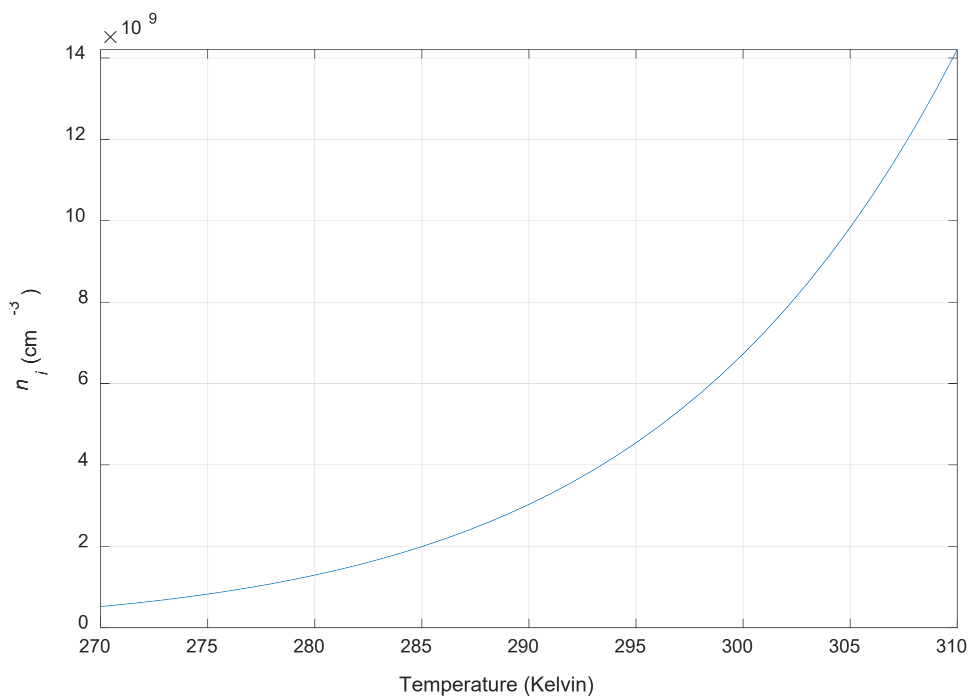
$$n_i = \sqrt{BT^3 e^{-(\frac{E_G}{kT})}} \qquad \text{cm}^{-3}$$

where:

n_i	number of intrinsic carriers	# free electrons & electron holes	cm^{-3}
B	material parameter	Silicon: 1.08×10^{31}	$\text{K}^{-3}\text{cm}^{-6}$
T	absolute temperature	Kelvin	K
E_G	band gap energy	Silicon: 1.12 at 300K	eV
k	Boltzmann's constant	8.62×10^{-5}	eV / K

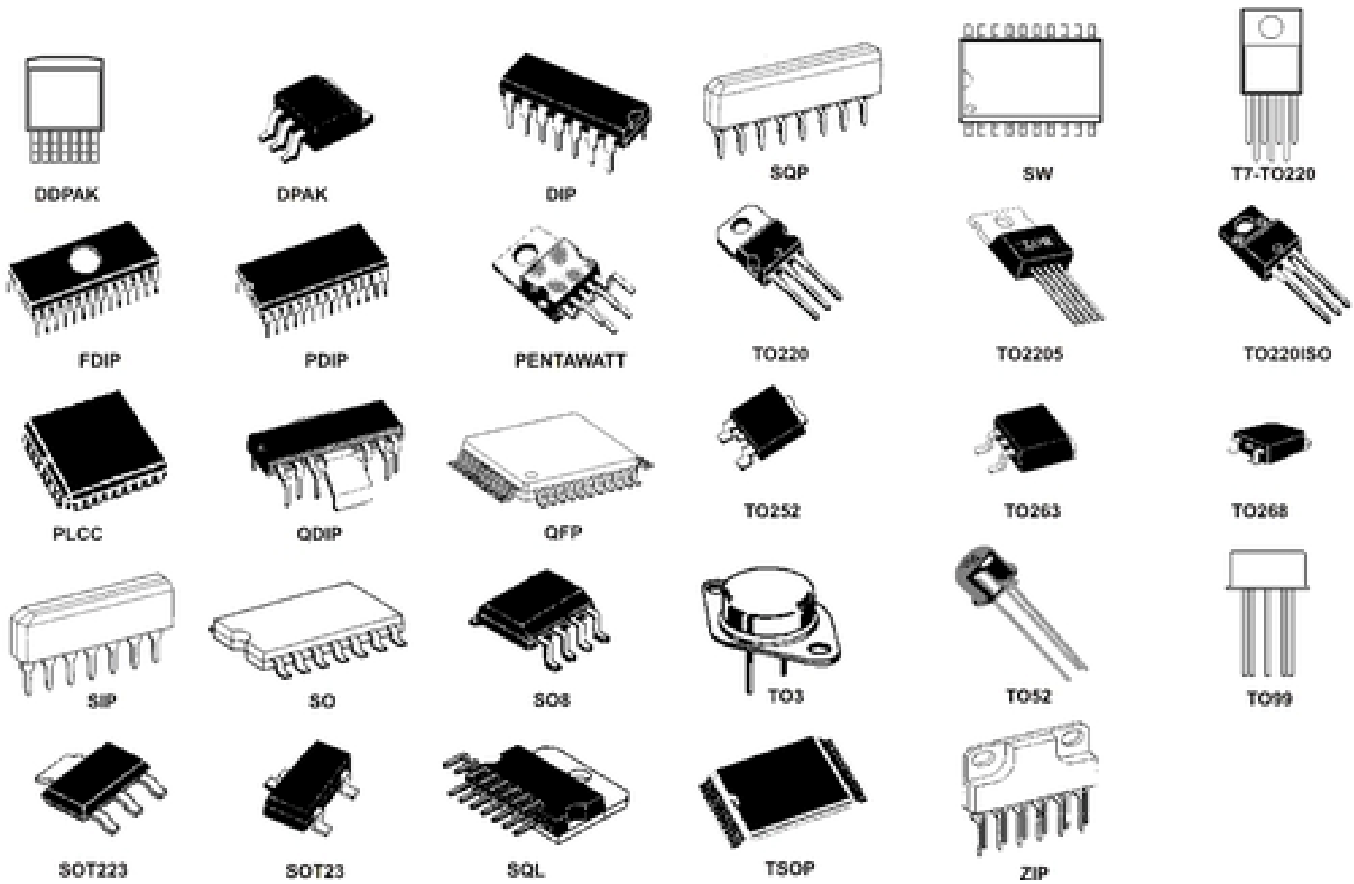
The higher n_i is, the better the current flow. Temperature greatly affects intrinsic carrier concentration due to the T^3 term. Semiconductor doping impurities through diffusion allows current to flow in only one direction. The n-type doping elements are found in Group 5A of the Periodic Table of Elements. The p-type doping elements are found in Group 3A of the Periodic Table of Elements.

The plot below shows the number of intrinsic carriers n_i for Silicon, plotted vs. absolute temperature in Kelvin. The temperature range covers approximately -3° C to 37° C (26° F to 98° F).



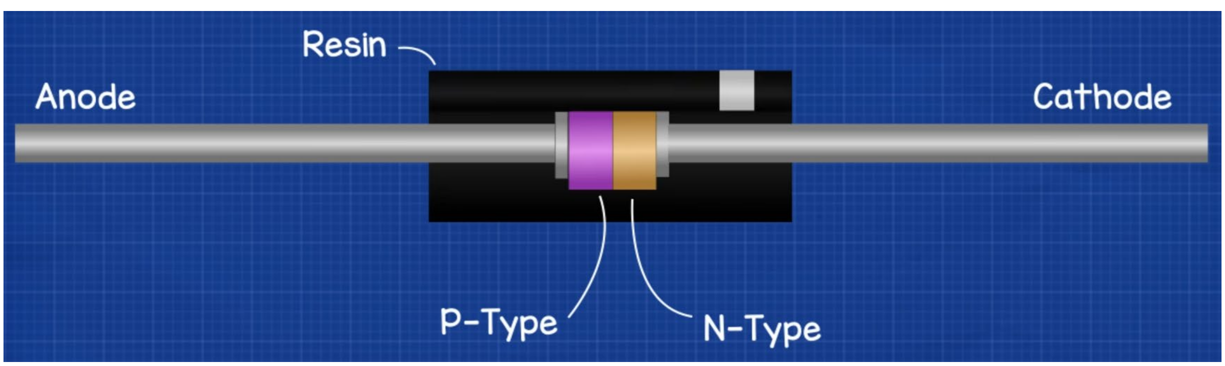
Intrinsic Carrier Concentration vs. Absolute Temperature T

We see the dominating cubed relationship between intrinsic carriers concentration n_i and the absolute temperature T in Kelvin.



Example Integrated Circuits

The p-n junction is the foundation of integrated circuits (diodes, transistors, MOSFETs).



Typical Diode Construction

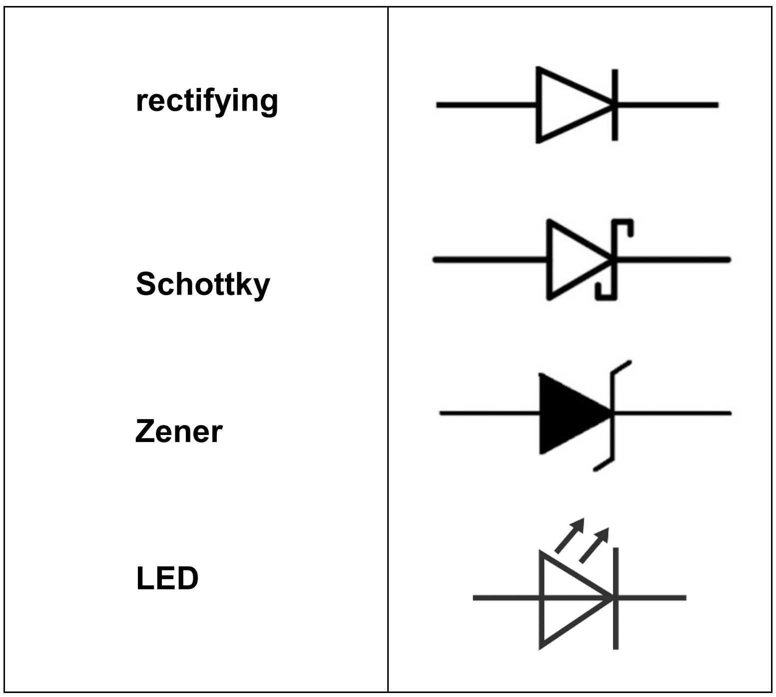
A **rectifying diode** is a semiconductor diode used to change alternating current (AC) to direct current (DC).

A **Schottky diode** is a semiconductor diode formed by the junction of a semiconductor with a metal. Compared to conventional diodes, this allows faster response times and reduced power loss, due to the lack of a depletion layer.

A **Zener diode** allows current to flow backwards when the reverse Zener voltage is reached. With a Zener diode, the current becomes almost vertical vs. voltage at breakdown.

A **light-emitting diode (LED)** emits light when forward-biased (current flows) and doesn't emit light when reverse-biased (current doesn't flow).

Diodes may be used to turn current on and off (in place of mechanical switches).



Diode Types and Schematic Symbols

A **thyristor** is a solid-state semiconductor device (invented in 1956) that is a highly robust and switchable diode. The most common type of thyristor is the silicon-controlled rectifier (SCR). It is used in high-power applications such as inverters and radar generators. A thyristor usually consists of 4 layers of alternating p- and n-type materials. It acts as a bi-stable switch (latch).

Thyristor - Wikipedia

4.3 Transistors

A transistor is a semiconductor device with generally three terminals used to amplify or switch electrical signals and power. The transistor is a basic building block of electronics and thus is considered one of the most important inventions of the 20^{th} century.

In some cases, transistors are visible, but often they are deep within an integrated circuit. In discrete amounts, transistors are used as electronic switches, digital logic, and for signal amplifying. In huge quantities (billions), transistors are implemented in computer chips for computer memory, microprocessors, and other integrated circuits. Nowadays, an integrated circuit the size of an adult human fingernail can have up to 2 billion transistors, with the capability to manipulate individual electrons!

Transistors come in many shapes and sizes. Lower power transistors often have a resin case. Higher power transistors have a metal and resin case; the metal, often attached to a heat sink, is used to reject the heat caused by the higher power. From the part numbers stamped on the transistor, one can find the maximum allowable voltage and current.

Resin Case

Resin/Metal Case

www.electronica-max.com

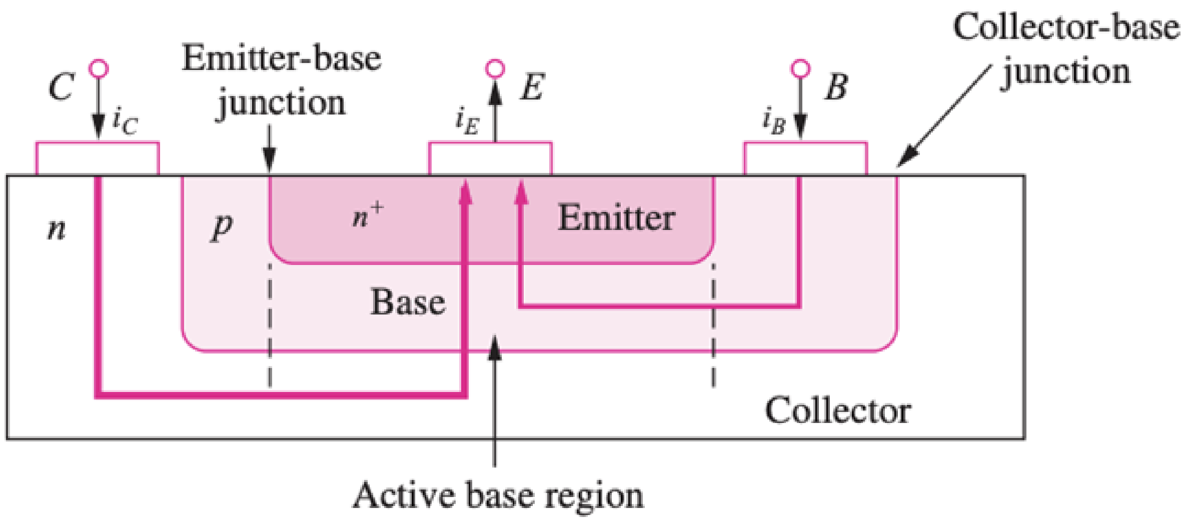
Transistors have three pins, **E** (**emitter**), **B** (**base**), and **C** (**collector**). Often (but not always), these are arranged left-to-right when the flat side is facing upwards. A small voltage of 0.7 V is required to activate the transistor, i.e. to allow the current to pass through. A small voltage and current to the transistor can be used to activate a larger voltage and current in the circuit, so a transistor can be an amplifier (e.g. 1 mA to 100 mA). The current gain is called β:

$$\beta = \frac{i_C}{i_B}$$

Where i_C is the collector current and i_B is the base current.

Transistors come in two major categories, **Bipolar Junction Transistors (BJTs)** and **Field-Effect-Transistors (FETs)**.

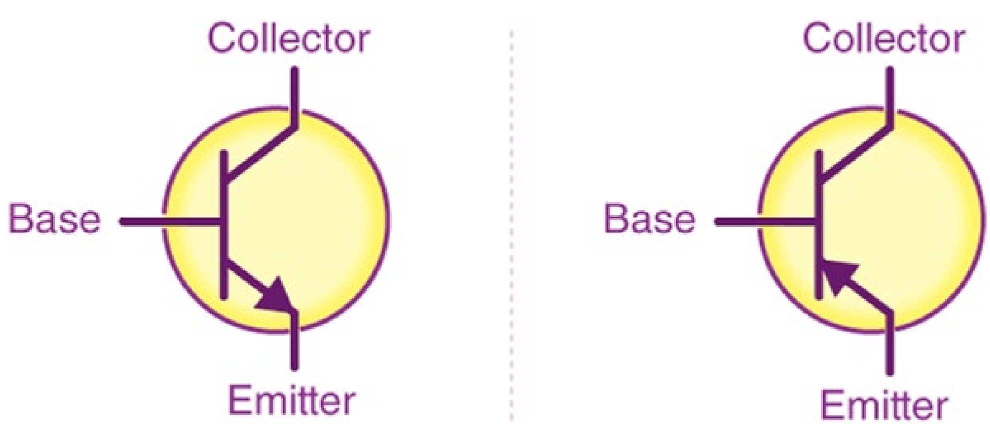
A **Bipolar Junction Transistor (BJT)** is a current-driven three-terminal semiconductor device consisting of two p-n junctions, able to amplify an electrical signal. A BJT uses both electrons and electron holes as charge carriers. In contrast, a unipolar transistor such as a field-effect transistor (FET) uses only one type of charge carrier.



BJT Construction (NPN-type)

One of the pins is biased (usually the base), allowing current to flow from the emitter to the collector. The emitter valence electrons occupy the holes in the base material. The remaining valence electrons with no holes (the base is thin) flow to create the current.

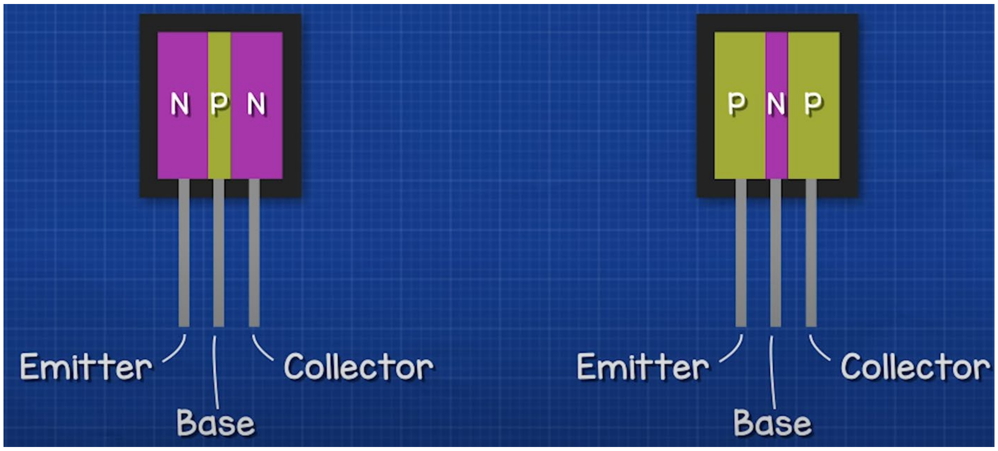
BJTs come in two major categories, **Negative-Positive-Negative (NPN)** and **Positive-Negative-Positive (PNP)**. In the diagrams below the arrow is on the emitter, showing the direction of conventional current flow. In Electrical Engineering, **conventional current** means the current flows from positive to negative. In reality the flow of current is opposite, i.e. from negative to positive!



NPN **PNP**

BJT Schematic Symbols, NPN vs. PNP

In an NPN BJT, the emitter and collector pins are connected to the N-material, while the base pin is connected to the sandwiched P-material. In contrast, In a PNP BJT, the emitter and collector pins are connected to the P-material, while the base pin is connected to the sandwiched N-material.

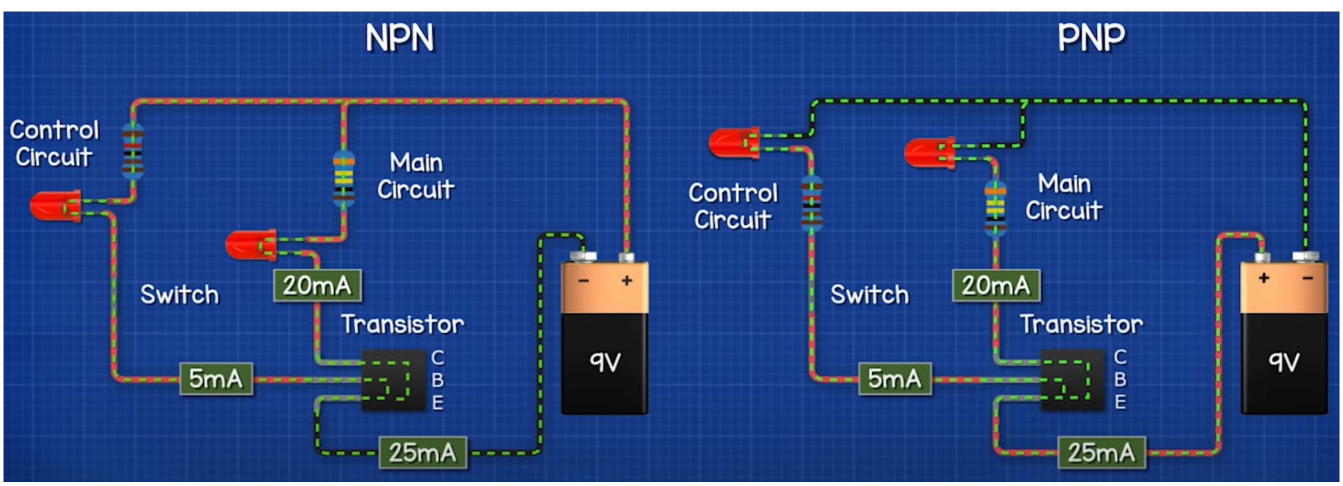


NPN **PNP**

Bing Videos

With an NPN BJT, the main circuit and control circuit are both connected to the positive side of the battery. For example, a current of 5 mA flowing into the base pin from the control circuit combines with a current of 20 mA from the main circuit flowing into the collector pin to yield a current of 25 mA leaving the emitter pin. In an NPN BJT, the currents thus combine.

With an PNP BJT, the emitter is connected to the positive side of the battery. For example, if a current of 25 mA flows into the emitter pin from the battery, and a current of 5 mA flows out of the base pin into the control circuit, this leaves a current of 20 mA to flow from the collector emitter pin to the main circuit. In an PNP BJT, the currents thus separate.

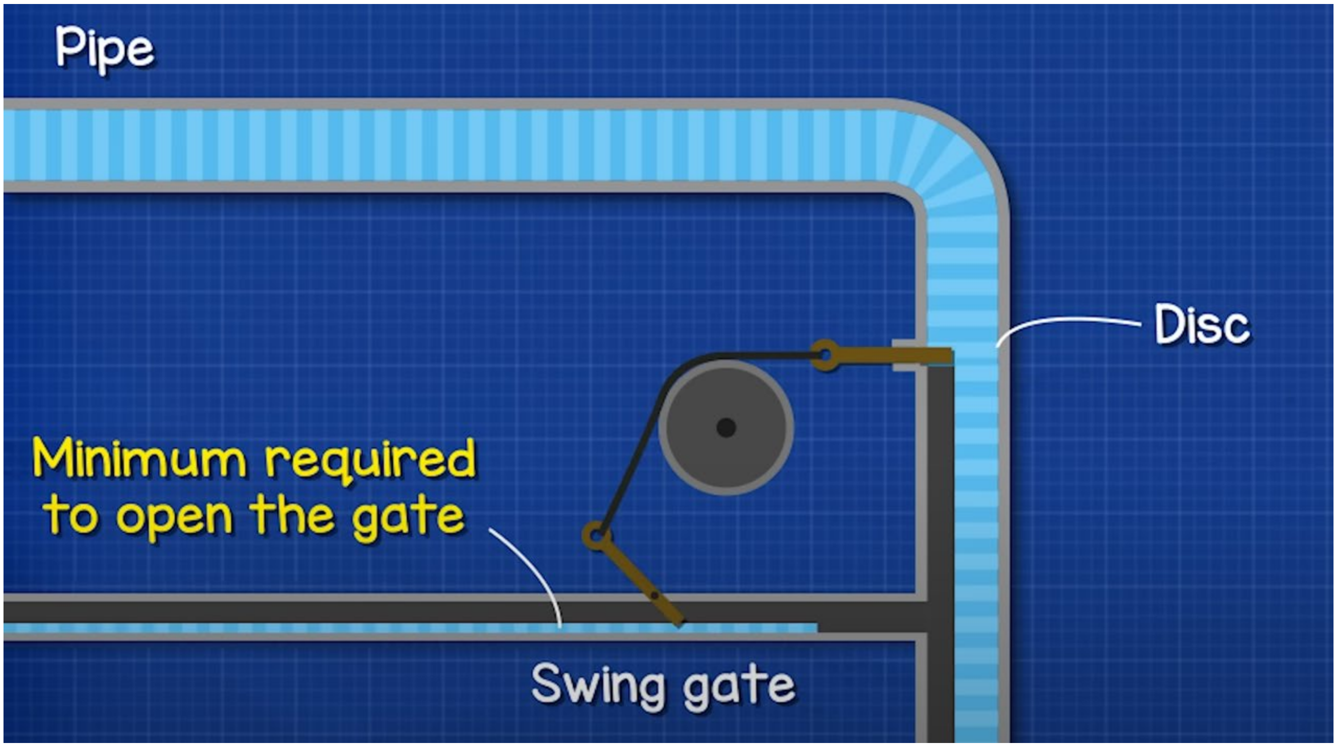


NPN vs. PNP BJT Current Flow
(NPN anti-clockwise, PNP clockwise; conventional current used)

Bing Videos

Hydraulic Analogy of an NPN BJT

Consider the fluid flow of water through a main pipe. The flow is smooth until a disc blocks the flow. Now add a smaller pipe with a swing gate downstream and connected by a cable/pulley to the disc. A small amount of flow in the smaller pipe will not be able to open the swing gate, due to inertia. If the flow in the smaller pipe exceeds a certain minimum, the swing gate will start to open, also opening the blocking disc. The higher the flow in the smaller pipe, the more the swing gate will open, opening the blocking disc more and more, allowing more flow in the main pipe (up to a maximum).



Fluid Flow / NPN BJT Analogy

This is analogous to the operation of an NPN BJT in an electrical circuit. Again, fluid flow rate is analogous to electrical current. The smaller flow in the smaller pipe with swing gate is analogous to electrical current flowing into the base pin of an NPN BJT. As the swing gate opens and allows more and more flow into the main pipe (downstream), this is analogous to a larger flow into the collector pin of an NPN BJT. The two flows combine downstream of the smaller pipe, analogous to the larger current leaving the emitter pin in an NPN BJT.

Bipolar Transistor Operation Regions

Base-Emitter Junction	Base-Collector Junction	
	Reverse Bias	**Forward Bias**
Forward Bias	forward-active region (normal-active region) (good amplifier)	saturation region* (closed switch)
Reverse Bias	cutoff region (open switch)	reverse-active region (inverse-active region) (poor amplifier)

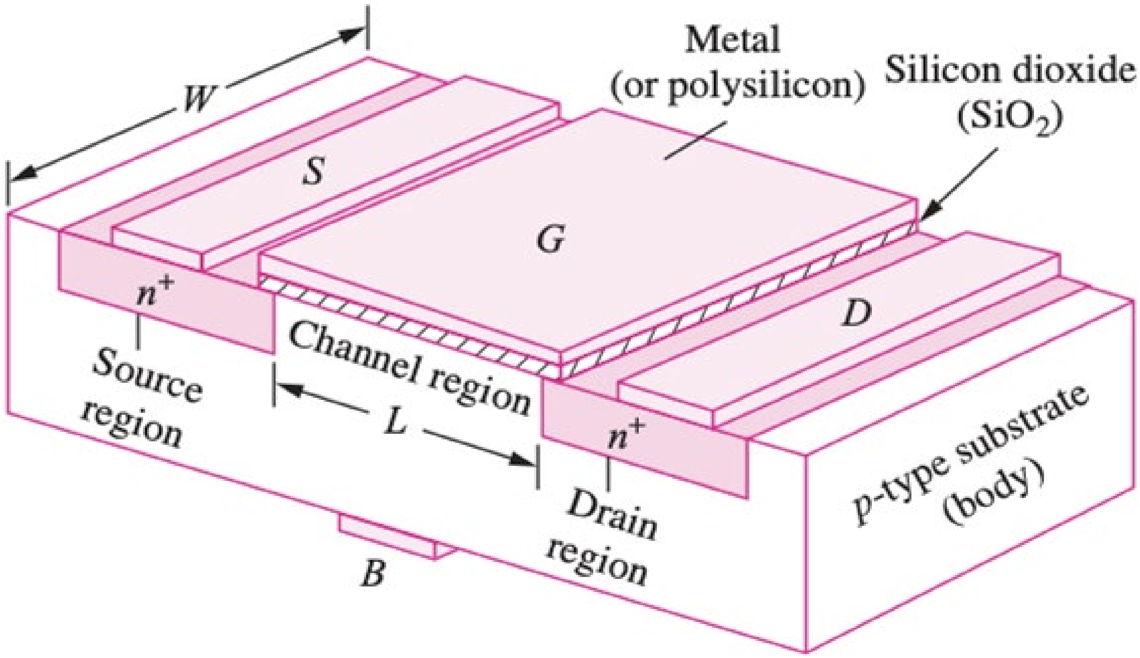
*Note: the saturation region of the bipolar transistor does not correspond to the saturation region of the FET.

BJTs can be used as switches or amplifiers. When used as a switch, BJTs use a small current to turn on a larger current, via a threshold voltage.

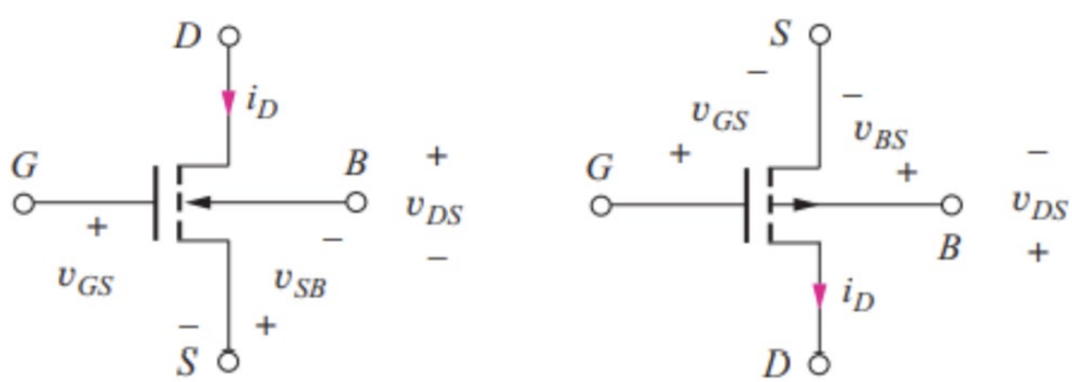
For both NPN and PNP:

- Forward Bias – current flows into the Base and out the Emitter (or Collector)
- Reverse Bias – current flows into the Emitter and out the Base (or Collector)

Metal-Oxide-Semiconductor Field-Effect-Transistors (MOSFETs) are simpler in construction and operation than BJTs. They are constructed as n-channel-MOSFETs or p-channel-MOSFETs.



MOSFET Construction (cutaway)



NMOS Transistor **PMOS Transistor**

MOSFET Schematic Symbols, N vs. P

G – gate, *D* – drain, *S* – source; *B* – body

FET Operation Regions

- Cut-off region FET acts like an open switch
- Triode region FET acts like a linear resistor (like amplifier in BJT; like potentiometer)
- Pinch-off region FET acts like a closed switch

NMOS Transistor Equations Summary

All Regions

$$K_n = K_n^{'} \frac{W}{L} = \mu_n C_{ox}^{"} \frac{W}{L} \qquad\qquad i_G = i_B = 0$$

Cutoff Region

$$i_D = 0 \qquad \text{for} \qquad v_{GS} \le V_{TN}$$

Triode Region

$$i_D = K_n \left[v_{GS} - V_{TN} - \frac{v_{DS}}{2} \right] v_{DS} \qquad \text{for} \qquad (v_{GS} - V_{TN}) \ge v_{DS} \ge 0$$

Saturation Region

$$i_D = \frac{K_n}{2}(v_{GS} - V_{TN})^2 (1 + \lambda v_{DS}) \qquad \text{for} \qquad v_{DS} \ge (v_{GS} - V_{TN}) \ge 0$$

Threshold Voltage

$$V_{TN} = V_{TO} + \gamma(\sqrt{v_{SB} + 2\phi_F} - \sqrt{2\phi_F})$$

where v_{SB} is the source body voltage (zero), and usually $V_{TN} = V_{TO}$.

<u>PMOS Transistor Equations Summary</u>

All Regions

$$K_p = K_p' \frac{W}{L} = \mu_p C_{ox}'' \frac{W}{L} \qquad\qquad i_G = i_B = 0$$

Cutoff Region

$$i_D = 0 \qquad \text{for} \qquad v_{GS} \geq V_{TP}$$

Triode Region

$$i_D = K_p \left[v_{GS} - V_{TP} - \frac{v_{DS}}{2} \right] v_{DS} \qquad \text{for} \qquad (v_{GS} - V_{TP}) \leq v_{DS} \leq 0$$

Saturation Region

$$i_D = \frac{K_p}{2} (v_{GS} - V_{TP})^2 (1 + \lambda |v_{DS}|) \qquad \text{for} \qquad v_{DS} \leq (v_{GS} - V_{TP}) \leq 0$$

Threshold Voltage

$$V_{TP} = V_{TO} - \gamma(\sqrt{v_{SB} + 2\phi_F} - \sqrt{2\phi_F})$$

where v_{SB} is the source body voltage (zero), and *K*' is the amplification constant. In the saturation region a PMOS acts as a high-current switch.

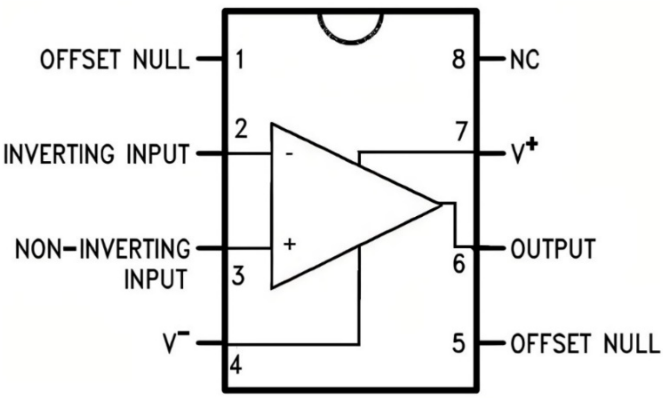
MOS Transistor Parameters

	NMOS	PMOS
V_{TO}	$+0.75V$	$-0.75V$
γ	$0.75\sqrt{V}$	$0.5\sqrt{V}$
$2\phi_F$	$0.60V$	$0.60V$
K' µA/V^2	100	40

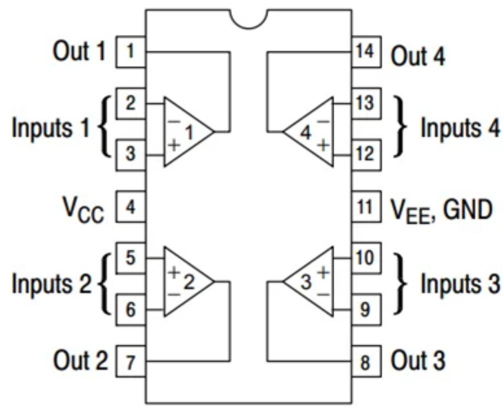
4.4 Op-Amps

An **Op-Amp** (operational amplifier) is a direct-coupled (physical contact vs. inductor or capacitive non-contact coupling) voltage amplifier with two differential inputs, a single-ended output, and an extremely-high gain. The gain may be considered to be infinite in most applications; the gain value is 10,000 and up. Op-amps have a very high (infinite) input impedance and a very low (zero) output impedance. Op-amps are applied to a wide variety of applications, including amplifiers, mixers, filters, oscillators, differentiators, and integrators.

The op-amp output is proportional to the difference between the two inputs. The non-inverting input is labeled + and preserves the phase of the input signal. The inverting input is labeled – and reverses the phase of the input signal by 180°. If the two input signals are identical, then there will be no output (zero, the common mode rejection ratio). Positive and negative power rails must be supplied. Often they are not shown in the diagram, to reduce clutter, because they are assumed to exist.

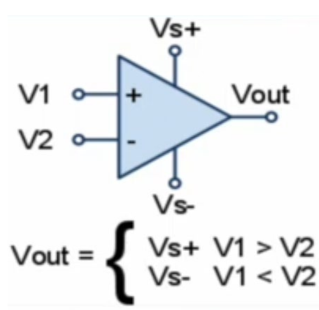


LM741 Op-Amp Pinout Diagram



LM324 Quad Op-Amp

The Offset Null is used to zero any small DC value. The output waveform will not surpass the supply voltage (5V / 0 on VCC / ground). VCC is the acronym for **voltage at the common collector**. The output waveform shows only the top half of the input sine wave.



Voltage Comparator

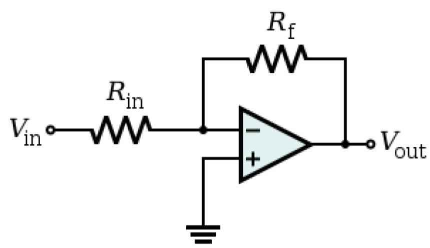
The voltage comparator performs the inequalities shown above. Here the voltage inputs have high impedance. They largely don’t affect the circuitry feeding them a signal voltage, so there is no current to/from the inputs.

Since op-amps have extremely high gains, they are generally used with feedback.

- Inputs are driven to zero
- Op-amps are limited by gain-bandwidth product
- Op-amps have linear operation

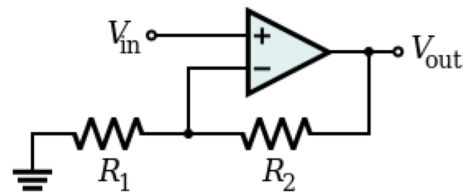
Various Op-Amps and their gains *G*

$$G = -\frac{R_f}{R_{in}} \qquad V_{out} = GV_{in}$$



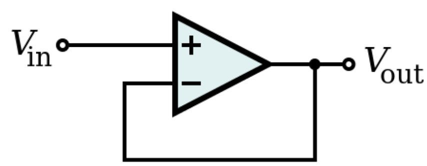
Inverting Op-Amp

$$G = 1 + \frac{R_2}{R_1} \qquad V_{out} = GV_{in}$$



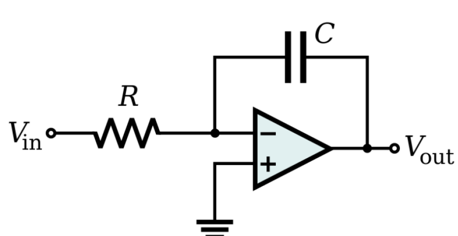
Non-Inverting Op-Amp

With an inverting amplifier, the voltage output inverts the input voltage (converts the input voltage to 180° out of phase) and amplifies the input voltage by the gain R_f / R_{in}. With a non-inverting amplifier, the voltage output maintains the input voltage phase and amplifies it by the gain $1 + (R_2 / R_1)$.



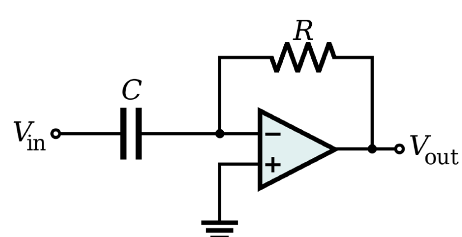
Unity Gain (*G* = 1) Op-Amp (Voltage Follower, $V_{out} = V_{in}$)

$$V_{out} = -\frac{1}{RC}\int_0^t V_{in}\,dt$$



Integrating Op-Amp

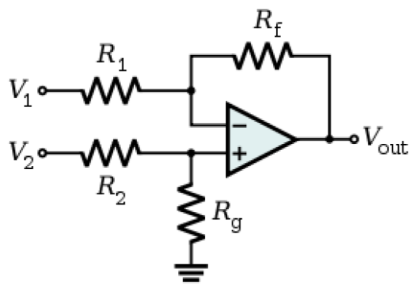
$$V_{out} = -RC\frac{dV_{in}}{dt}$$



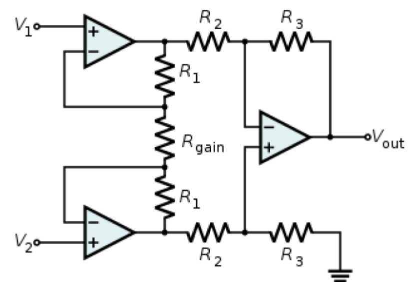
Differentiating Op-Amp

$$R_1 = R_2$$
$$R_f = R_g$$
$$V_{out} = \frac{R_f}{R_1}(V_2 - V_1)$$

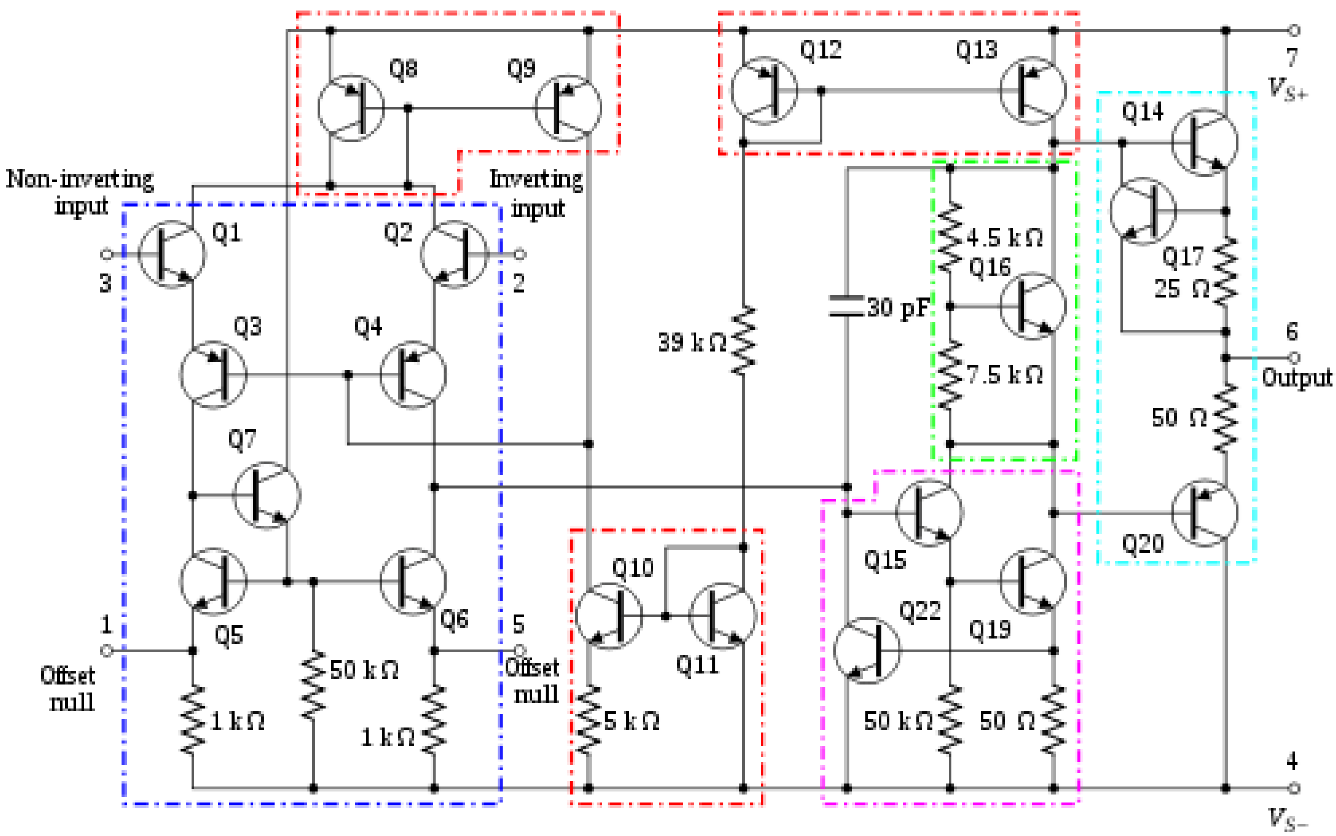
$$G = \left(1 + \frac{2R_1}{R_{gain}}\right)\frac{R_3}{R_2}$$
$$\frac{V_{out}}{V_2 - V_1}$$



Differential Op-Amp



Instrumentation Op-Amp



Op-Amp Example (or, why Dr. Bob became an ME)

4.5 Logic Circuits

Logic Circuits (aka **Logic Gates**) form the basis for computation in digital computers. In a digital computer, when there is a voltage across a wire, with an electrical current flowing through, this represents the binary digit '1'. When there is no voltage across a wire, and no electrical current flowing through, this represents the binary digit '0'. Logic gates manipulate the '1s' and '0s' in a computer. Digital electronics is heavily based on Boolean logic, with two states T / F (1 / 0). Please note that 1 / 0 are not integers, but represent the states true / false. Therefore, normal integer algebra does not apply. The table below shows there are many names for these two logical states.

1	T	Hi	On	5 V
0	F	Lo	Off	0 V

In most basic logic gates there are two inputs and one output (except for the NOT, inverter gate, which has one input and one output).

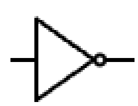
Logical Operators

Operator	Logical Operation	Notation	Output
NOT	negation	NOT x, $\sim x$	**0** if x=1; **1** if x=0
AND	conjunction	x AND y	**1** if x=y=1; **0** otherwise
OR	disjunction	x OR y	**0** if x=y=0; **1** otherwise

The following four tables presenting the output for all possible inputs for the various logic gates are called **Truth Tables**.

Logical Operator NOT (Inverter)

x	NOT x
0	**1**
1	**0**



NOT
Schematic Symbol

Logical Operators AND & OR

x	*y*	*x* AND *y*	*x* OR *y*
0	0	**0**	**0**
1	0	**0**	**1**
0	1	**0**	**1**
1	1	**1**	**1**

AND **OR**
Schematic Symbols

Logical Operators NAND & NOR

x	*y*	*x* NAND *y*	*x* NOR *y*
0	0	**1**	**1**
1	0	**1**	**0**
0	1	**1**	**0**
1	1	**0**	**0**

NAND **NOR**
Schematic Symbols

Logical Operators Exclusive OR & Exclusive NOR

x	*y*	*x* XOR *y*	*x* XNOR *y*
0	0	**0**	**1**
1	0	**1**	**0**
0	1	**1**	**0**
1	1	**0**	**1**

Exclusive OR **Exclusive NOR**
Schematic Symbols

Why is there no such thing as an **XAND** (or **XNAND**) gate?

Logic Gates

- There are 5-15 transistors inside each logic gate.
- A 2-pin representation is used for clarity – physical logic gates have more pins, at minimum two more pins, ground and power.

Example

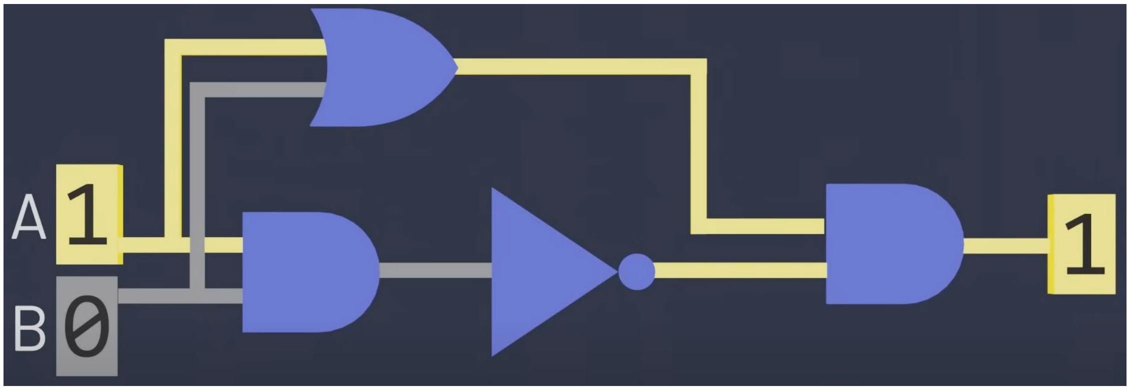
Given two inputs A and B, we wish to determine if exactly one of these is a 1.

Solution:

Either A or B must be a 1. But both A and B cannot both be a 1.

Logically, using Boolean Algebra, we have:

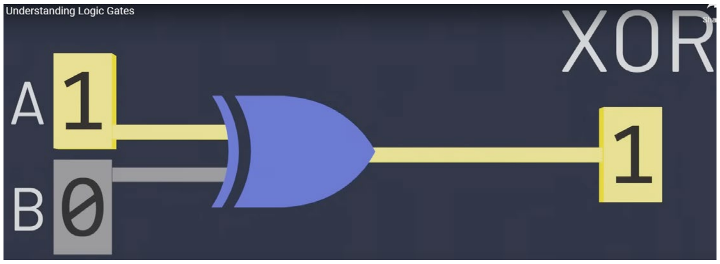
(A OR B) AND (NOT (A AND B))



Circuit with Logic Gates for the Example

Bing Videos

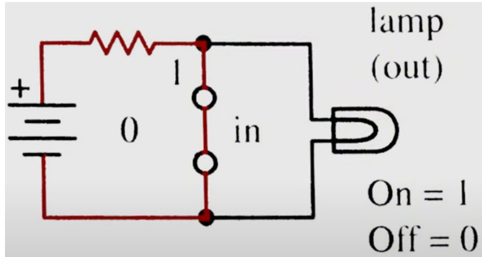
This is useful, but complicated, so engineers have developed a logic gate to do the same determination: the Exclusive OR (XOR) gate presented earlier.



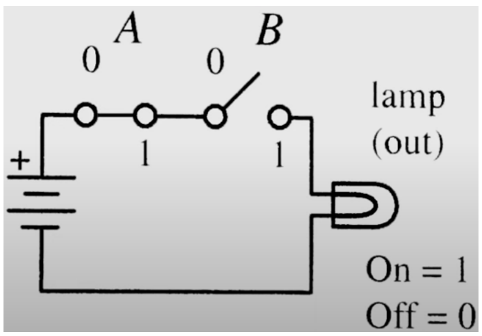
Example concludes.

Analog Circuits to Represent Logic Gates

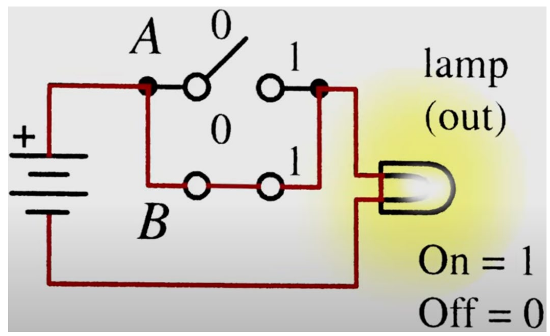
https://www.youtube.com/watch?v=lXWpWNKwYbo



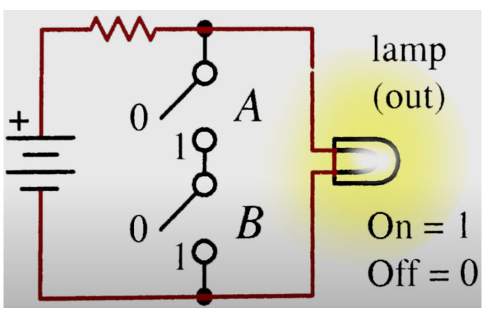
NOT



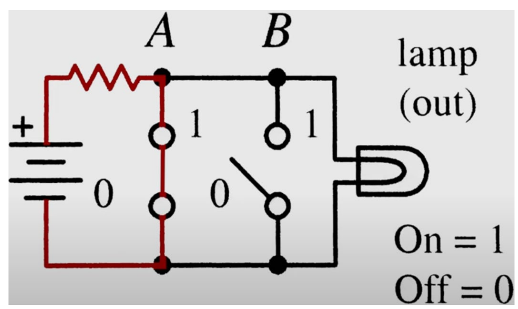
AND



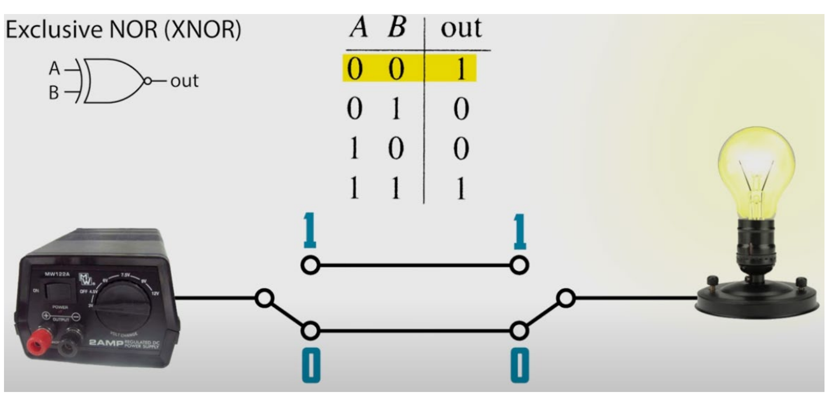
OR



NAND



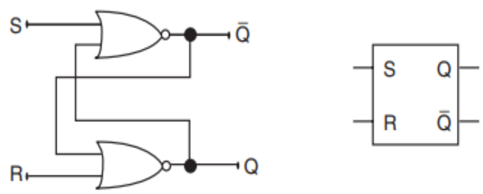
NOR



NOR

S-R Flip-Flops

A flip-flop is an electronic circuit with two stable states, used to store binary data. It has one or more control inputs and one or two outputs. Flip-flops circuits are essential in computer memory for reliably storing 1s and 0s. Staying in the last state it was assigned, the circuit 'remembers' the last state. **S** stands for set and **R** stands for reset.



S-R Flip-Flop based on NOR Gates

S	R	Q	Q̄
0	0	no change	
0	1	0	1
1	0	1	0
1	1	undefined	

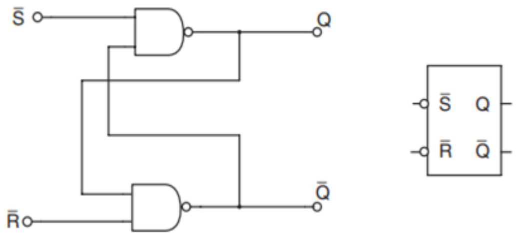
Truth Table for S-R Flip-Flop

An S-R flip-flop may be Active-HIGH or Active-LOW as seen in the figure below.

Active-HIGH vs. Active-LOW S-R Flip-Flops

https://www.youtube.com/watch?v=Hi7rK0hZnfc

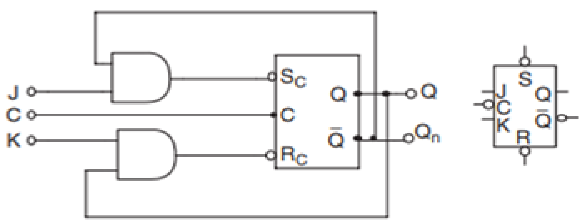
Again, the **overbar** notation means **opposite of**. An alternative S-R flip-flop can use NAND Gates in place of the above NOR Gates (see below).



Alternative S-R Flip-Flop, NAND Gates

S	R	Q	$\bar{Q}$
0	0	no change	
0	1	1	0
1	0	0	1
1	1	undefined	

Truth Table, Alternative S-R Flip-Flop

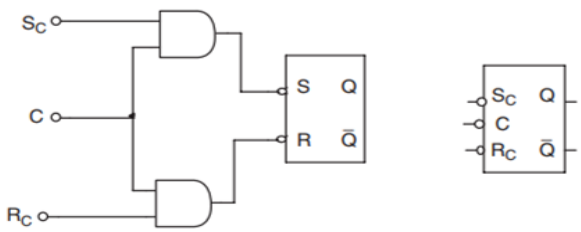


J-K Flip-Flop based on AND Gates

J	K	C	Q	$\bar{Q}$
0	0	*p*	no change	
0	1	*p*	0	1
1	0	*p*	1	0
1	1	*p*	toggle	

Truth Table for J-K Flip-Flop

The J-K flip-flop is named for its inventor, Jack Kilby. C is the clock-pulse pin (not Capacitor). The microcontroller is operating at a certain frequency that can be tuned. This is called the clock rate and is based on a vibrating quartz crystal. The J-K flip-flop solves the undefined problem in S-R flip-flops when both **S** and **R** are 1.

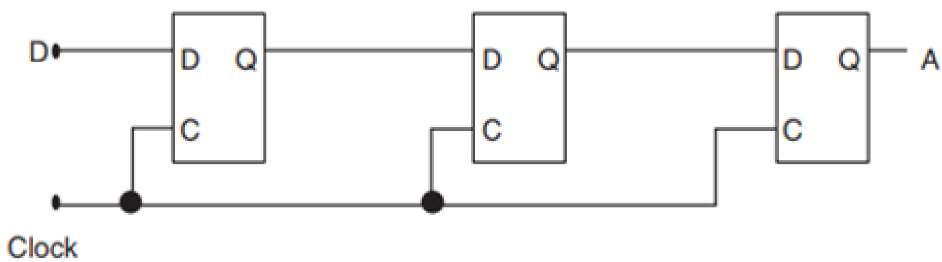


S_c	R_c	C	Q	$\bar{Q}$
X	X	0	no change	
0	0	1	no change	
0	1	1	0	1
1	0	1	1	0
1	1	1	undefined	
0	0	*p*	no change	
0	1	*p*	0	1
1	0	*p*	1	0
1	1	*p*	undefined	

D Flip-Flop (Clocked S-R Flip-Flop) **Truth Table for Clocked S-R Flip-Flop**

Shift Registers

A shift register is a sequential logic circuit capable of storage and transfer of data. It is a digital circuit using a cascade of flip-flops. The output of one flip-flop is connected to the input of the next flip-flop. All flip-flops share a single clock signal, causing the stored data to shift from one location to the next.



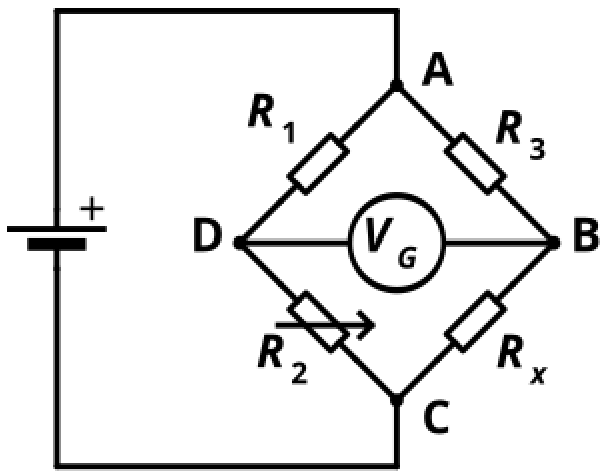
Shift Register

Shift registers are common in digital computers. They are stateful devices one can string data into. Shift registers are used to store bytes of data. Plain flip-flops can only store a single bit. Shift registers and flip-flops use logical voltages: low *v* and small current *i*, just enough power to reliably indicate 1s and 0s.

4.6 Bridge Circuits

4.6.1 Wheatstone Bridge

A Wheatstone Bridge is an electrical circuit used to measure an unknown electrical resistance by balancing two legs of a bridge circuit, one leg of which has a variable resistance, and the other includes the unknown component.



Wheatstone Bridge

Wheatstone bridge - Wikipedia

The 'Bridge' is leg **DB**. Three resistors are known, R_1, R_3, and the variable resistance R_2. The value of R_X is unknown and is to be found. The engineer changes R_2, balancing the bridge so that no current flows through the galvanometer V_G. (Galvanometer is an historic term for ammeter, a current-measuring sensor.) Then the unknown R_X can be found from the following equations. At the point of balance, both current and voltage along the bridge **DB** are zero. Therefore:

$$i_1 = i_2 \qquad i_3 = i_X \qquad V_D = V_B$$

Since $V_D = V_B$, then:

$$V_{DC} = V_{BC} \qquad V_{AD} = V_{AB}$$

so:

$$\frac{V_{DC}}{V_{AD}} = \frac{V_{BC}}{V_{AB}}$$

$$\frac{i_2 R_2}{i_1 R_1} = \frac{i_X R_X}{i_3 R_3} \qquad R_X = \frac{R_2 R_3}{R_1}$$

The Wheatstone Bridge is also the basis for Strain Gauges (electromechanical sensors used to measure mechanical strain in a material under load; see Section 4.8).

Wheatstone Bridges were developed in 1843, before multimeters existed. They allow voltage measurement of resistance. Wheatstone Bridges can be used with a potentiometer to balance $V_G = 0$. They are more flexible than a voltage divider circuit. Wheatstone Bridges allow +/– voltages for +/– strain. They are used to represent one resistor as a strain gauge: one strain gauge and three resistors are laid on the material to be measured. The Wheatstone Bridge output is differential. Temperature is also automatically compensated due to the symmetric nature of a Wheatstone Bridge.

The Wheatstone Bridge output voltage V_G equation is:

$$V_G = \left(\frac{R_2}{R_1 + R_2} - \frac{R_x}{R_x + R_3} \right) V_S$$

where V_S is the single DC voltage source.

For an on-line animated Wheatstone Bridge simulator, please see:

https://tinyurl.com/2n2aypjz

For more information on Wheatstone Bridges, please see:

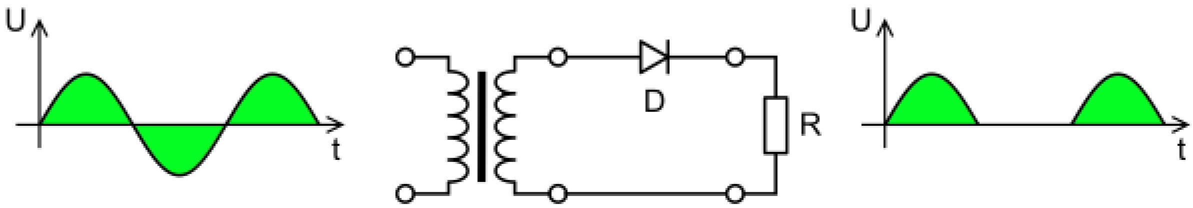
https://www.hbm.com/en/7163/wheatstone-bridge-circuit

4.6.2 Bridge Rectifiers

Bridge rectifiers convert AC voltage to DC voltage. Half-wave and full-wave rectifiers are the subject of ME 3550 Lab #3, Diodes.

Half-Wave Rectifiers

A half-wave rectifier AC voltage input, circuit, and "DC" voltage output are shown left-to-right in the figure below.



Half-Wave Rectifier

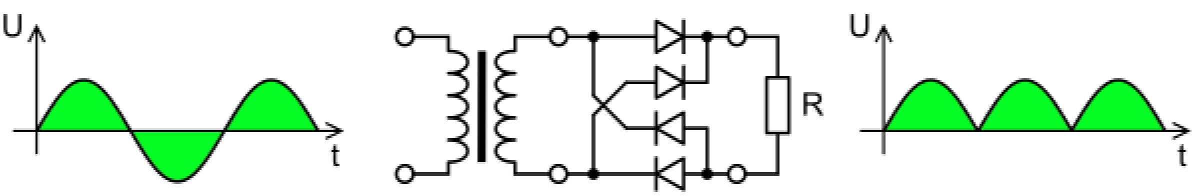
Rectifier - Wikipedia

The circuit consists of a transformer, a diode, and a resistor. Despite being always positive (or zero), the "DC" voltage output is not acceptable as a DC voltage source.

A **transformer** is a passive device that transfers electrical energy from one circuit to another via electromagnetic induction. The voltage across circuits may be increased or decreased without changing the AC supply frequency.

Full-Wave Rectifiers

A full-wave rectifier AC voltage input, circuit, and "DC" voltage output are shown left-to-right in the figure below.

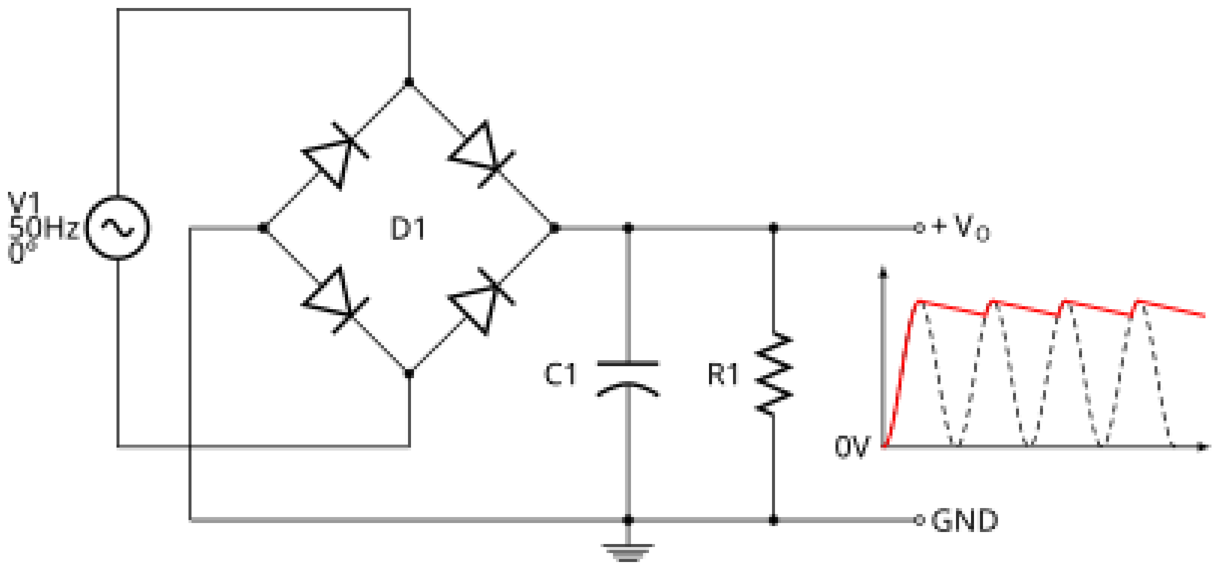


Full-Wave Rectifier

Rectifier - Wikipedia

The circuit consists of a transformer, four diodes, and a resistor. The "DC" voltage output is improved compared to the half-wave rectifier due to the middle positive peak. However, this is still unacceptable as a DC voltage source.

The "DC" voltage output could be improved by using multi-phase AC voltage input. However, as we learned in the ME 3550 Lab #3, judicious inclusion of a capacitor in the four-diode full-wave rectifier circuit can also improve the DC voltage output, as shown in the figure below.

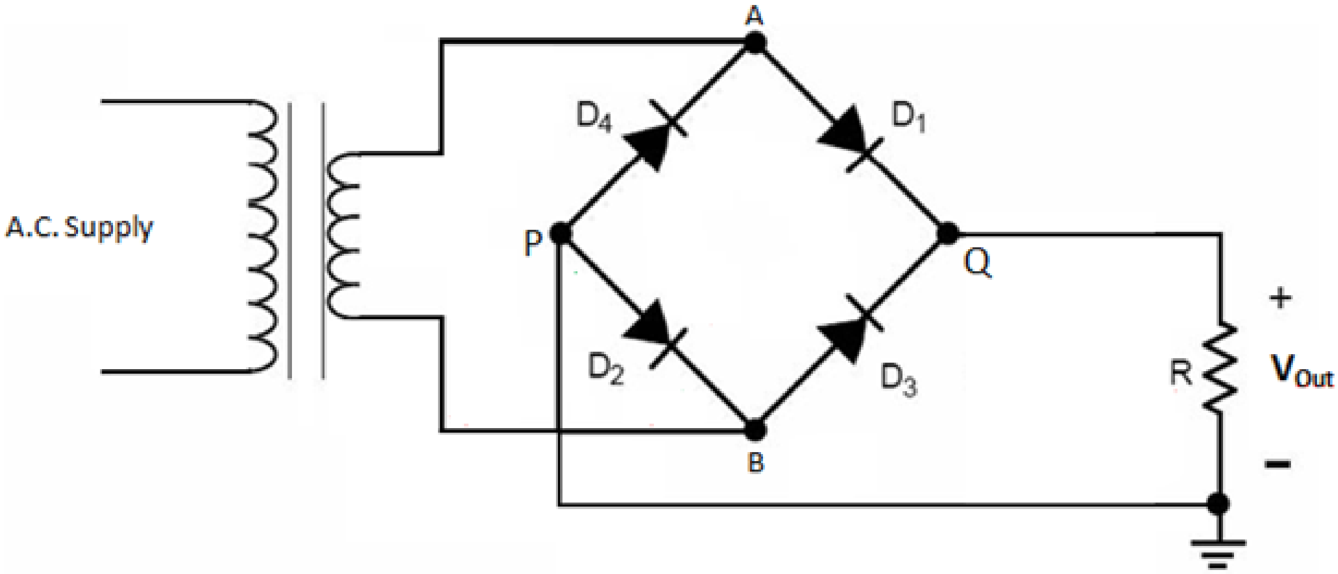


Full-Wave Rectifier with Capacitor

Rectifier - Wikipedia

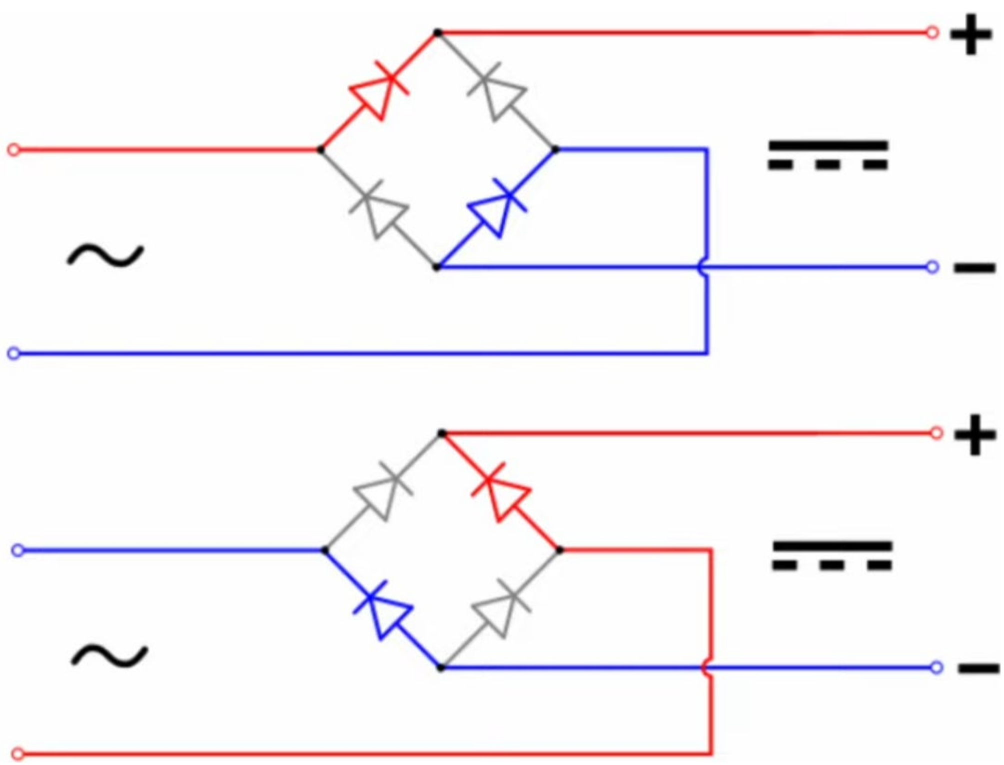
Thanks to the addition of the capacitor in parallel with the resistor, now the DC voltage output is closer to a constant positive value, as seen in the right plot above.

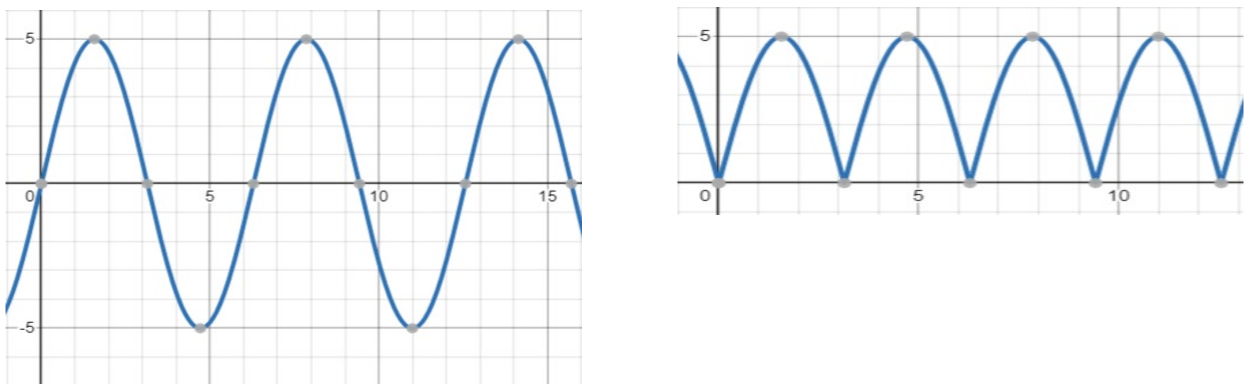
The figure below shows a full-wave bridge rectifier, going back to the non-capacitor case. Why is called a Bridge-Rectifier? (The possible bridge PQ is missing, and hence this is not a good name.) The schematic symbol to the right of the AC Supply indicates a Transformer.



Full-Wave Bridge Rectifier

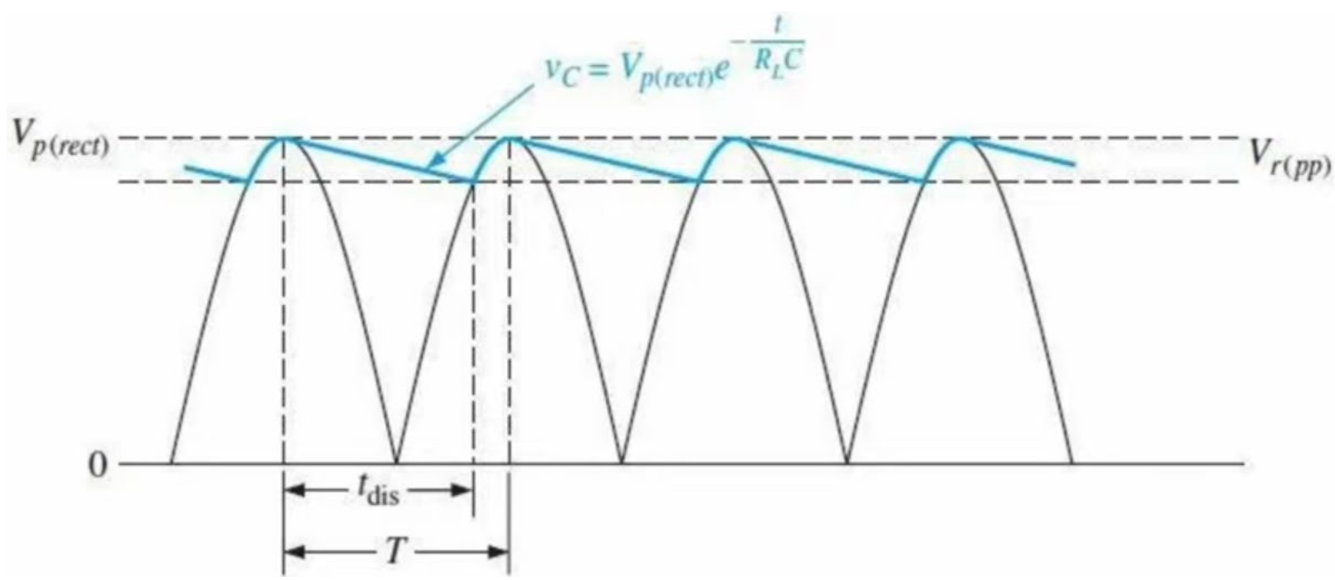
The top half of the AC voltage sine wave is allowed to flow through. Also, the bottom half of the AC voltage sine wave is allowed to flow through in the opposite direction, thus creating a + DC current.





AC input / DC Output Full-Wave Rectifier

The output voltage signal on the right above is unacceptable for DC power generation. Therefore, a smoothing capacitor is used, again in parallel with the resistor.



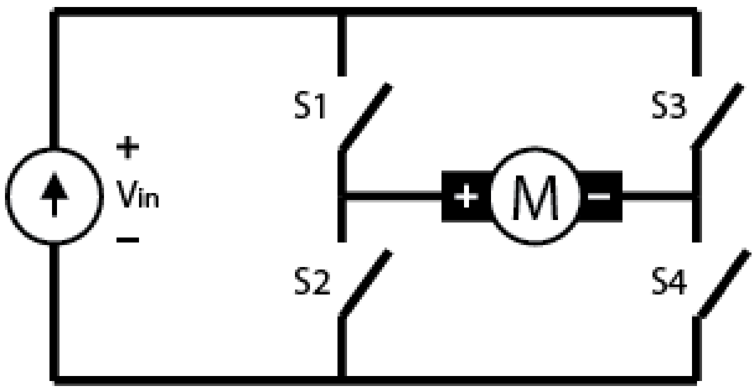
Smoothing Capacitor

$$C = \frac{i}{2fV_{\text{ripple}}}$$

This equation is approximate. For f = 60 Hz, use a 120 Hz ripple voltage V_{ripple}.

4.6.3 H-Bridge

H-Bridges can be used as brushed DC motor drivers and hence see wide application in robotics, machinery, vehicles, and manufacturing. In particular, H-Bridges can reverse the input voltage polarity so that the rotating brushed DC motor shaft may be driven in both directions, **+** (say **CCW**) and **–** (say **CW**). H-Bridges are efficient in that they require only a single unipolar power supply. An H-Bridge circuit diagram is shown below.



H-Bridge Diagram

digilent.com/blog/what-is-an-h-bridge

An H-Bridge can control variable electric motor speed when using pulse-width modulation (PWM; see Section 6.1).

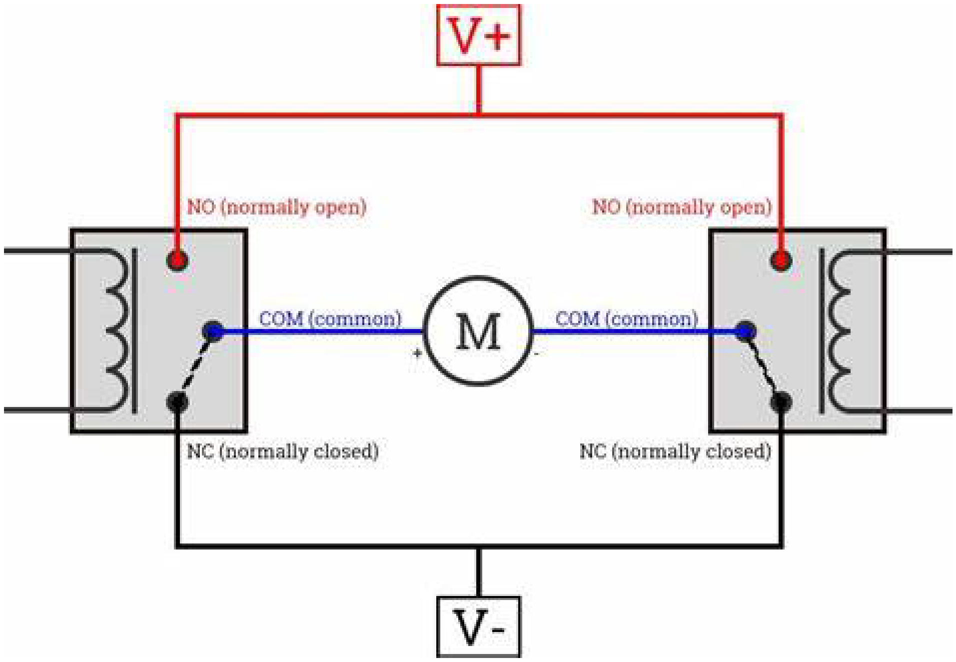
An H-Bridge consists of four simple on / off switches **S1**, **S2**, **S3**, and **S4**, arranged on the arms and legs of a capital **'H'** (hence the name H-Bridge). By opening (**0, off**) or closing (**1, on**) the four switches in various combinations, different motor motions can be obtained. The table below shows the motor action effect of the 11 combinations of these four switches.

S1	S2	S3	S4	Action
1	0	0	**1**	motor rotates **+**
0	**1**	**1**	0	motor rotates **–**
1	0	0	0	motor coasts; will slow down
0	**1**	0	0	
0	0	**1**	0	
0	0	0	**1**	
0	0	0	0	
1	0	**1**	0	motor brakes; power braking
0	**1**	0	**1**	
1	**1**	X	X	short circuit (shoot-through)
X	X	**1**	**1**	

H-Bridge Switch Combinations

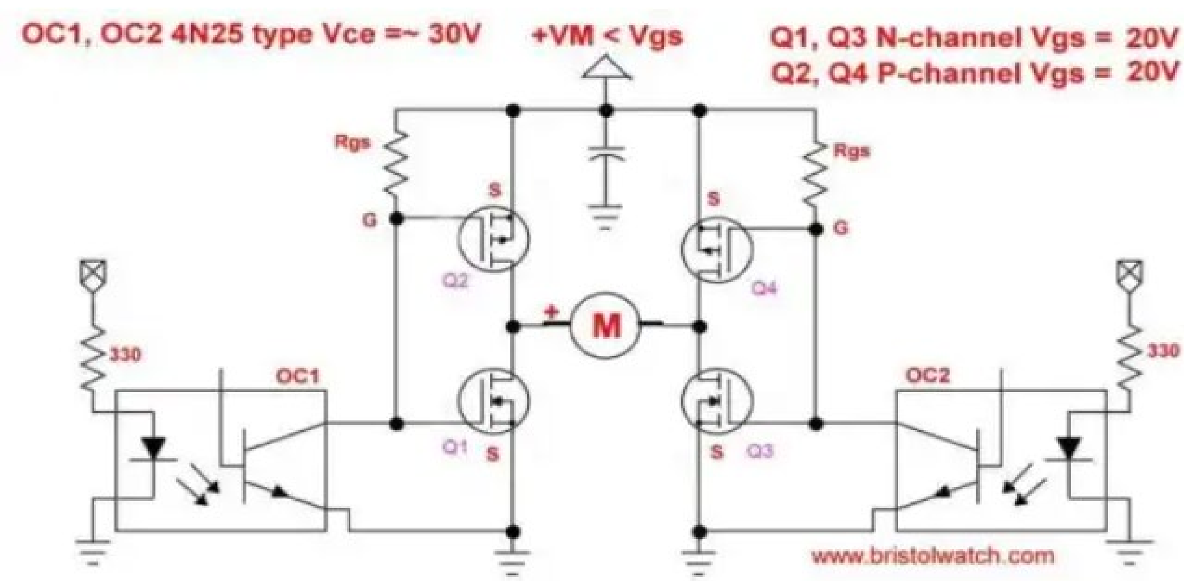
adapted from H-bridge - Wikipedia

Obviously the short circuit combinations (shoot-through) must be avoided to avoid harm to humans and damage to equipment. MOSFETs or Transistors can be used to automate the H-Bridge so there is no need for two switches' activation. This automation can also protect against shoot-through.



H-Bridge for Bi-Directional Motor Control

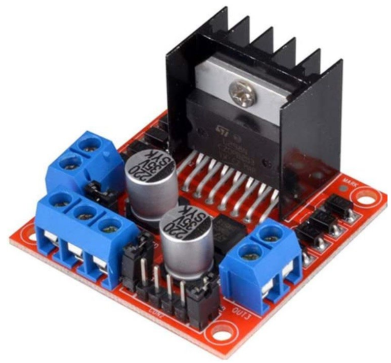
deepbluembedded.com/dc-motor-speed-control-l293d-motor-driver



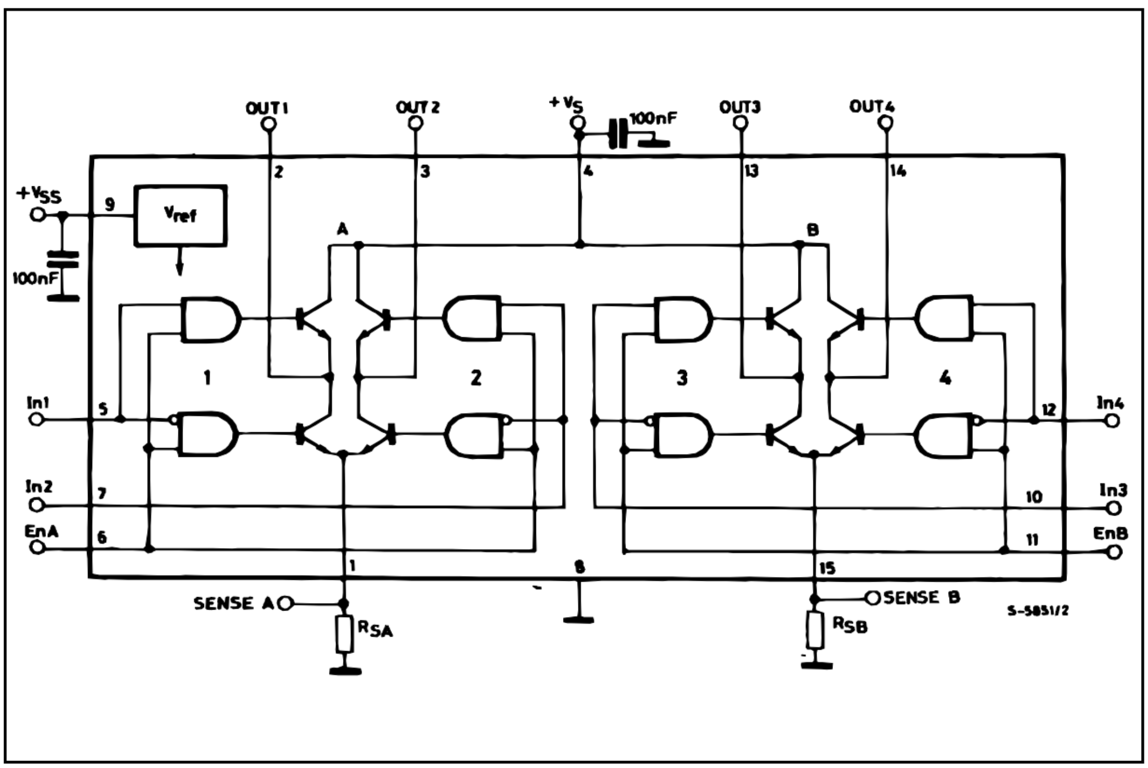
MOSFET H-Bridge

H-Bridge switching should be constructed of MOSFETs, not relays, for the four switches. MOSFETs operate in the MHz range, while relays have much slower activation times of 100 – 200 msec (5 – 10 HZ!). This can help avoid shoot-through. H-Bridges can also convert DC to AC current.

H-Bridge Integrated Circuit



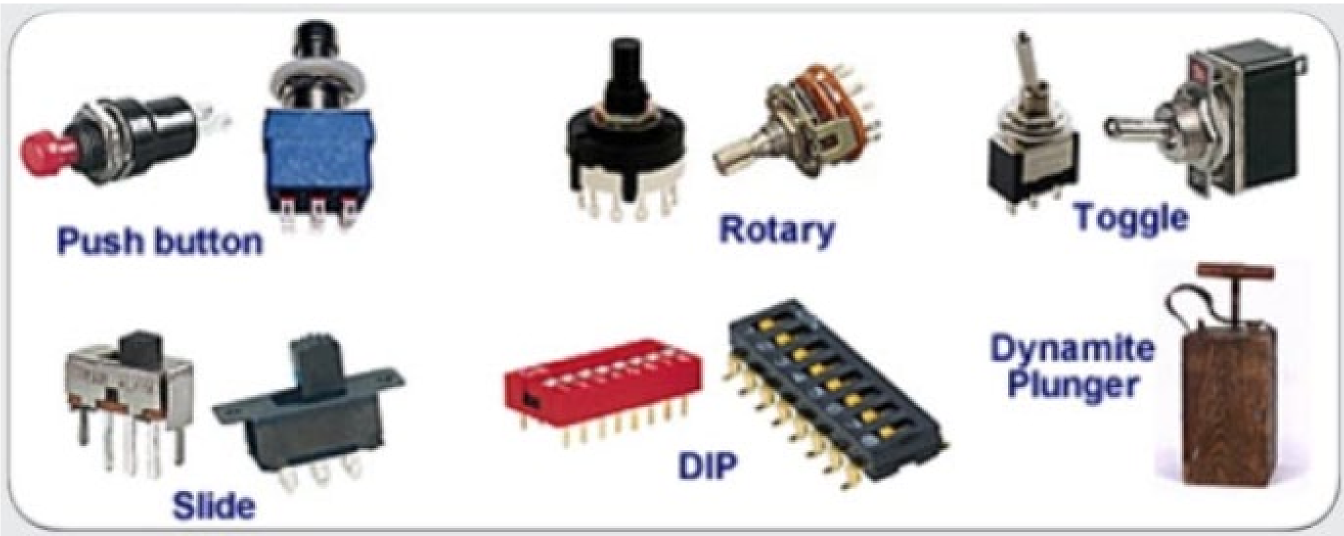
- L298N
- Max: 2A (peak 3A)
- Power: 25W
- 2x H-Bridges
- IN1/IN2+ENA
- IN3/IN4+ENB
- OUT1/2
- OUT3/4



L298N H-Bridge Integrated Circuit Diagram

4.7 Switches and Relays

Switches and relays are electrical devices that control the flow of electrical current in electrical circuits. Switches open or close the circuit. Relays use electromagnets to control the flow of electricity. Relays isolate the low voltage from the high voltage circuit. Relays use a relatively small current to turn on/off a much higher current.



Switch Types

Switches can be mechanical, electronic, or a combination of these two.

Mechanical Switches use physical contact and manual actuation. They using moving parts such as levers, buttons, and/or sliders.

- Toggle Switches
- Push-Button Switches
- Rocker Switches
- Rotary Switches
- Slide Switches

Electronic Switches use semiconductor devices such as transistors, diodes, and MOSFETs, to manage current flow electronically. In these cases, there are no moving parts so the switching can be at a high rate.

Poles Number of electrically-isolated switches in one mechanical package

Throws Number of terminals the common pin can be connected to electrically

State Behaviour of a switch or relay in its idle and active operation modes

- NO – normally-open
- NC – normally-closed

Now we add another column to the table from Section 4.5 Logic Gates:

1	T	Hi	On	5 V	**closed**
0	F	Lo	Off	0 V	**open**

Switches are also categorized by their actuation behaviour.

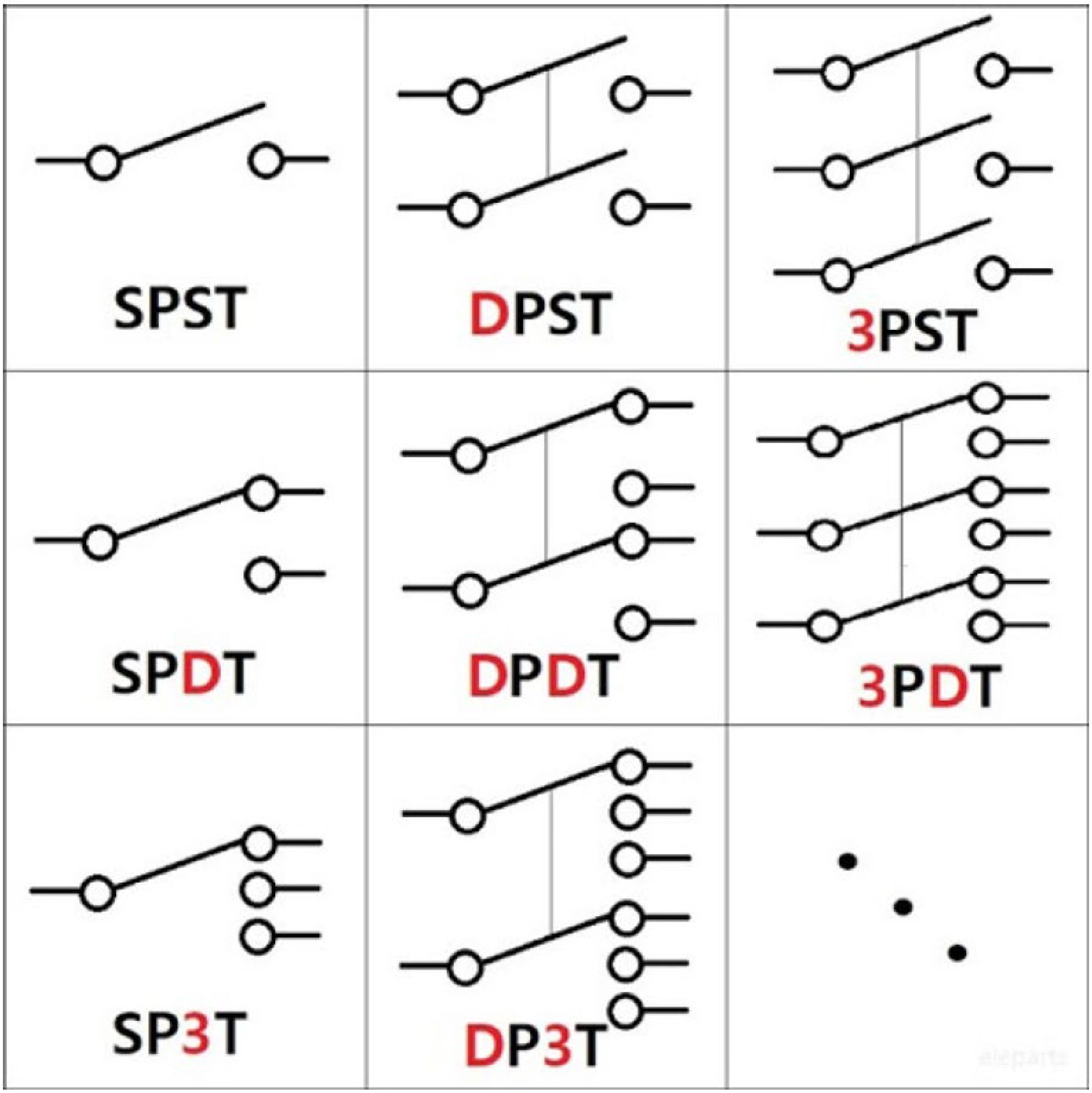
Normally-Open (NO) Switches are open by default and close upon activation. **Normally-Closed (NC) Switches** are closed by default and open upon activation.

Maintained Switches, also called **Latching Switches**, keep their state upon actuation. Toggle and rocker switches are examples of maintained switches. **Momentary Switches** revert to their default state when they are released. Example momentary switches include push-button switches, doorbells, and keyboard keys. A momentary switch from robotics is the **Deadman Switch**, which must be held actively to allow power to operate the robotic system. When the switch is released, all power is shut off, for safety.

Changeover Switches allow both NO and NC connections. Upon activation, these switches change from NO to NC or vice-versa.

Multi-Throw Switches (see next page) have the following two possibilities. **Make-Before-Break (MBB) Switches** ensure a new contact is established before the old contact is disconnected. **Break-Before-Make (BBM) Switches** breaks the old contact before a new contact is enabled.

The figure below shows the classification of switches according to number of Poles (**P**) and number of Throws (**T**). The classification of relays is the same.

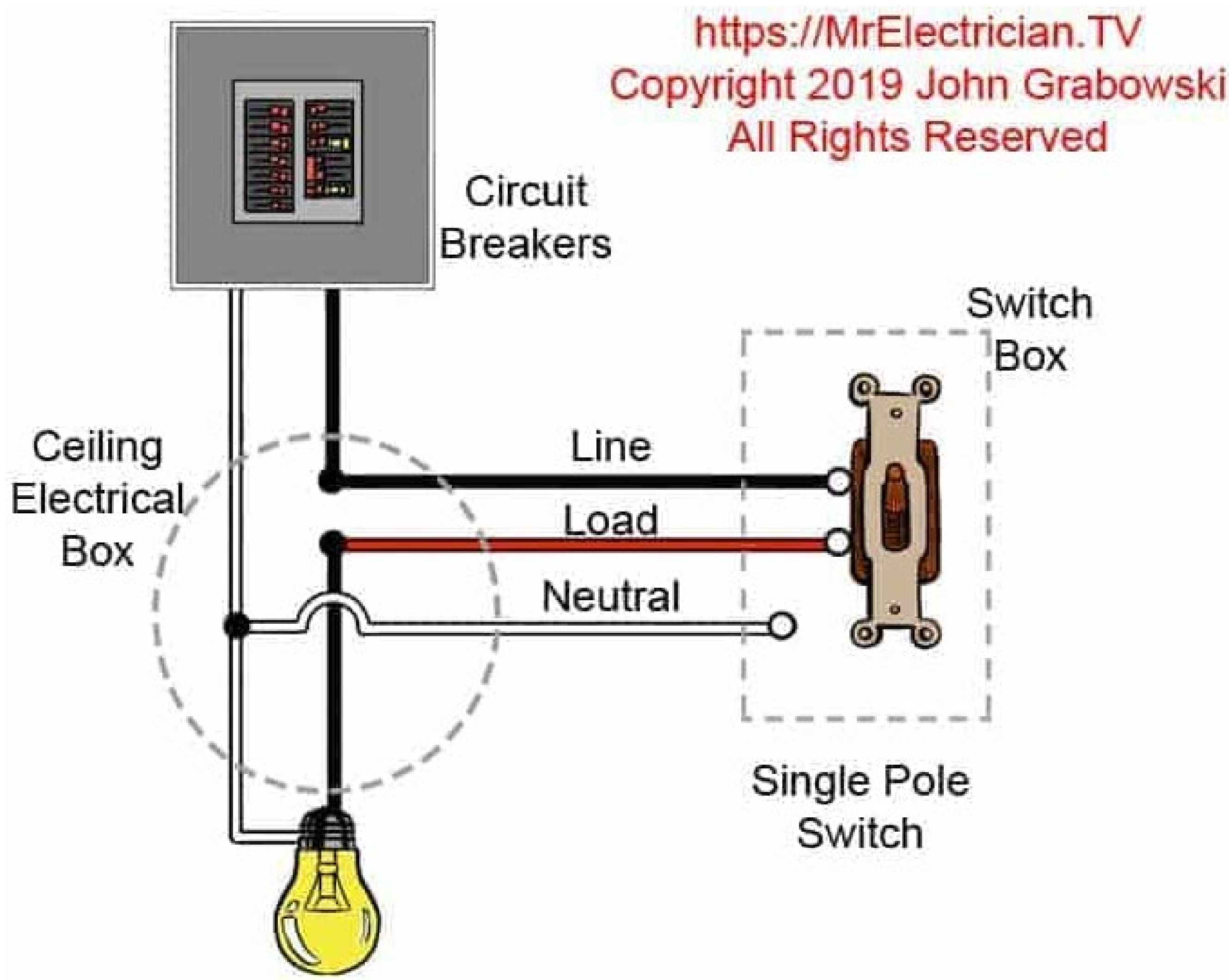


Classification of Switches / Relays

The top row types in this figure above (**SPST**, **DPST**, and **3PST**) are all normally-open (NO).

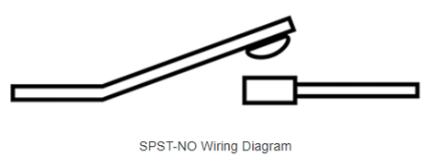
4.7.1 Switches

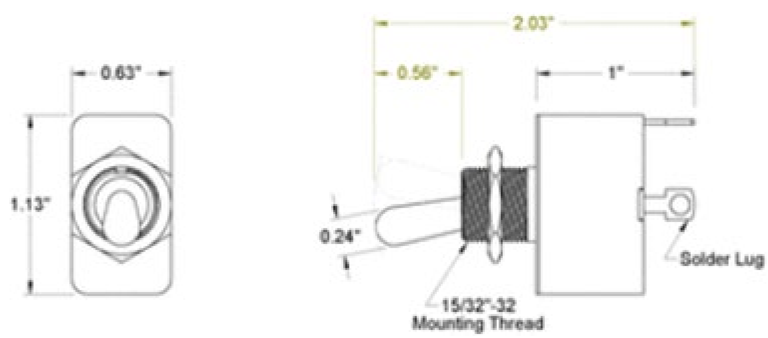
A switch is an electromechanical component that connects or disconnects the current path in an electrical circuit. A switch can enable or interrupt the flow of current, or it can divert current to another circuit. The most familiar example of a switch is the **SPST** (**NO**) switch on the wall for turning on household lights (see the figure below).



Household Light Switch Wiring Diagram

Switch Type	Manual
Actuator Style	Toggle
Toggle Style	Rounded
Illumination	Not Illuminated
Number of Positions	2
Number of Circuits Controlled	1
Switch Starting Position	1 Off (Normally Open)
Switch Action	Stays Switched (Maintained)
Number of Terminals	2
Industry Designation	SPST-NO
Position Designation	On-Off
Switching Current @ Voltage	6 A @ 125 V AC/28 V DC
Maximum Voltage	28V DC 250V AC
Mounting Location	Panel
For Panel Cutout Diameter	1/2"
For Maximum Panel Thickness	0.13"
Depth Behind Panel	1"





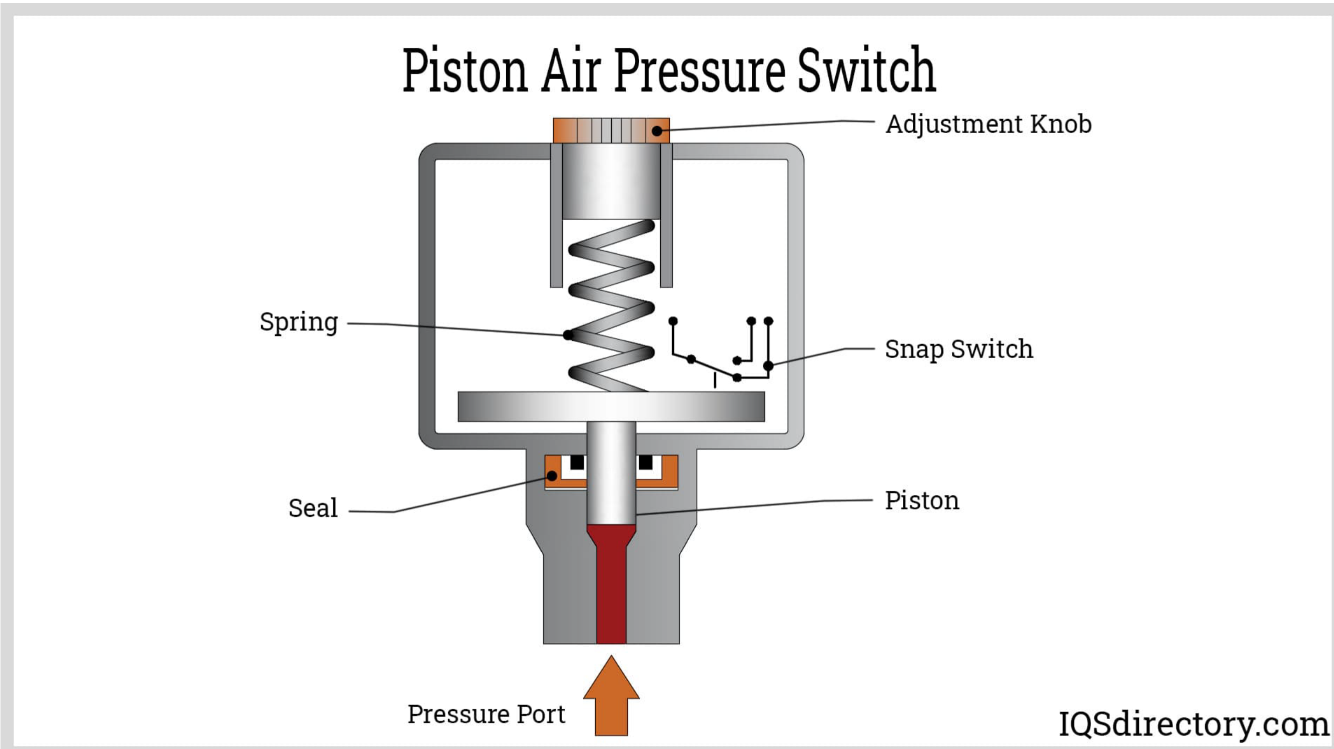
Toggle Switch Product Example

In the above product table, why can the AC voltage be so much higher than the DC voltage (maximum 250 V AC vs. 28 V DC)? The 60 Hz AC supply (from a wall plug) current crosses zero 120 times per second. It is much easier to throw a switch then, compared to the DC voltage/current where the power never goes to zero like that.

Pressure Switches

A pressure switch is a type of switch that enables electrical contact when a given fluid pressure is reached. The fluid pressure could be either pneumatic (air) or hydraulic (liquid). Pressure switches can be either normally-open (activated on pressure rise) or normally-closed (activated on pressure drop).

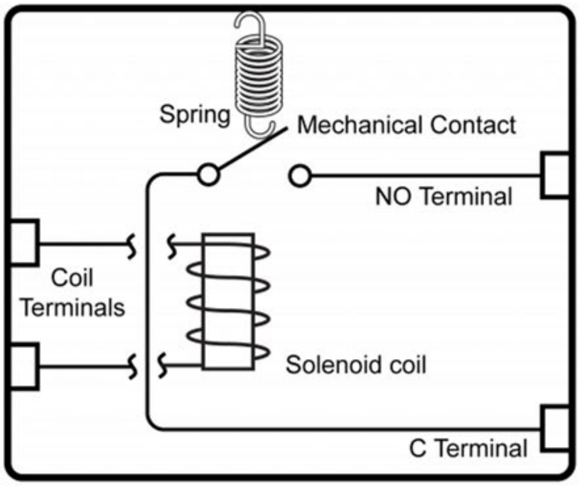
The figure below shows the diagram for a pneumatic piston pressure switch.



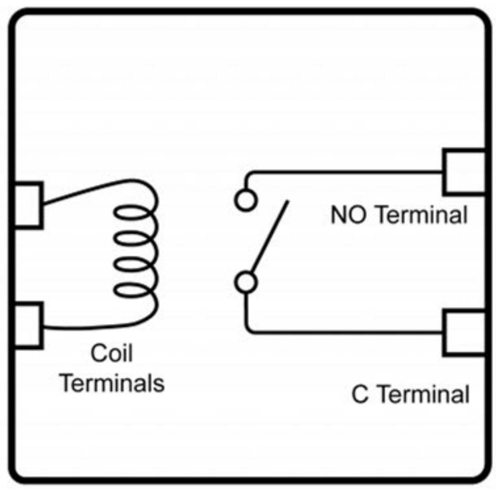
Another kind of pressure switch is a pressure-sensitive mat, that can read weight (force) and automatically open a door when a person is on the mat. Such a pressure-sensitive floor is also used in building security systems.

4.7.2 Relays

- A relay is an electrically-actuated mechanical switch.
- A coil moves the switch.
- A relay uses power to hold in the ON position.
- There are many different designs (single and multiple contacts). The same arrangements exist as for switches given above (SPST, SPDT, DPST, DPDT).
- There is a 100 – 200 msec actuation time. This is very slow compared to MOSFETs.
- Repeated usage fatigue will cause failure at some point (the number of switchings at max load).
- Relays use relatively low current to turn on a much higher current to the load.



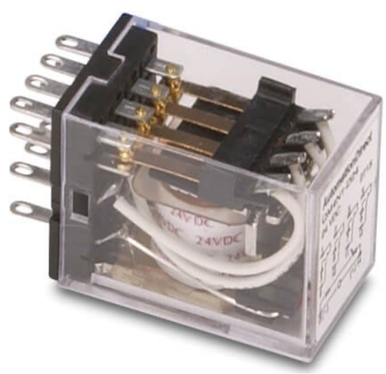
Physical



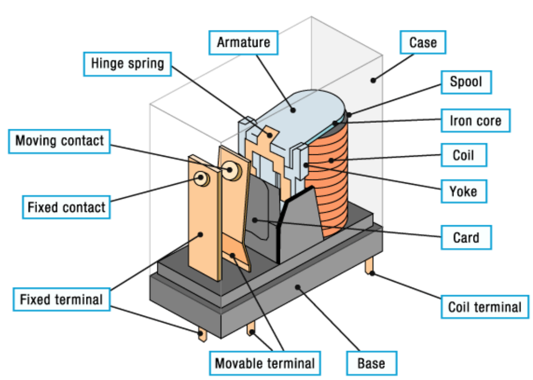
Electrical

Relay Diagrams (SPST NO)

In the left diagram below (the physical relay), when the coil is de-energized, the spring opens the switch.

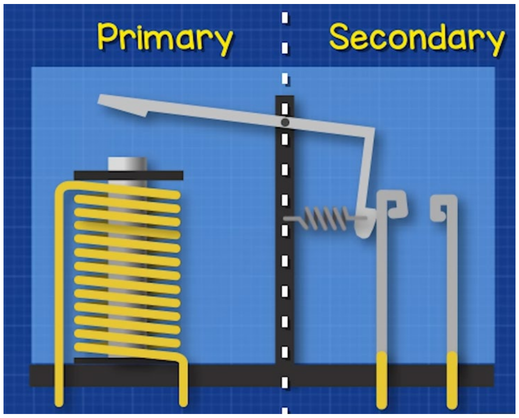


Relay Product



Relay Details

Relays have two circuits, the Primary (low-current) Circuit providing the signal to operate the relay, via a manual switch, thermostat, or other appropriate sensor. The primary circuit is usually powered by a low-voltage DC supply. The Secondary (high-current) Circuit is connected to the load to be activated. This circuit can be DC or AC to drive various loads (light, fan, pump, compressor, etc).

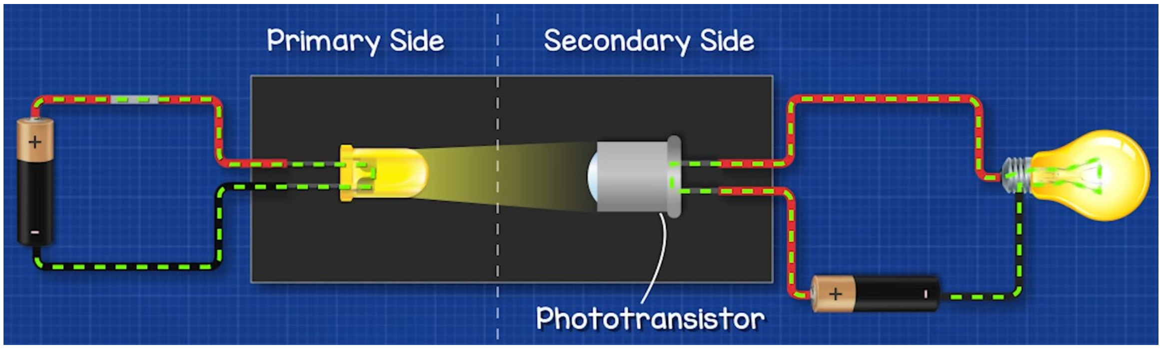


Primary and Secondary Relay Circuits

www.youtube.com/watch?v=n594CkrP6xE

` Like switches discussed above, relays can be either of the normally-open (NO) or normally-closed type.

Conventionally relays use an electromagnet powered by an electric field to operate the mechanical switch. Conversely, solid-state relays use purely electronic means to active the switch. Solid-state relays use the electrical and optical properties of semiconductors to perform the switching, as shown below.



Solid-State relay using an LED

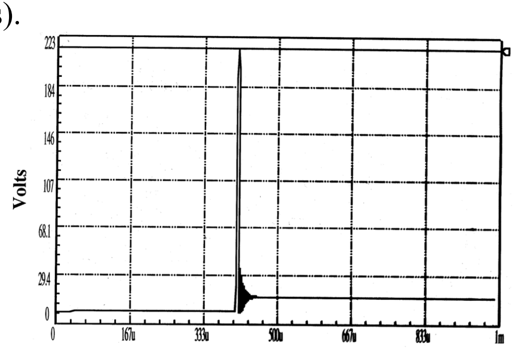
www.youtube.com/watch?v=n594CkrP6xE

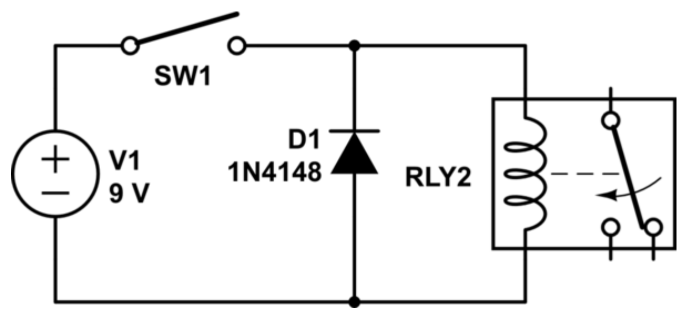
A Latching Relay maintains either switch contact position with no power applied to the Primary Circuit coil. An example is in elevators – when the user presses the button, the button should light up until the elevator car arrives (even though the user finger withdraws). Latching Relays have positional memory: once activated, they remain in the last position without the need to current input on the Primary side.

Relay Type	Standard
Number of Circuits Controlled	1
Switch Starting Position	1 Off (Normally Open) or 1 On (Normally Closed)
Switch Action	Springs Back (Momentary)
Number of Terminals	5
Industry Designation	SPDT
Input Voltage	60V DC
Control Current	3 mA
Switching Current @ Voltage	6 A @ 240 V AC
Maximum Switching Voltage	250V AC
Operation Type	Mechanical
Mechanical Life Cycles	10,000,000

Relay Product Example

Relays are activated more frequently than switches, hence the fatigue rating (Mechanical Life Cycles).





Inductive Kickback Voltage Spike **Flyback Diode**

The left figure above shows the inductive kickback voltage spike resulting when the applied voltage across an inductor is switched off abruptly. This spike in voltage (Faraday's Law) is obviously bad for circuits. A flyback diode prevents current flowing through its branch, so the relay is activated by the current.

Voltage Regulator

A voltage regulator is an electronic component for the purpose of automatically maintaining a constant AC or DC voltage. A simple voltage/current regulator design is constructed of a resistor in series with a diode. An electromechanical voltage regulator uses relays for switching.

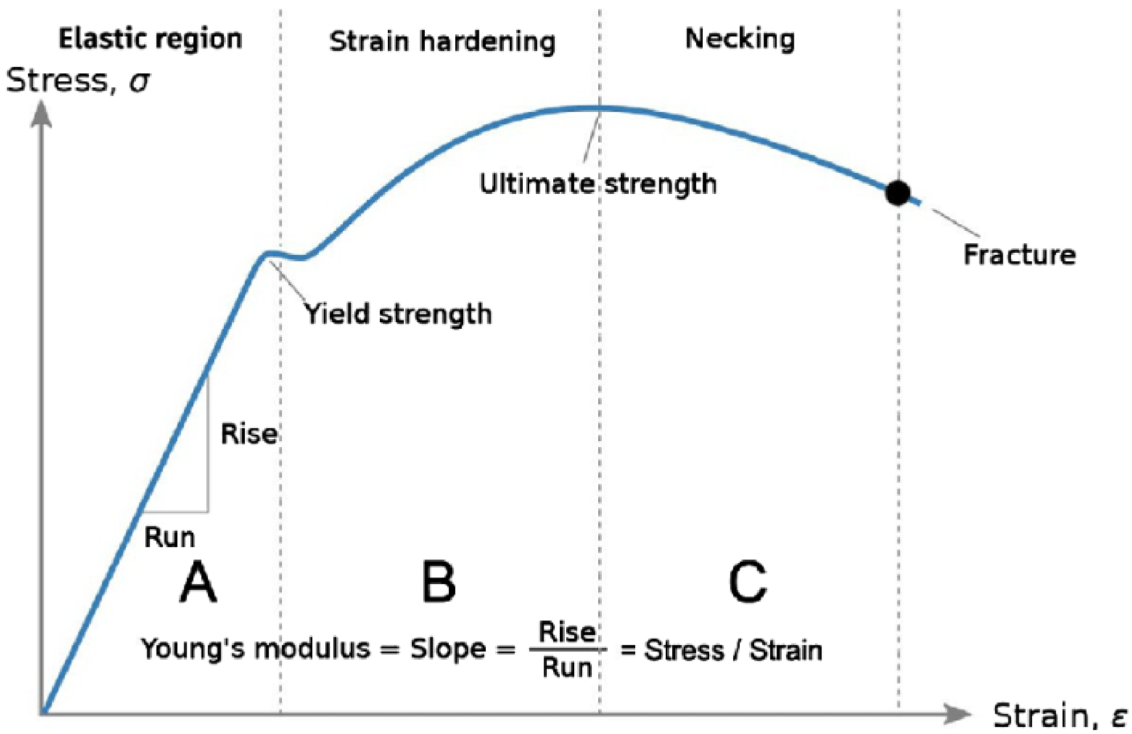
4.8 Strain Gauges and Load Cells

Strain gauges are very important as structures and machines are developed to be ever lighter and more efficient, while still maintaining safety, durability, and functionality. To achieve this balance, engineers use strain gauges to measure the deflection and stress limits of materials. A strain gauge is a device for measuring strain (change of length divided by original length) of an object. Strain gauges can thus monitor the amount of surface stress in the operation of a device.

Here is a quick overview of strain and stress in engineering materials. In the simplest manner, strain ε is defined to be the change in length under load divided by the original unloaded length (see the left equation below). Hence strain is unitless (m/m). Again in the simplest manner, tensile or compressive stress σ is defined to be an applied force divided by the area bearing the force. The units of stress are N/m^2 (Pascals, Pa, often MPa for engineering materials; see the right equation below).

$$\varepsilon = \frac{\Delta L}{L} \qquad\qquad \sigma = \frac{F}{A}$$

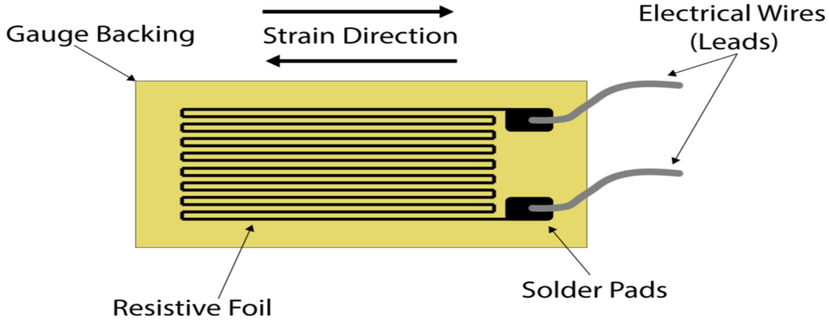
Stress-strain plots are used to express the expected behaviour of any engineering material. Here is a typical stress-strain plot for linearly elastic engineering materials:

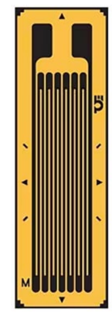


Typical Stress-Strain Plot

A typical stress-strain curve [5]. | Download Scientific Diagram

The slope of the linear portion of this curve is called Young's Modulus E (units MPa), so we have $\sigma = E\,\varepsilon$. E varies widely for different engineering materials. You can Google E values as easily as I could put a table in at this point in our narrative.





Strain Gauge Diagram **Example Strain Gauge**

Strain gauges are generally constructed with three layers:

- A protective laminate top layer;
- A thin metallic sensing element; and
- A plastic film base.

When a strain gauge is bonded to a surface which is then placed under stress it will flex in unison with that surface. This causes a change in electrical resistance linearly proportional to the strain experienced on the surface of the part under sensing. A Wheatstone Bridge formula is then applied to convert this change in resistance to an accurate strain reading.

A strain gauge works by converting mechanical strain to electrical resistance. This resistance change is measured using a Wheatstone Bridge, which always has a nominal resistance (e.g. 150 or 300 Ω). A strain gauge is a very sensitive analog device that can detect minor changes in resistance and thus strain.

Strain gauges can be used to measure any type of strain:
- Direct load
- Torque
- Use multiple gauges for multiple directions of strain

Strain gauges come in a variety of configurations. The correct strain gauge for an application is dependent on the direction of the primary strain, what type of strain is to be measured, and the target measuring area.

The Gauge Factor (GF) of a strain gauge is defined as the ratio of the relative change in electrical resistance R to the mechanical strain ε. Thus, we see gauge factor is unitless:

$$\mathrm{GF} = \frac{\Delta R / R}{\varepsilon}$$

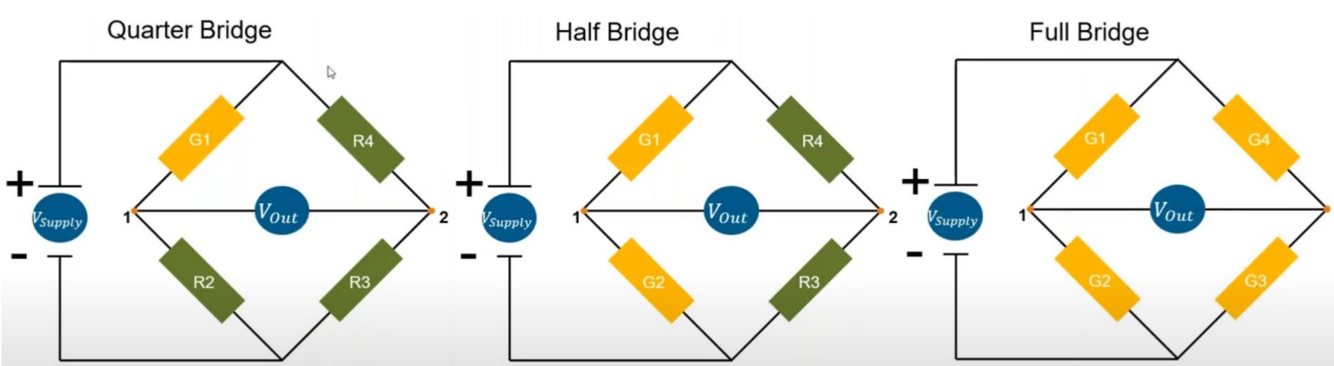
The Wheatstone Bridge voltage / resistance equation is:

$$V_O = \left(\frac{R_3}{R_3 + R_4} - \frac{R_2}{R_1 + R_2} \right) V_S$$

where V_O is the output voltage and V_S is the supply voltage. V_O is generally very small, in μV. Therefore, strain gauges may be susceptible to electrical noise, which must be managed.

$$\frac{V_O}{V_S} = \frac{\mathrm{GF} N \varepsilon}{4}$$

where N is the number of strain gauges replacing a resistor in your Wheatstone Bridge arrangement. Larger N leads to higher output voltages. The figure below shows a Quarter Bridge ($N = 1$), a Half Bridge ($N = 2$), and a Full Bridge ($N = 4$),



Introduction to Strain Gauges

Strain Gauge Equation from a Wheatstone Bridge

For a load cell, assume resistors R_1 and R_4 & R_2 and R_3 are equal and opposite and all base R_i values are identical.

$$R_1 = R_2 = R_3 = R_4 \qquad \Delta R_1 = \Delta R_4 \qquad \Delta R_2 = \Delta R_3$$

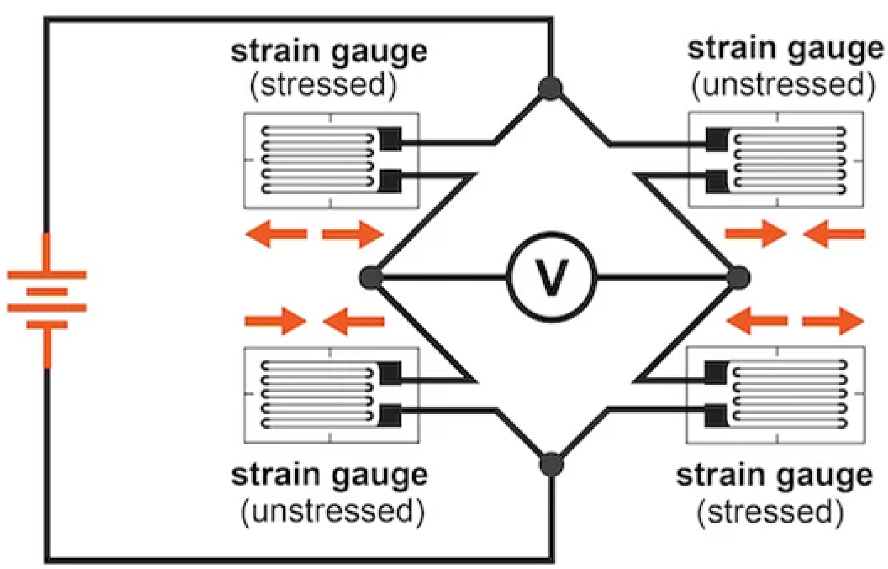
$$V_O = \frac{V_S}{(R_1 + R_2)(R_3 + R_4)}\left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} - \frac{\Delta R_3}{R_3} + \frac{\Delta R_4}{R_4}\right)$$

$$\text{GF} = k \qquad \frac{\Delta R_i}{R_i} = k\varepsilon$$

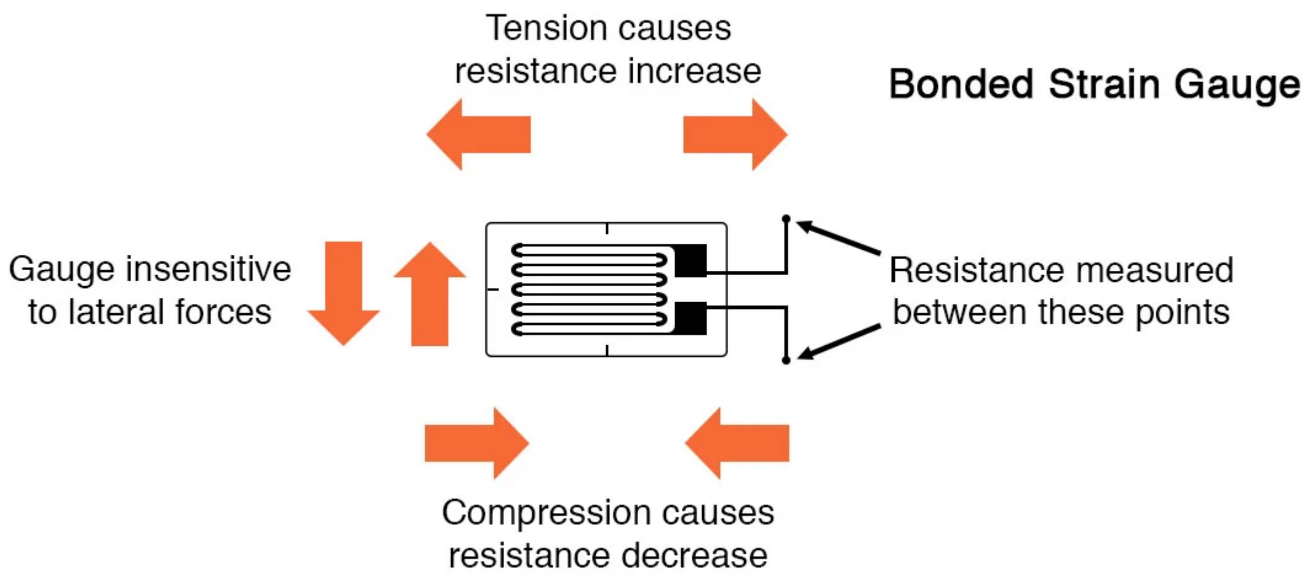
$$V_O = \frac{kV_S}{2AE}(1+\upsilon)$$

where:

σ	stress	Pa (N/m^2)
ε	strain	unitless
A	cross-sectional area	m^2
E	Young's modulus (σ / ε)	Pa (N/m^2)
υ	Poisson's ratio for the material	unitless



Full-Wheatstone-Bridge Strain Gauge Circuit



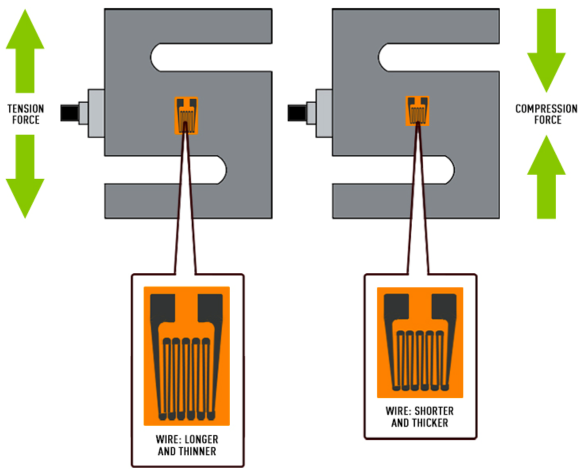
Bonded Strain Gauge Characteristics

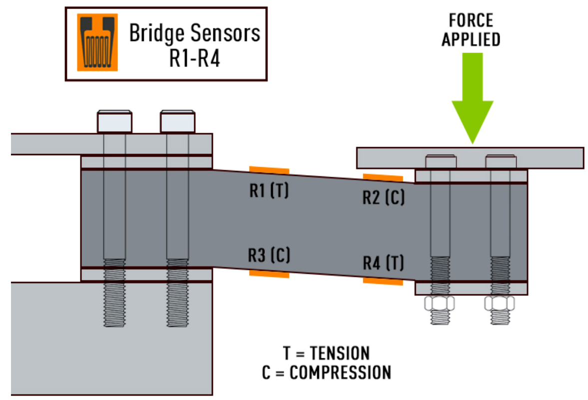
<u>KFH-10-120-C1-11L1M2R</u>

- From Omega.com
- Nominal R = 120 Ω
- Size 6 x 10 x 3 mm
- Maximum displacement of ΔL = 50,000 μm
- Gauge Factor GF = 2

Load Cells

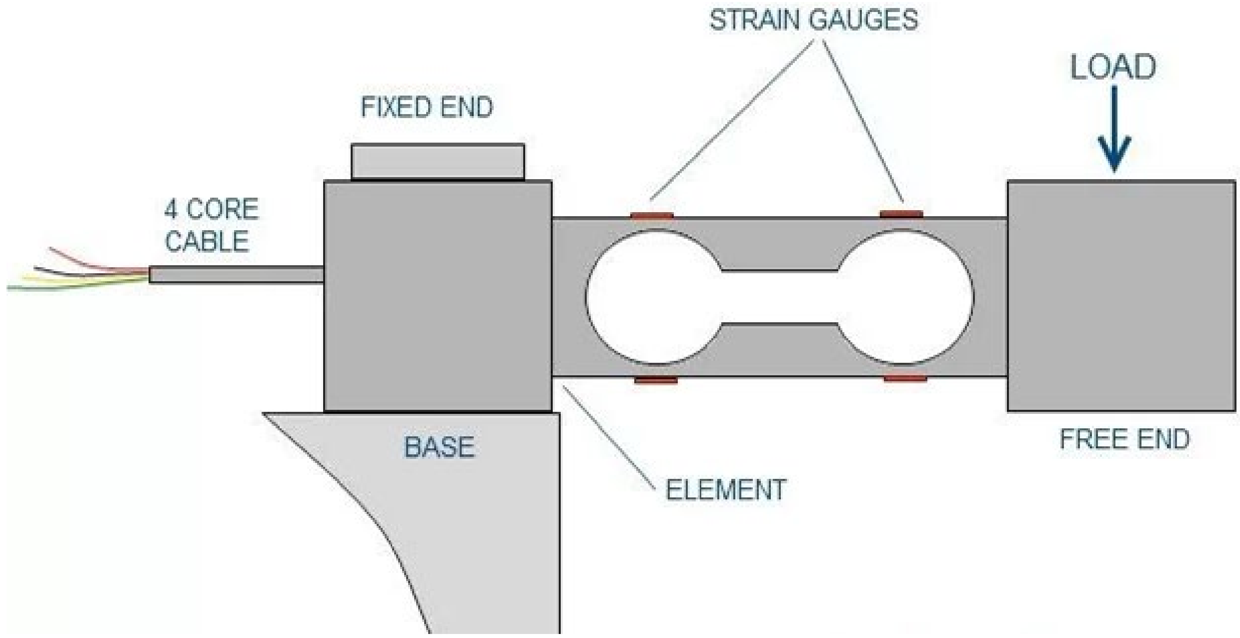
Load Cells are electromechanical mechatronic sensors that measure force. These are integrated products using strain gauges. Load cells may instead be based on pneumatic or hydraulic systems. Strain-gauge-based load cells has a metal body to which strain gauges in a Wheatstone bridge arrangement are affixed to the surface. The metal body (aluminum or stainless steel) is sturdy but also elastic, i.e. it is called a spring element to measure the strain and the output is force in units of Newtons.





- Strain Gauge + Package
- Easier to use than strain gauges directly
- Requires inline measurement (just like current)
- 4 wires: Vs(+/–) and strain (+/–)

Quarter-Bridge Wheatstone Bridges come in two-wire and three-wire configurations. Half-Bridge Wheatstone Bridges can be either temperature-compensated (and electromagnetic noise-compensated), normal, or a Poisson configuration. Using a Full-Bridge Wheatstone Bridge, different loading types may be canceled out (e.g. a bending-measurement-only full bridge).



Binocular Beam Load Cell

Beam-Type Load Cell **Cell-Type Load Cell** **S-Type Load Cell**

- 6 kg max
- ±0.02% Linearity
- 2 mV/V Output

4.9 Electronic Filters

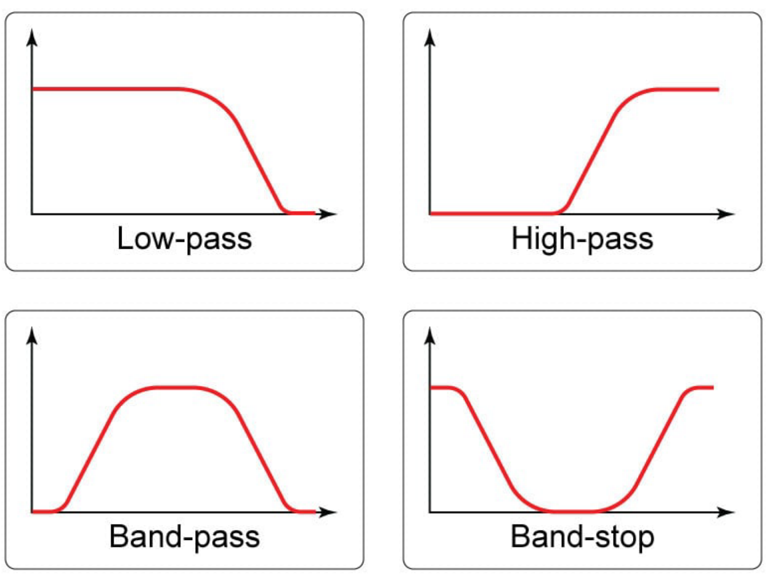
Electronic filters are a basic building block in day-to-day use products, including telecommunications systems, home audio systems, and telephones. An electronic filter is a linear circuit that removes unwanted noise, distortion, interference, and other signals of certain frequencies. Filter circuits are comprised of resistors *R*, capacitors *C*, and inductors *L*, singularly or in combination.

An electronic filter is a circuit that can be designed to:

- Modify, reshape, or reject undesired frequencies of an electrical signal.
- Alter the amplitude and/or phase characteristics of an electrical signal with respect to frequency.
- Select certain bands of frequencies to pass while rejecting other bands of frequencies.
- Separate some frequencies from others in mixed-frequency electrical signals.

Electronic filters are classed into passive and active filters. Passive filters are composed of the passive elements resistors, capacitors, and inductors. Active filters employ transistors and op-amps in addition to these passive components. Frequency-based electronic filters are also categorized according to functionality:

- Low-pass filters
- High-pass filters
- Band-pass filters
- Band-stop filters



Electronic Filters Amplitude vs. Frequency Characteristics

low pass filter graph - Search Images

Recall the impedance of a circuit changes with frequency. This provides alternative routes for electricity through a circuit. Electronic filter performance is evaluated by Bode Plots of gain attenuation (dB) vs. frequency (Hz).

Introduction to Bode Plots

Bode Plots show how the amplitude and phase angle of a system change with changing driving frequency.

Consider a simple mechanical vibrational system (forced, frictionless, mass-spring device) exhibiting harmonic motion.

The output displacement is represented by the particular solution:

$$x(t) = A\sin(\omega t + \phi)$$

where ω is the driving frequency of the input force. The Bode Plot:

shows that the output amplitude A cannot be maintained as the driving frequency increases. Also, the phase angle changes (initially in phase with $\phi = 0$ for low frequencies), in this example going completely out of phase ($\phi \rightarrow 180^{\circ}$) as the driving frequency increases.

The above discussion applies to Vibrations Engineers. Controls Engineers use frequency methods and the Bode Plot to model and understand dynamic systems, and to design controllers for them. Electrical Engineers use frequency methods and the Bode Plot for AC circuit analysis and design.

The Decibel Unit

The **decibel** is a standard relative unit comparing the power of two electrical signals with each other. The decibel is also a standard unit is acoustics for expressing relative sound energy.

The unit decibel, one-tenth of a bel, is named after Alexander Graham Bell. The decibel expresses the ratio of two electrical power values, on a base-10 logarithmic scale. Two electrical signals differing by 1 decibel have a power ratio of:

$$10^{0.1} \approx 1.259$$

When expressing a relative power ratio, the decibel is defined as 10 times the base 10 logarithm of the power ratio:

$$\text{dB} = 10\log_{10}\left(\frac{P_2}{P_1}\right)$$

where P_i represents the power of signal i in Watts.

From this equation we can conclude that a 10x change in power level corresponds to 10 dB change. Power increases from 1 to 2 yield a positive dB, while power losses (attenuation) from 1 to 2 yield a negative dB. Given the two power values, or the power ratio, it is easy to calculate the dB from the above equation. In fact, Excel uses `log` as the base-10 log. However, in MATLAB, one must explicitly use `log10` since the plain `log` is associated with the natural logarithm base *e*.

If we are given the dB value, we must invert the above equation to solve for the power ratio:

$$\frac{P_2}{P_1} = 10^{(\text{dB}/10)}$$

Electrical engineers have settled on a standard of –3 dB power attenuation to describe a known loss in power. A 3 dB loss (attenuation) corresponds to losing approximately 50% power:

$$\text{dB} = 10\log_{10}\left(\frac{50}{100}\right) = -3.01$$

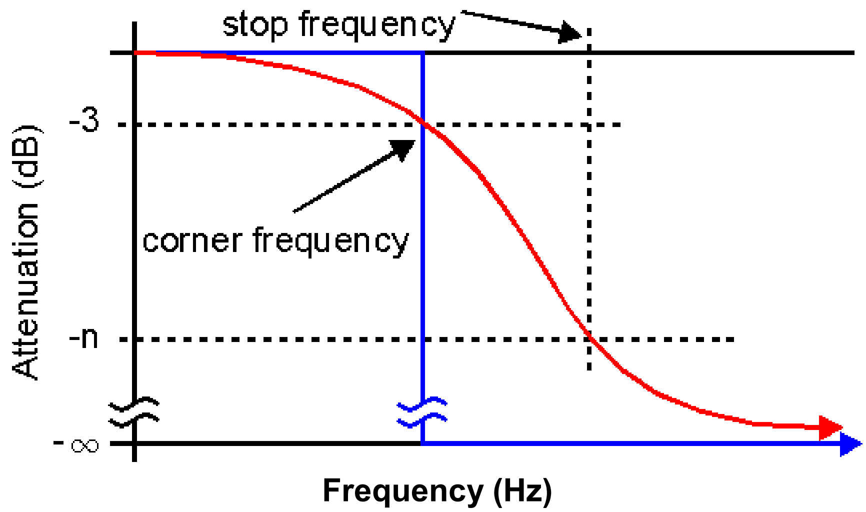
Or, turning this around, a power ratio of 50.12% corresponds more exactly to a –3 dB power attenuation. We can ignore this tiny error and conclude that a 3 dB power loss (attenuation) corresponds to a 50% loss in power from 1 to 2.

$$\frac{P_2}{P_1} = 10^{(-3/10)} = 0.5012$$

Note: another usage of decibels relate to root-power ratios, not discussed here.

Low- and High-Pass Filters

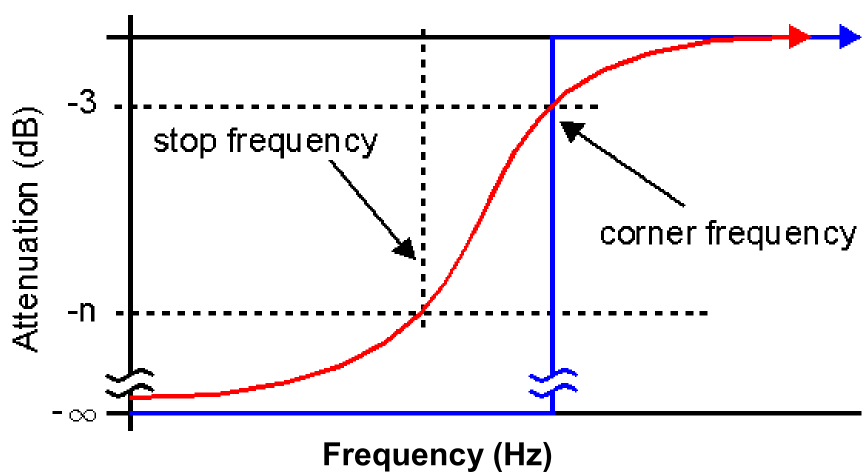
A low-pass filter allows signals of lower frequencies to pass through, while blocking signals of higher frequencies. The **corner frequency** corresponding to a loss (signal amplitude attenuation) of –3 dB is called the **cutoff frequency**. At a higher frequency, one can specify a desired attenuation –n in dB. The slope of the red curve can be made steeper by choosing a **stop frequency** closer to the **cutoff (corner) frequency**.



Low-Pass Filter Characteristics (blue ideal, red actual)

Entering LP Filter Data

The high-pass filter behaves opposite to the low-pass filter. It blocks signals of frequencies lower than the cutoff frequency, while allowing signals of higher frequencies to pass through. Again, the **corner frequency** corresponding to a loss (signal amplitude attenuation) of –3 dB is called the **cutoff frequency**. At a higher frequency, one can specify a desired attenuation –n in dB. The slope of the red curve can be made steeper by choosing a **stop frequency** closer to the **cutoff (corner) frequency**.



High-Pass Filter Characteristics (blue ideal, red actual)

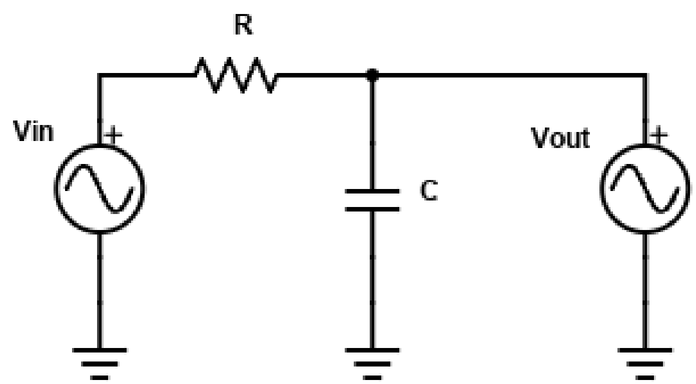
Entering HP Filter Data

(Show the **Stop Band** and **Pass Band** on the above figures.)

First-Order Filters (*RC* circuits)

Low-Pass *RC* Filter

The low-pass *RC* circuit diagram is shown in the figure below.



Low-Pass Filter *RC* Circuit

Low-pass filters have a cutoff frequency that is 3 dB (down, negative) of the filter gain. This 3 dB drop is equivalent to half (50%) of the gain. This 3 dB drop is used as a design point and is called the cutoff frequency f_C (Hz). The role of the low-pass filter is to nearly preserve the amplitudes of signals with frequency less than the cutoff frequency, while blocking the amplitudes of signals with frequency higher than the cutoff frequency.

The formulae for the cutoff frequency f_C and ω_C are presented below, in Hz and rad/sec.

$$f_C = \frac{1}{2\pi RC} \quad \text{Hz} \qquad\qquad \omega_C = \frac{1}{RC} \quad \text{rad/sec}$$

Recall from Section 2.2 that $\tau = RC$ is the time constant (sec) for an *RC* circuit. Recall from Section 2.7 that the real impedance due to resistance (Ω) and the imaginary capacitor reactance (Ω) are:

$$Z_R = R \qquad\qquad X_C = \frac{1}{j\omega C}$$

Note Section 2.7 used capacitor reactance; this is identical to capacitor impedance ($Z_C = X_C$), which is the imaginary part of the total *RC* impedance. The complex-number low-pass filter gain gn_L is:

$$gn_L = \frac{Z_C}{Z_R + Z_C}$$

For the Bode plot magnitude, this complex *gn* ratio must be expressed as a real number *GN* in dB (decibel) units:

$$GN_L = 10\log_{10}(\text{Re}(gn_L))$$

Low-Pass *RC* Filter Design

A low-pass *RC* filter circuit can be designed by specifying the cutoff frequency f_C (the frequency at which the system has a 3 dB, i.e. 50%, amplitude loss). Choose the value for *R* or *C*; then the applicable equation is:

$$f_C = \frac{1}{2\pi RC} \text{ Hz}$$

Solve for the missing element value *C* or *R*, respectively.

$$C = \frac{1}{2\pi R f_C} \text{ F} \qquad \text{or} \qquad R = \frac{1}{2\pi C f_C} \ \Omega$$

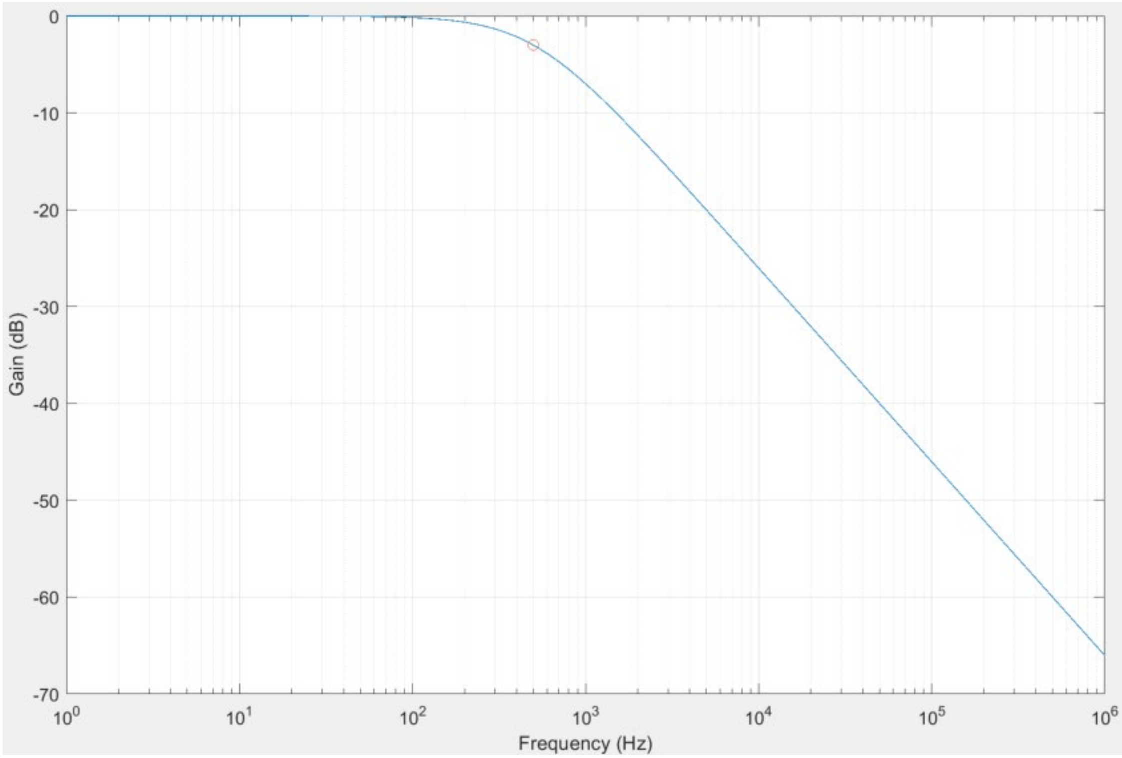
Low-Pass *RC* Filter Design Example

Design a low-pass filter that has a 3 dB loss at a cutoff frequency of $f_C = 500$ Hz. Choose a constant resistance $R = 50\ \Omega$. Then the capacitance constant *C* is calculated from this *R* and from the given cutoff frequency:

$$C = \frac{1}{2\pi R f_C} = \frac{1}{2\pi(50)(500)} = 6.37 \ \mu\text{F}$$

Low-Pass Filter Design and Evaluation MATLAB Code

```
clc; clear;

fCutoff = 500;             % cutoff frequency (Hz)
R = 50;                    % constant resistance (Ohms)
C  = 1/(2*pi*R*fCutoff);   % constant capacitance (F); calculated from fCutoff and R
f = logspace(0, 6, 100);   % independent variable, cyclical frequency (Hz)
w = 2*pi*f;                % circular frequency (rad/sec)
Zr = R;                    % real impedance (Ohms)
Zc = 1./(j*w*C);           % imaginary capacitor impedance (Ohms)
gn  = Zc./(Zr+Zc);         % filter gain

figure;
semilogx(f,10*log10(real(gn))); grid;  % Bode plot (magnitude part)
xlabel('Frequency (Hz)'); ylabel('Gain (dB)');
hold on;
scatter(fCutoff,-3);
```



Low-Pass Filter Bode Plot (Gain portion)

What are the low-pass filter gains (in dB) at frequencies of 100 and 2000 Hz?

–0.1703	and	–12.3045	respectively.
(96.2%	and	5.9%	of original power, respectively)

Example concludes.

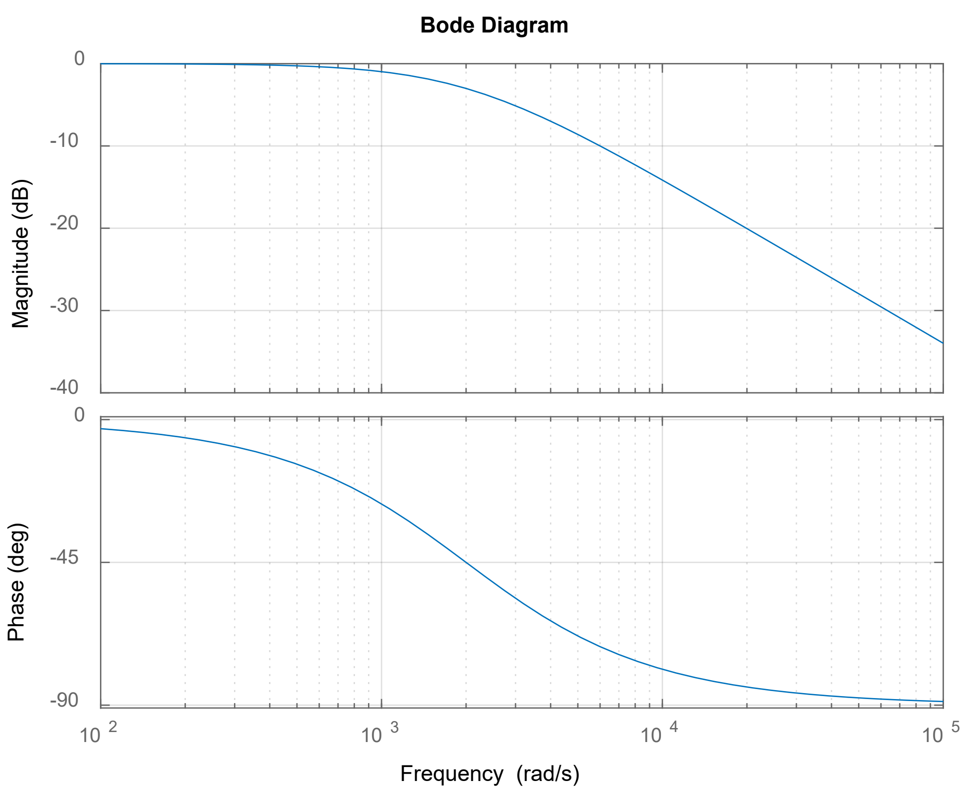
First-Order RC Low-Pass Filter

First-Order RC Low-Pass Filter transfer function $G(s)$:

$$G(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{\omega_0}{s + \omega_0} \qquad \text{where } \omega_0 = \frac{1}{RC} \text{ rad/sec}$$

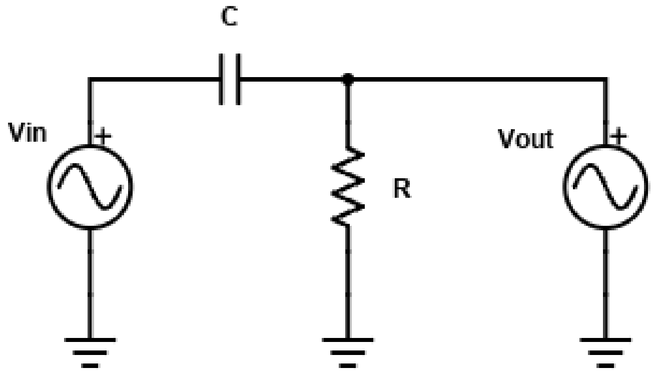
First-Order RC Low-Pass Filter MATLAB Code

```
clc; clear;

R   = 500;              % constant resistance (Ohms)
C   = 1e-6;             % constant capacitance (Farads)
w0  = 1/(R*C);          % cutoff frequency (rad/sec)

num = [w0];             % transfer function G(s)
den = [1 w0];
G   = tf(num,den);

figure;                 % Low-pass filter Bode Plots
bode(G); grid;
```



High-Pass *RC* Filter

The high-pass *RC* circuit diagram is shown in the figure below.



High-Pass Filter *CR* Circuit

High-pass filters have a cutoff frequency that is 3 dB (down, negative) of the desired high-pass filter gain. This 3 dB drop is equivalent to half of the gain (i.e. 50%, amplitude attenuation). This 3 dB drop is used as a design point, and is called the cutoff frequency f_C (Hz). The role of the high-pass filter is to block the amplitudes of signals with frequency less than the cutoff frequency, while nearly preserving the amplitudes of signals with frequency higher than the cutoff frequency.

The formulae for the cutoff frequency f_C and ω_C are presented below, in Hz and rad/sec. The next two lines of equations below are identical to the equations for the low-pass *RC* filter.

$$f_C = \frac{1}{2\pi RC} \quad \text{Hz} \qquad\qquad \omega_C = \frac{1}{RC} \quad \text{rad/sec}$$

Recall from Section 2.2 that $\tau = RC$ is the time constant (sec) for an *RC* circuit. Recall from Section 2.7 that the real impedance due to resistance (Ω) and the imaginary capacitor reactance (Ω) are:

$$Z_R = R \qquad\qquad X_C = \frac{1}{j\omega C}$$

Note Section 2.7 used capacitor reactance; this is identical to capacitor impedance ($Z_C = X_C$), which is the imaginary part of the total *RC* impedance. The complex-number high-pass filter gain gn_H is different than the low-pass filter gain gn_L:

$$gn_H = \frac{Z_R}{Z_R + Z_C}$$

For the Bode plot magnitude, this complex *gn* must be expressed as a real number *GN* in dB (decibel) units (this equation is identical to the that for the low-pass *RC* filter, using *H* in place of *L*):

$$GN_H = 10\log_{10}(\mathrm{Re}(gn_H))$$

High-Pass *CR* Filter Design

A high-pass *CR* filter circuit can be designed by specifying the cutoff frequency f_C (the frequency at which the system has a 3 dB, i.e. 50% amplitude loss (attenuation)). Choose the value for *R* or *C*; then the applicable equation is:

$$f_C = \frac{1}{2\pi RC} \text{ Hz}$$

Solve for the missing element value *C* or *R*, respectively.

$$C = \frac{1}{2\pi R f_C} \text{ F} \qquad \text{or} \qquad R = \frac{1}{2\pi C f_C} \ \Omega$$

Note: these equations are identical to the Low-Pass filter design ones.

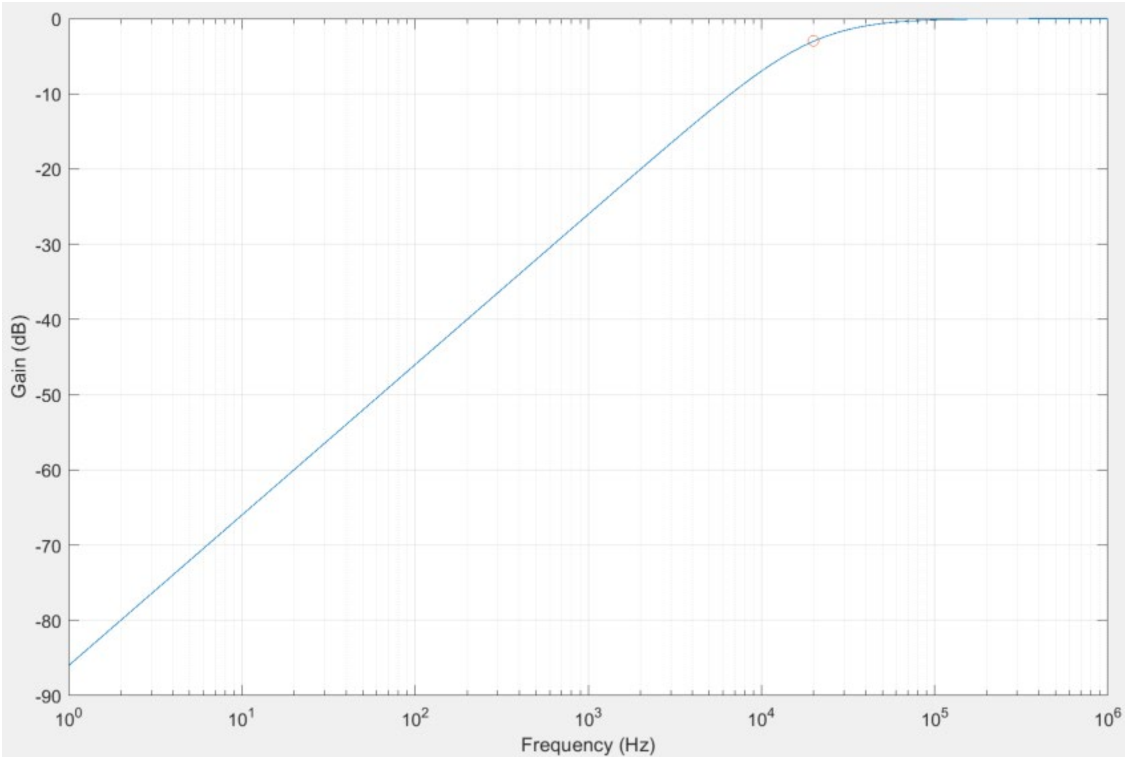
High-Pass *RC* Filter Example

Design a high-pass filter that has a 3 dB loss at a cutoff frequency of f_C = 20 kHz. Choose a constant resistance *R* = 50 Ω. Then the capacitance constant *C* is calculated from this *R* and from the given cutoff frequency:

$$C = \frac{1}{2\pi R f_C} = \frac{1}{2\pi(50)(20000)} = 0.159 \ \mu\text{F}$$

High-Pass Filter Design and Evaluation MATLAB Code

```
clc; clear;

fCutoff = 20000;           % cutoff frequency (Hz)
R = 50;                    % constant resistance (Ohms)
C  = 1/(2*pi*R*fCutoff);   % constant capacitance (F); calculated from fCutoff and R
f = logspace(0, 6, 100);   % independent variable, cyclical frequency (Hz)
w = 2*pi*f;                % circular frequency (rad/sec)
Zr = R;                    % real impedance (Ohms)
Zc = 1./(j*w*C);           % imaginary capacitor impedance (Ohms)
gn  = Zr./(Zr+Zc);         % filter gain

figure;
semilogx(f,10*log10(real(gn))); grid;  % Bode plot (magnitude part)
xlabel('Frequency (Hz)'); ylabel('Gain (dB)');
hold on;
scatter(fCutoff,-3);
```



High-Pass Filter Bode Plot (Gain portion)

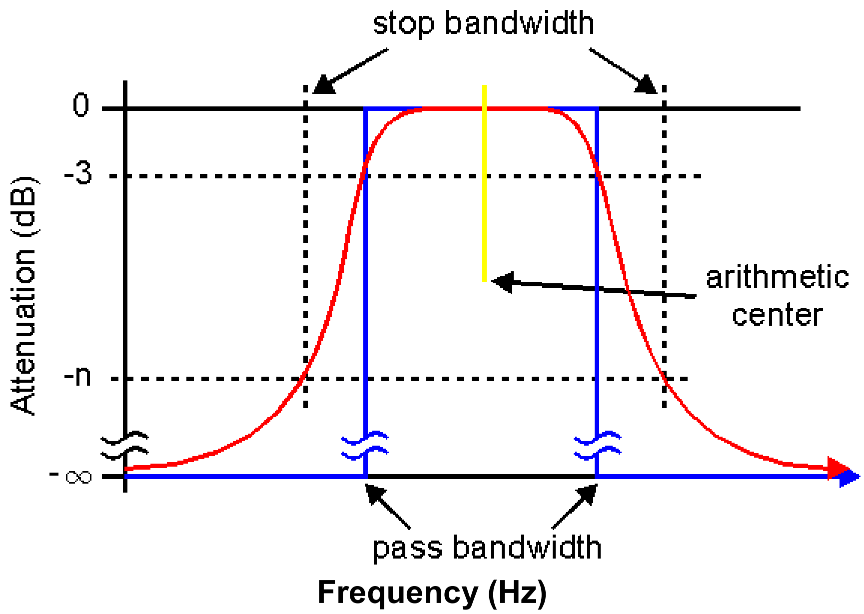
What are the high-pass filter gains at frequencies of 500 and 40,000 Hz?

–32.0439	and	–0.9691	respectively.
(0.062%	and	80.0%	of original power, respectively)

Example concludes.

Band-Stop and Band-Gap Filters

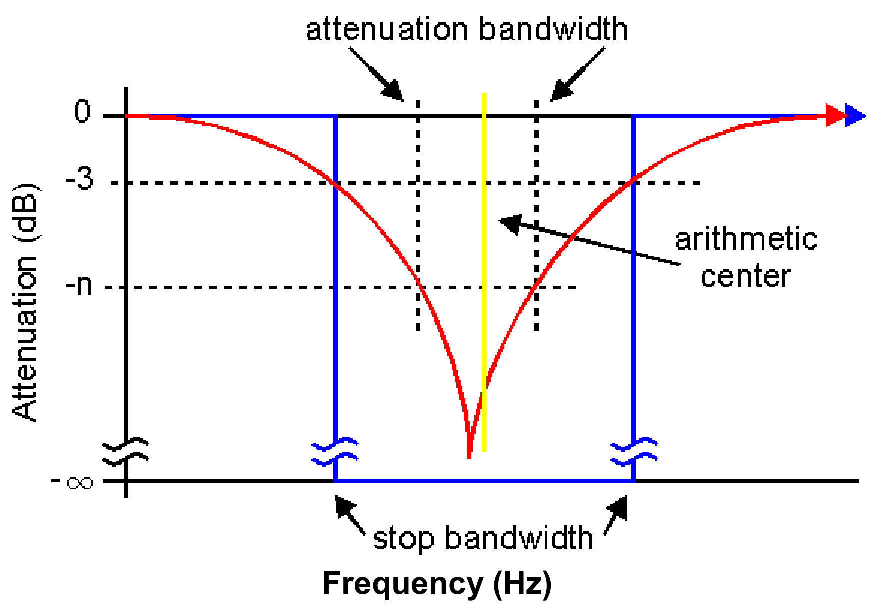
Band-Pass Filters allow signals of intermediate frequencies to pass through without much output amplitude attenuation, while blocking frequencies in lower and higher stop bands.



Band-Pass Filter Characteristics (blue ideal, red actual)

Entering BP Filter Data

Band-Stop Filters work opposite to Band-Pass Filters, i.e. they block signals of intermediate frequencies while allowing frequencies in lower and higher regions to pass through without much output amplitude attenuation.



Band-Stop Filter Characteristics (blue ideal, red actual)

Entering Band-Stop Filter Data

Butterworth Filter

The Butterworth Filter (S. Butterworth, 1930) is an analog signal processing filter designed to have a flat frequency response in the passband. It can also be implemented digitally. Butterworth stated:

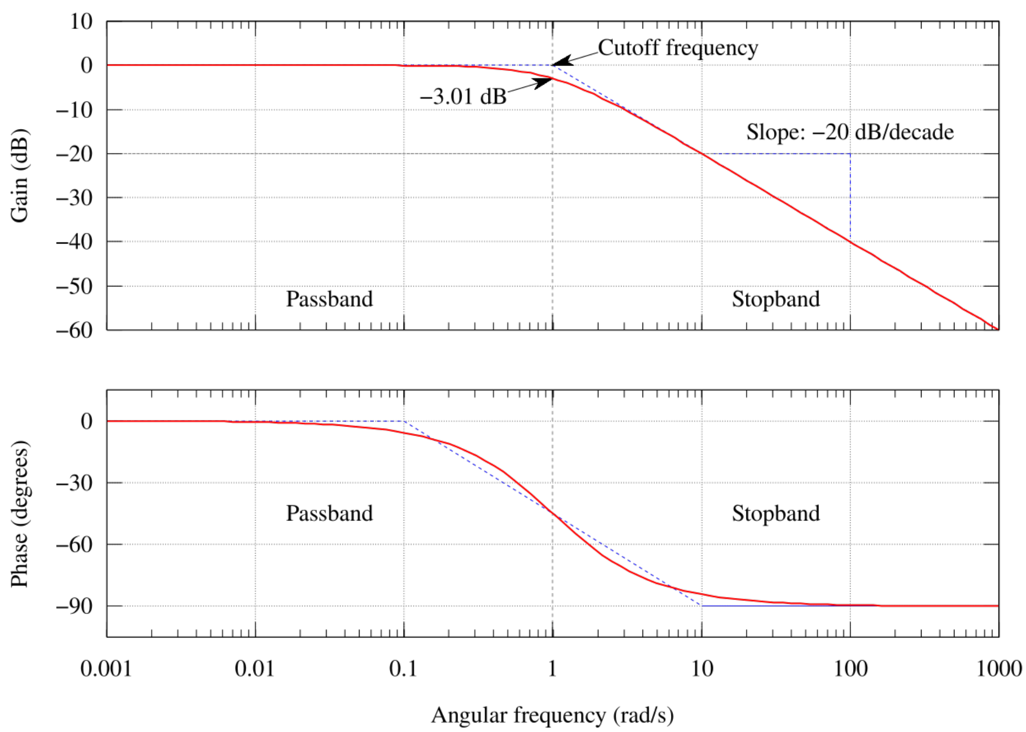
> "An ideal electrical filter should not only completely reject the unwanted frequencies but should also have uniform sensitivity for the wanted frequencies".

This ideal filter is not achievable in reality, but an approximation is possible using an analog circuit. The low-pass filter gain G (not open-loop transfer-function from Controls) as a function of angular frequency ω (rad/sec) is:

$$G(\omega) = \frac{1}{\sqrt{1+\omega^{2n}}}$$

where n is the number of poles (roots of the transfer function denominator) in the filter (equal to the number of reactive elements in a passive filter). The cutoff frequency is normalized to $\omega = 1$ rad/sec. The cutoff frequency is the half-power point, approximately –3 dB, a voltage gain of $1/\sqrt{2}$.

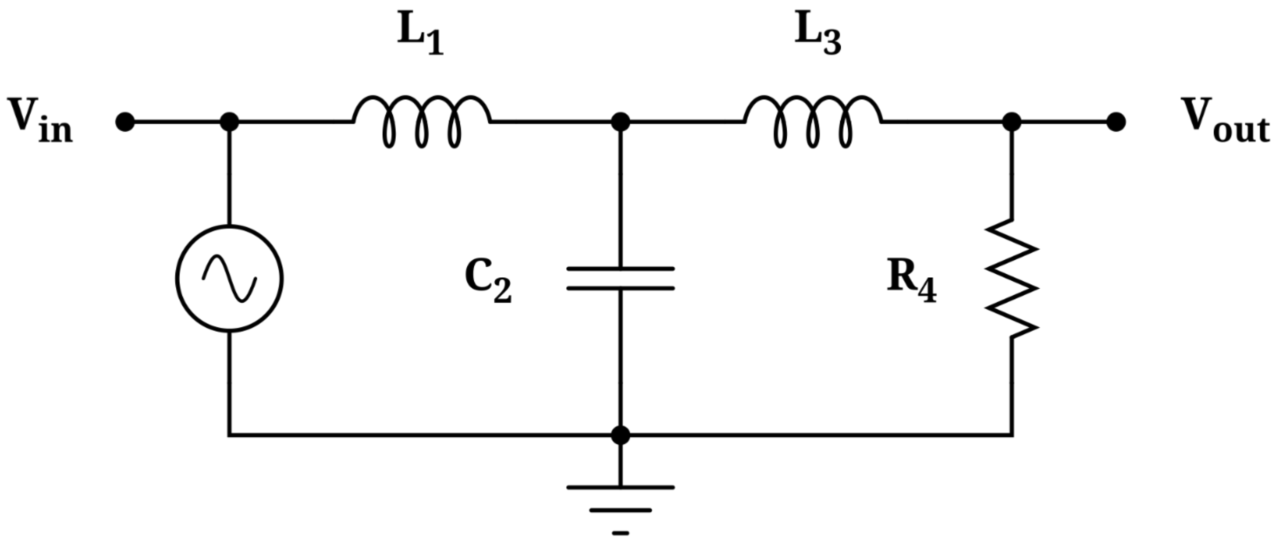
The frequency response of the Butterworth Filter is flat (no ripples) in the passband and rolls off towards zero in the stopband. The Bode Plot of a First-Order Low-Pass Butterworth Filter is shown below.



First-Order Low-Pass Butterworth Filter Bode Plot
(dashed blue is ideal)

Butterworth filter - Wikipedia

Example: Third-Order Low-Pass Butterworth Filter



Third-Order Low-Pass Butterworth Filter

The transfer function is:

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R_4}{(L_1 C_2 L_3)s^3 + (L_1 C_2 R_4)s^2 + (L_1 + L_3)s + R_4}$$

For example, assume the following is given:

$$L_1 = \frac{3}{2}\ \text{H} \qquad C_2 = \frac{4}{3}\ \text{F} \qquad L_3 = \frac{1}{2}\ \text{H} \qquad R_4 = 1\ \Omega$$

Then the numerical transfer function is:

$$H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

whose poles are:

$$s_1 = -1 \qquad s_{2,3} = -0.5 \pm 0.866i$$

The magnitude of the frequency response gain is:

$$G(\omega) = \left|H(j\omega)\right| = \frac{1}{\sqrt{1+\omega^6}}$$

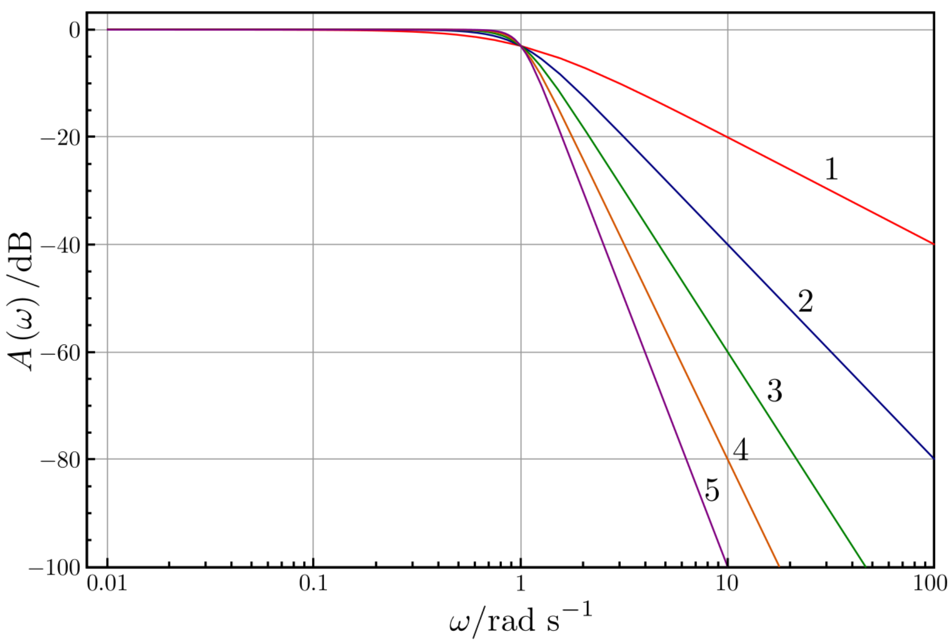
Example concludes.

Finally, the gain of an nth-order Low-Pass Butterworth Filter is:

$$G^2(\omega) = \left|H(j\omega)\right|^2 = \frac{G_0^2}{1+\left(\dfrac{\omega}{\omega_c}\right)^{2n}}$$

where n is the filter order, ω_c is the cutoff frequency (approximately the –3 dB frequency), and G_0 is the DC gain (i.e. the gain at zero frequency).

The figure below shows the magnitude characteristics of the Low-Pass Butterworth Filter of orders n = 1 through n = 5.



Low-Pass Butterworth Filter Magnitudes, *n* = 1 – 5

If n could approach infinity (it cannot in practice), then there would be a perfect rectangle above wherein frequencies below the cutoff frequency ω_c would be passed perfectly with DC gain G_0 and frequencies above the cutoff frequency ω_c would be blocked perfectly.

Approximate Derivative Low-Pass Filter

It is well-known that numerical time differentiation via the discrete back-stepping method:

$$\dot{x}(t) = \frac{dx(t)}{dt} \cong \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}$$

is unreliable and may yield numerically-unstable results with errors and high-frequency spikes that are garbage, leading to poor results in simulation. This can be true even when the discrete time step $\Delta t = t_i - t_{i-1}$ is small. However, the Simulink PID block requires a numerical derivative; here is how MATLAB implements it to avoid this issue.

The exact time derivative transfer function is $G(s) = s$; however, this is an improper transfer function since the order of the numerator is greater than the order of the denominator. This exact differentiating transfer function can be replaced by an approximate differentiating transfer function:

$$s \cong \frac{s}{\varepsilon s + 1}$$

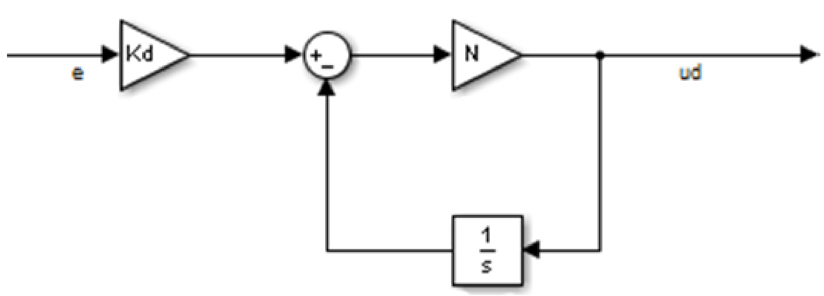
where ε is a small, positive, real number, $\varepsilon \ll 1$. MATLAB/Simulink defines a derivative divisor N for this approximate differentiating transfer function:

$$s \cong \frac{s}{\frac{s}{N} + 1} \qquad \text{where} \qquad N = \frac{1}{\varepsilon} \gg 1$$

Note:

$$G_{\text{approx}}(s) = \frac{s}{\frac{s}{N} + 1} = \frac{Ns}{s + N} = \frac{N}{1 + \frac{N}{s}}$$

is called a **Low-Pass Filter** and is implemented for the D term in a PID controller as shown below. The user need only specify N in the Simulink PID block, rather than repeating the diagram below. Larger values of N lead to greater accuracy of the resulting numerical derivative. However, numerical problems may also be introduced for high N, so try to keep the default value of $N = 100$. $N = \inf$ (the theoretically-perfect time derivative) is specifically disallowed by MATLAB.



Low-Pass Filter

4.10 Accelerometers

An **accelerometer** is an electromechanical device that measures an object's translational acceleration (m/sec^2, time rate of change of velocity, second time rate of change of position) with respect to an inertial reference frame. The gravity vector must be subtracted, and corrections must be made for the rotation of the earth relative to the inertial frame. In order to find velocity and then position from an accelerometer, the accelerometer output must be integrated over time twice.

Since position $x(t)$ is a function of time, we can take successive time derivatives to find the velocity $v(t)$ and acceleration $a(t)$ formulae if we know $x(t)$:

$$v(t) = \dot{x}(t) = \frac{dx(t)}{dt} \qquad\qquad a(t) = \ddot{x}(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2}$$

But for accelerometer output we need to go the opposite way, i.e. integration. We start with the accelerometer reading of $a(t)$ and integrate to obtain velocity $v(t)$ and then integrate $v(t)$ to obtain position $x(t)$.

$$v(t) = v_0 + \int a(t)dt$$

$$\begin{aligned} x(t) &= x_0 + \int v(t)dt \\ &= x_0 + v_0 t + \iint a(t)dt \end{aligned}$$

Note in the above formulae we do not know an analytical function for acceleration $a(t)$ since it is read from the accelerometer sensor many times per second. Instead, only numerical integration can be used. Unlike numerical differentiation which is known to be noisy and unreliable (via the finite difference formula, see the previous page), numerical integration (using a rectangular or trapezoidal rule) is a reliable, noise-free, maths process.

Maths Example

This example doesn't apply well to accelerometers since we don't know the formula for $a(t)$. In this maths example, we simply assume the acceleration is constant: $a(t) = a$. This comes from the simplest mechanical dynamics problem in the world, i.e. 1-dof translational acceleration a of a point mass m with no friction, no spring, nor damping, and a constant force f. From Newton's Second Law we have $f = ma$ and so $a = f / m$ (constant).

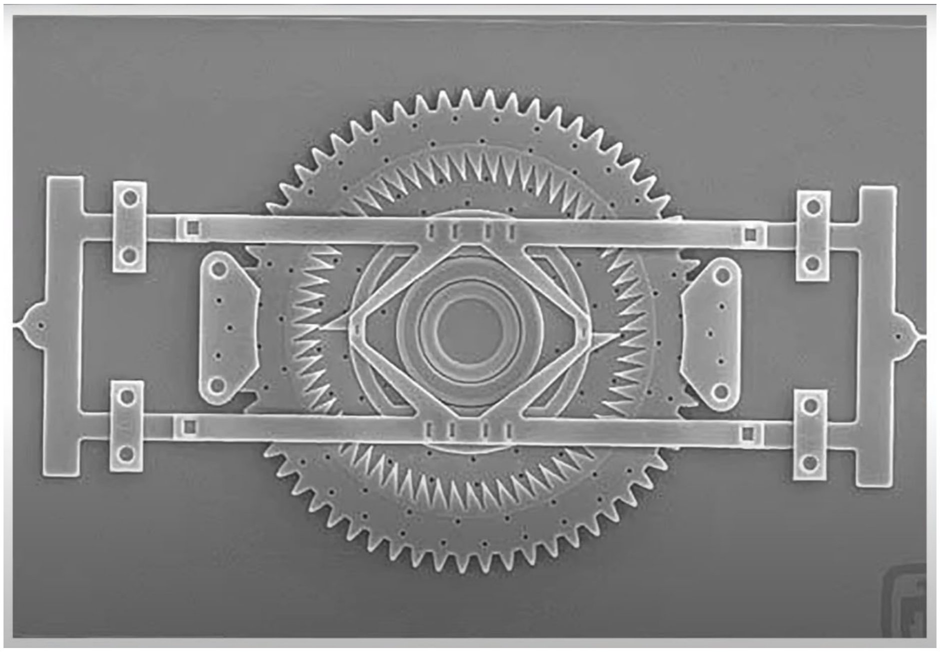
$$
\begin{aligned}
a(t) &= a \\
v(t) &= v_0 + \int a dt = v_0 + at \\
x(t) &= x_0 + \int v(t) dt = x_0 + v_0 t + \frac{1}{2} a t^2
\end{aligned}
$$

It is instructive to plot these results (recall the graphical interpretation of time derivatives and integrals):

Example concludes.

Gyroscopes, not presented here, can measure angular velocities (in rad/sec). Often accelerometers and gyroscopes appear together in one sensor, the Inertial Measurement Unit (IMU).

Modern accelerometers can be constructed in MEMS. MEMS are micro-electromechanical systems, combining mechanical and electrical components at the micron scale (10^{-6} m). This allows for portable accelerometer applications.

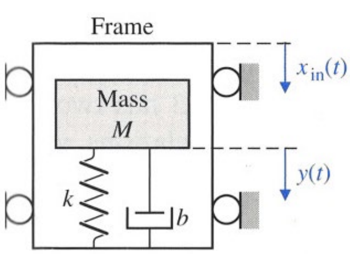


MEMS Accelerometer

Bing Videos

Accelerometers are used in inertial navigation systems for aircraft and missiles, in unmanned aerial vehicles (UAVs), smartphones, cameras, video game controllers, and vibration detection sensors for machinery and earthquakes. Accelerometers are also deployed to measure vibrations in automobiles, machines, buildings, and process control systems.

Accelerometers may be single-axis or multi-axis (usually multiple single axis units mounted along different axes). The multi-axis accelerometers can measure vector acceleration. As shown in the left figure below, a simple single-axis mechanical accelerometer is based on a damped proof mass on a spring.



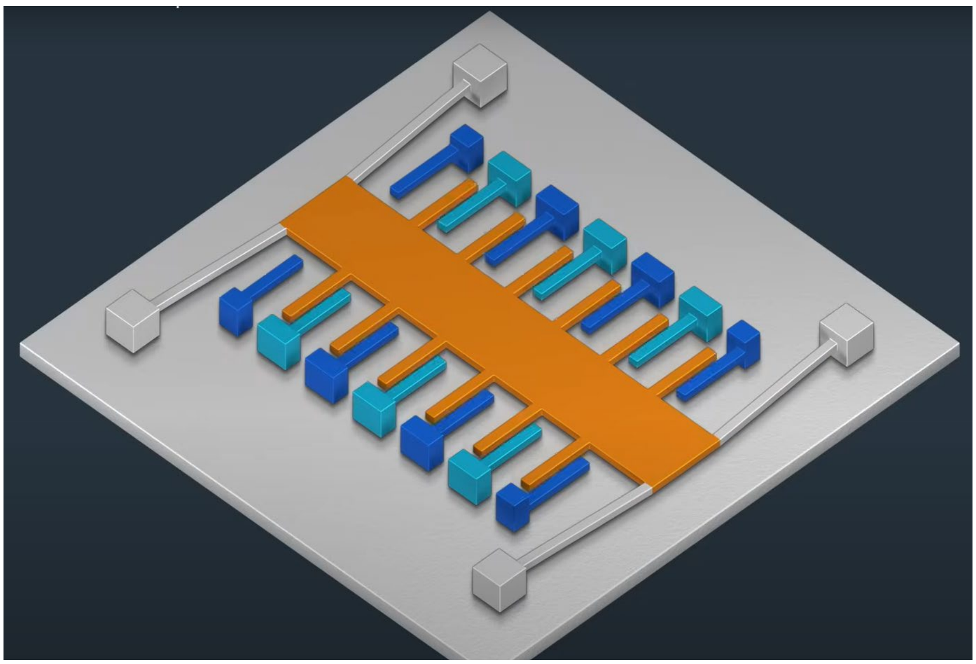
Accelerometer Diagram

Commercial Accelerometer

Accelerometer - Wikipedia

Accelerometers are constructed of a proof mass (aka seismic mass), a multiple-H-shaped structure with sense fingers extending from it. The proof mass is tethered to a substrate at both ends and can vibrate side-to-side. Electrodes are stationary in the substrate and interlace with the moving H structures of the proof mass. The proof mass never physically touches the electrodes, but forms a comb-like structure. This creates a capacitance C across a gap of d width; the closer the surfaces come together without touching, the greater the capacitance C (see Section 2.2 Capacitors to recall this inverse relationship between capacitance C and the gap distance d). When they move farther apart, increasing d, the capacitance C lessens.

Consider two stationary and one moving metal plate sandwiched between them. This is called a differential capacitor. The charges between each plate are sensed. As the proof mass vibrates, the charges between the plates fluctuates, alternately increasing and decreasing the capacitance C as the appropriate d becomes less and more.



Differential Capacitance Accelerometer

Bing Videos

The differential capacitance output is run through charge amplification, signal conditioning, low-pass filtering, and is converted to a digital signal using an analog-to-digital converter (ADC). This signal is calibrated to physical units for acceleration. Capacitance-based accelerometers are just one type of accelerometer, but they are popular due to high accuracy, stability, low-power, rejection of electrical noise, and simple construction.

In order to detect general 3D translational accelerations $a(t) = \{\ddot{x}(t) \quad \ddot{y}(t) \quad \ddot{z}(t)\}^T$, it is necessary to mount three differential accelerometers mutually perpendicular to each other.

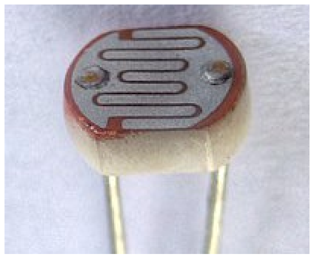
When the accelerometer experiences an acceleration, the spring's compression exerts a force on the proof mass to counteract the acceleration (according to Newton's Third Law). The spring force is linear with displacement (according to Hooke's Law, assuming a linear spring). Also, the spring stiffness and proof mass are constants; therefore, the spring's compression is directly related to the acceleration sensed. Damping is added to reduce oscillations which may interfere with the measurement of acceleration. This damping means the accelerometers have a limited frequency response, i.e. accurate readings of acceleration are not possible if the frequency of vibrations is too high.

Many animals including humans detect acceleration via sensory organs. In humans the labyrinths of the inner ear provide the sense of balance, orientation, and movements of the head. Humans are very sensitive to acceleration via semicircular canals in the inner ear. However, humans do not sense velocity (except by sight). In humans and many animals, the proof mass of their accelerometer sensors is one or more otoliths (Latin for ear stone, calcium carbonate crystals) acting against hairs connected to neurons. The hairs act as the spring and damping is provided by a fluid. Other invertebrates have accelerometers, but not in the inner ear. In those cases, the sensors are called statocysts.

Accelerometers are designed so that a circuit will sense a small motion, and then it controls a linear motor to push back on the proof mass to maintain small displacements of the proof mass. Such accelerometers are then steady (no oscillations), linear, and with a controlled frequency response.

4.11 Photoresistors

A **photoresistor** is a passive component that decreases in resistance with increasing luminosity on its light-sensitive surface. It is also called light-dependent resistor (LDR), or photo-conductive cell. Photoresistors do not have p-n junctions. Photoresistors are used in light-sensitive detector circuits and light- and dark-activated switching circuits, acting as a semiconductor resistance.



Commercial Photoresistor

Photoresistor Schematic Symbol

Photoresistor - Wikipedia

A photoresistor is made of a highly-resistant semiconductor material. Within a semiconductor, electrons have varying energy levels and they arrange themselves so similar energy levels are near each other. These levels are called energy bands. The valence band is the level with the lowest energy, where electrons move the least freely. The conduction band is the level with the highest energy, where electrons move the most freely. These two bands are separated by an area called the energy gap. The resistance of the semiconductor depends on the amount of electrons in the conduction band to carry electricity.

When light hits a photoresistor, the photons from the light excite the electrons in the valence band, increasing their energy levels, and allowing them to cross the energy gap to the conduction band. Because more electrons are now available to conduct electricity, the resistance of the photoresistor decreases.

In dark conditions, the photoresistor resistance is high, up to MΩ. In light conditions, the photoresistor resistance is low, as low as a few hundred Ohms.

When incident light on a photoresistor exceeds a given frequency, photons absorbed by the semiconductor give bound electrons enough energy to jump into the conduction band. This results in free electrons and holes, conduction of electricity, and thus lower resistance.

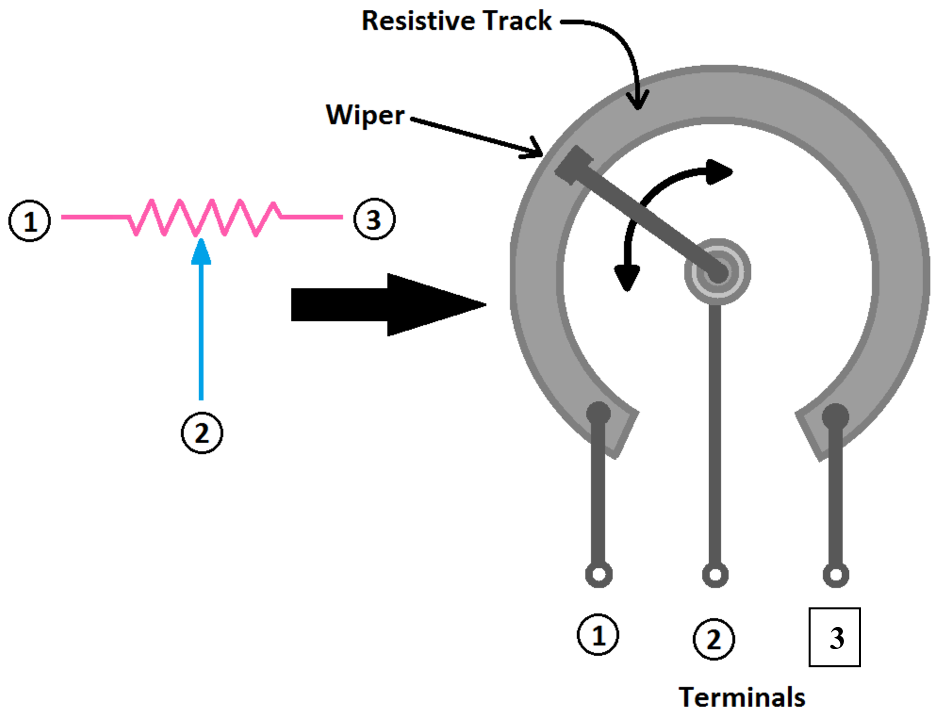
An intrinsic photoresistor is not an efficient semiconductor. An extrinsic photoresistor is doped with impurities, allowing the device to trigger more easily.

Economical photoresistors are deployed in a number of commercial applications, including streetlights, camera light meters, clock radios, nightlights, outdoor clocks, solar streetlamps, and solar road reflectors.

4.12 Potentiometers

A potentiometer is a three-terminal variable resistor with a rotating (or sliding) contact that comprises an adjustable voltage divider. Potentiometers can be either rotational or translational. This section focuses on rotational potentiometers.

A **potentiometer** is a variable resistor with a third adjustable terminal. The potential at the third terminal can be adjusted to give any fraction of the potential across the ends of the resistor. Or, if the third terminal is moved by an outside source, the potentiometer serves as a sensor for that motion.



Potentiometer Diagram

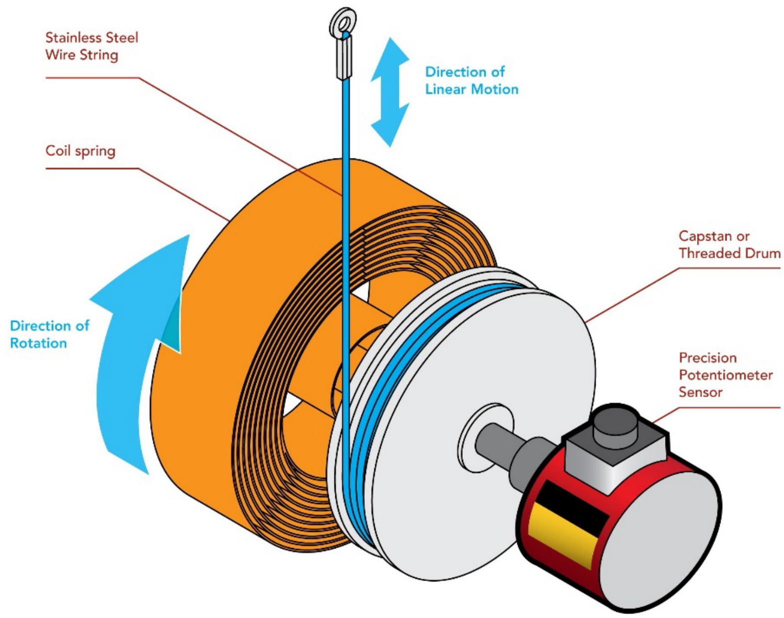
potentiometer+diagram.png (1015×1015) (bp.blogspot.com)

1. A potentiometer acts as a variable resistor.
2. The left pin (1) is grounded (0V), the right pin (3) has 5 V, and the center pin moves. There is a linear change in the output voltage given changes in the wiper position.

Potentiometer Schematic Symbol

String Pots

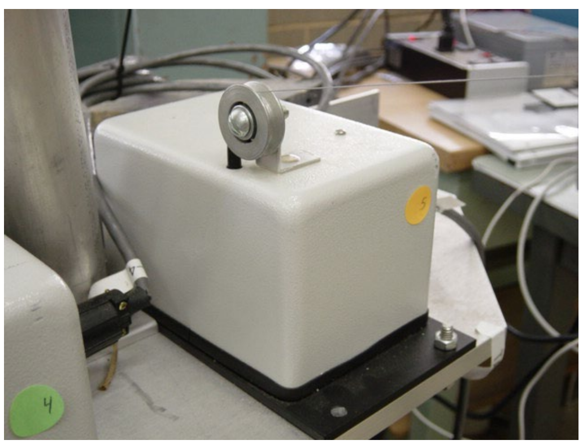
String Potentiometers (String Pots, Yo-Yo Sensors) are transducers used to measure length. A cable is wrapped around a shaft that drives a potentiometer (variable-resistance device). The length is linearly proportional to the change in resistance in the potentiometer.



String Pot Diagram

www.futek.com/string-potentiometer

The image below is of a single string pot (out of 6) employed in a research project at NIST (National Institute of Standards and Technology in Gaithersburg MD) during Dr. Bob's 2002-2003 sabbatical from Ohio University there. These string pots were used to measure lengths in large-scale cable-suspended robot projects in NIST's high-bay laboratory.



String Pot at NIST[3]

[3] R.L. Williams II, J.S. Albus, and R.V. Bostelman, 2004, "3D Cable-Based Cartesian Metrology System", Journal of Robotic Systems, 21(5): 237-257.

4.13 Thermal Sensors

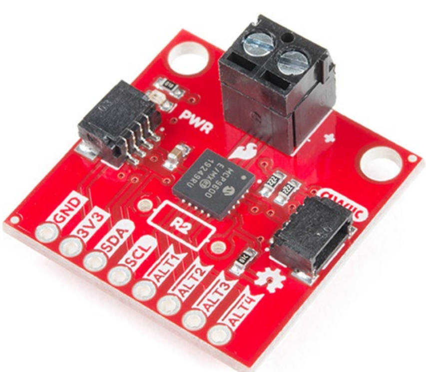
This section presents **thermocouples**, **thermistors**, **resistance temperature detectors** (RTDs), **temperature-to-digital converters**, and **thermal switches**. These are useful in mechatronics when heat and temperature are important aspects to measure and control.

Thermocouples

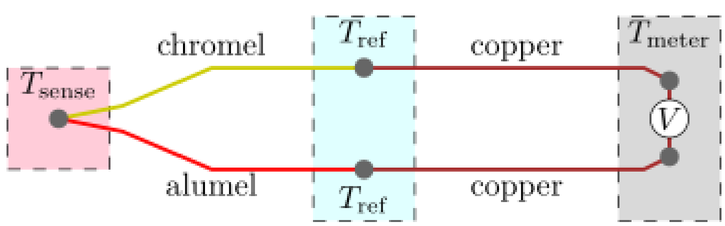
A **thermocouple** is a thermoelectric device commonly used for measuring temperature. It consists of two dissimilar conductors forming an electrical junction. A voltage is developed between the two junctions that is proportional to the temperature difference. This is due to the Seebeck effect: an electromotive force (emf) developing across two points of electrically-conducting material with a temperature difference.

Commercially-available thermocouples are inexpensive, interchangeable, have standard connectors, and measure a wide range of temperatures. Thermocouples are self-powered and thus require no external power. Accuracy is a main disadvantage of thermocouples, as it is hard to achieve less than 1° C error.

Thermocouples are applied in industrial kilns, gas turbine exhausts, diesel engines, as flame sensors in gas-powered appliances, and as the temperature sensors in thermostats for home and business heating-ventilation-air-conditioning (HVAC).



MCP9600 Converter

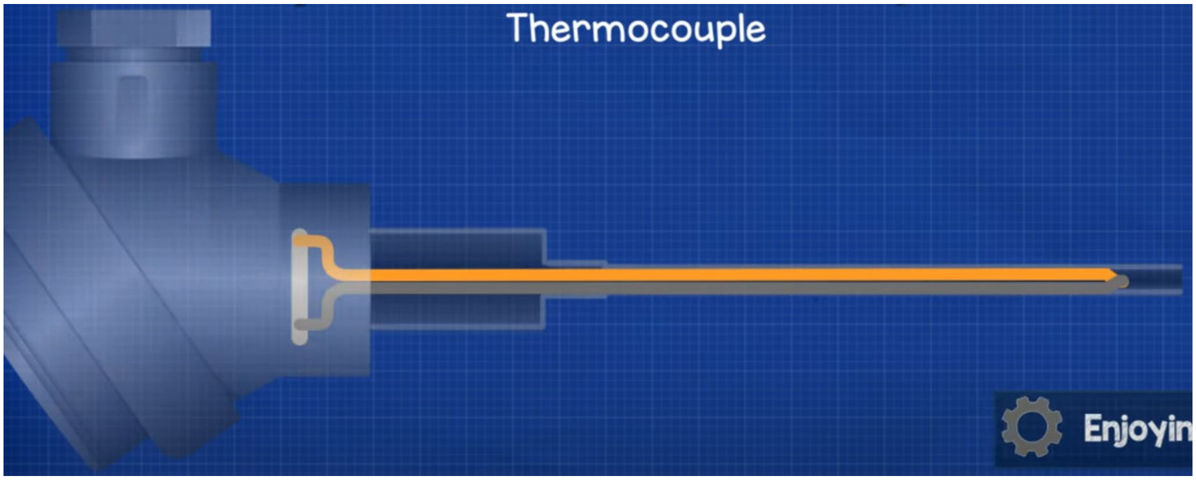


K-type Thermocouple

Thermocouple - Wikipedia

The MCP9600 is a full-integrated thermocouple emf to $^{\circ}$C converter with cold-junction compensation, supporting a variety of thermocouples.

K-Type Thermocouples are constructed with Chromel and Alumel as the two dissimilar conductors (ANSI/ASTM E230). They can read temperatures of $0\ ^{\circ}\mathrm{C}$ to $1200\ ^{\circ}\mathrm{C}$, with associated outputs of 0 to 49 mV. Chromel is a metal alloy of 90% Nickel and 10% Chromium. Alumel is a magnetic metal alloy of 95% Nickel, 2% Aluminum, 2% Manganese, and 1% Silicon.



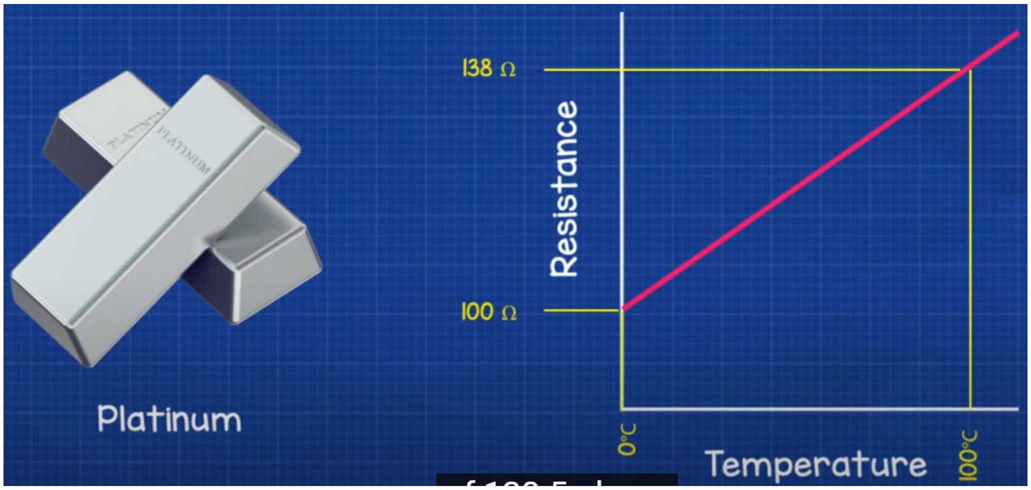
Industrial Thermocouple

[Temperature Sensors Explained - YouTube]

The typical rugged industrial thermocouple shown above consists of two different metal wires (e.g. copper and iron) connected on the distal end, and connected on the proximal end to a terminal block. A voltmeter is used to read the potential difference between the two wires. This voltage difference is generally tiny, in millivolts.

For best accuracy, the cold side of the thermocouple should be immersed in an ice bath of known temperature (0 $^\circ$C, 32 $^\circ$F). This is not practical outside a laboratory setting, so instead the cold junctions are at an equal ambient temperature. The temperature difference is compensated by measuring the temperature of the connection (using an RTD sensor, resistance temperature detector, presented later in this section) and applying a formula to offset the temperature error.

Different metals have different inherent resistances. Most conductors, especially metals, will increase in resistance with increasing temperature. As shown in the figure below, platinum has a linear change in resistance with increasing temperature.

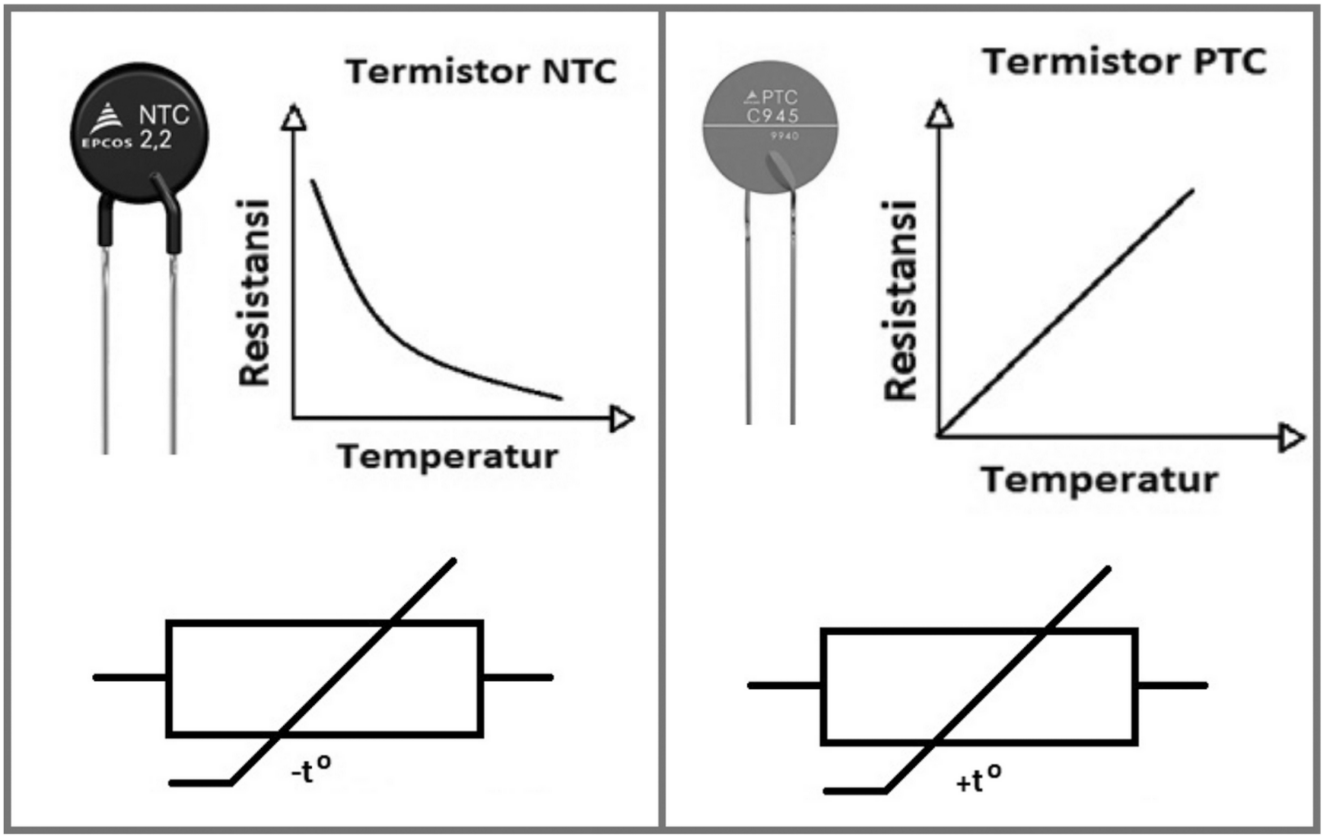


Platinum Resistance vs. Temperature

[Temperature Sensors Explained - YouTube]

Thermistors

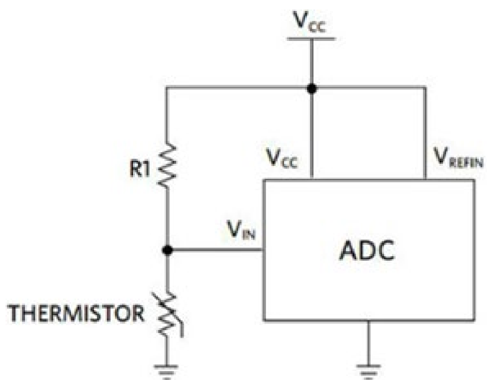
A **thermistor** (thermal resistor) is an electrical resistor whose resistance is greatly affected by heating. Thermistors are used in temperature measurement and control.

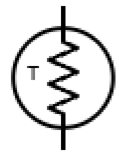


Thermistor Diagram

termistor-ntc-ptc.gif (1500×960) (bp.blogspot.com)

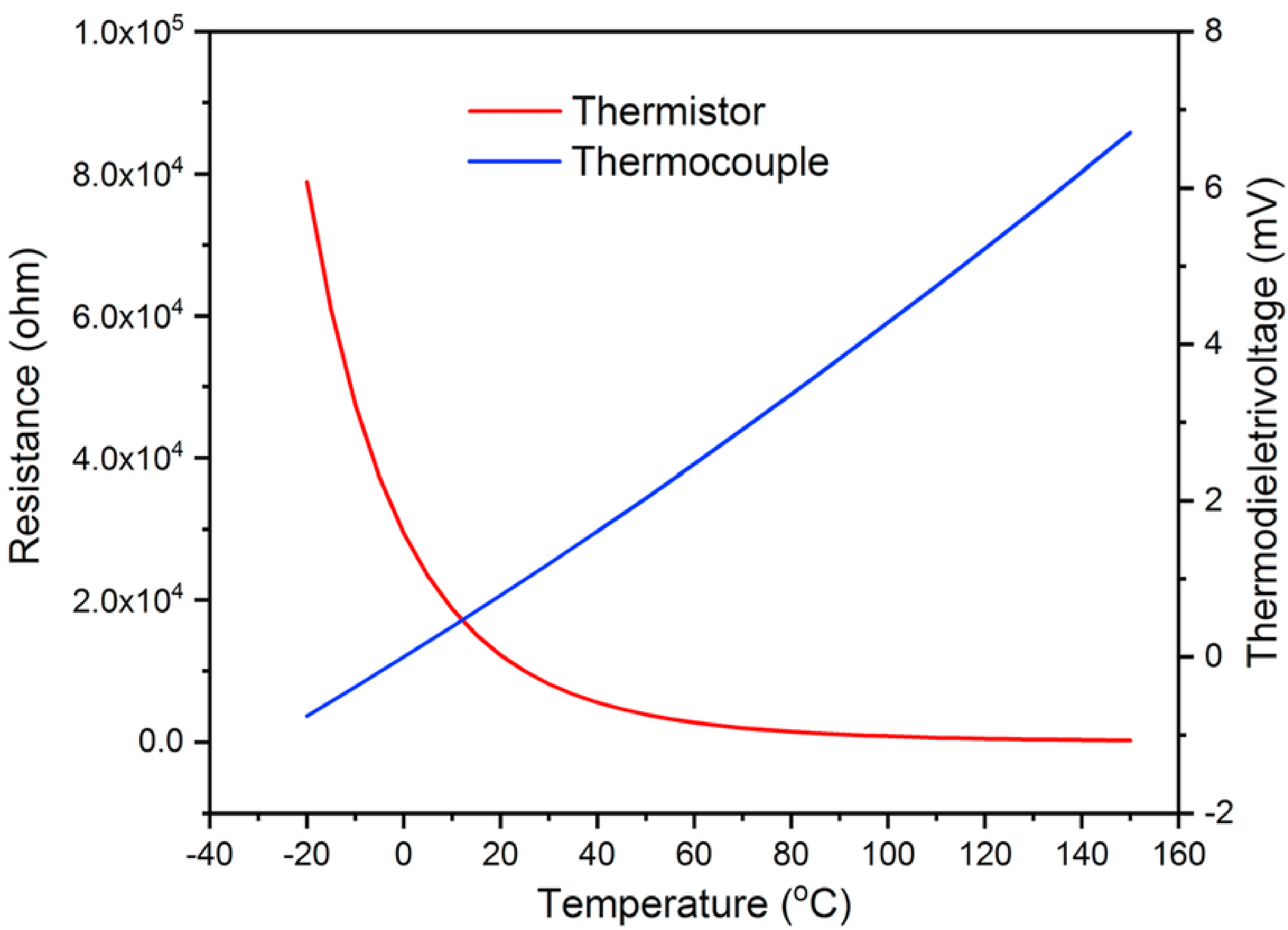
- Negative temperature coefficient (NTC) – resistance decreases inversely with temperature increase
- Positive temperature coefficient (PTC) – resistance increases linearly with temperature increase





Thermistor Circuit **Thermistor Schematic Symbol**

Commercial Thermistors



Comparison of Thermocouple and NTC Thermistor Resistance vs. Temperature

Resistance Temperature Detector (RTD)

A **resistance temperature detector (RTD)** is a temperature-measurement sensor that uses thermal expansion as opposed to heat in reading temperatures. An RTD is simply a wire, usually made of platinum, that has a known resistance at $0^\circ C$. RTDs are suitable for accurate temperature measurement, in applications where the temperature is not changing rapidly.

RTDs

These RTDs are made from platinum. The resistances are in powers of 10. There is a linear relationship for temperature vs. resistance.

The table below shows the relative advantages and disadvantages of thermocouples, RTDs, and thermistors. This can guide the choice of sensor for different needs in various applications.

No temperature is perfect for every application
The table shows general comparison

	Accuracy	Resolution	Cost	Range	Robustness
Thermocouple	Medium	Low	Low	-200°C to +1200°C	High
RTD	High	Medium	Medium / High	-200°C to +600°C	High
Thermistor	Low / Medium	High	Low	-80°C to +150°C	Low

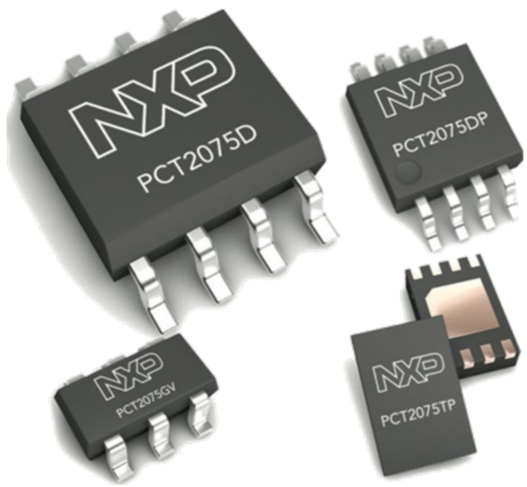
Comparison of Thermal Sensor Characteristics

[Temperature Sensors Explained - YouTube]

Temperature-to-Digital Converters

Temperature-to-Digital Converters are mechatronic components to take analog temperature signals and convert them into digital data for precise temperature monitoring and control. This type of analog-to-digital converter (ADC) has high resolution, fast sampling rates, and compatibility with various thermal sensors. They are deployed in a range of industries, from automotive to healthcare.

The **NXP PCT 2075** is an example temperature-to-digital converter on an integrated chip. It provides $\pm 1\ ^{\circ}C$ accuracy over a temperature range of $-25\ ^{\circ}C$ to $+100\ ^{\circ}C$.

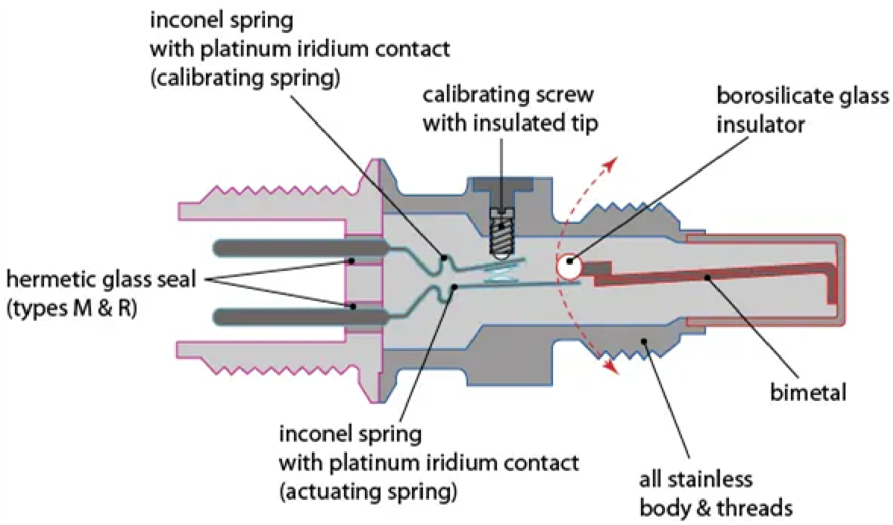


NXP PCT 2075 Chips

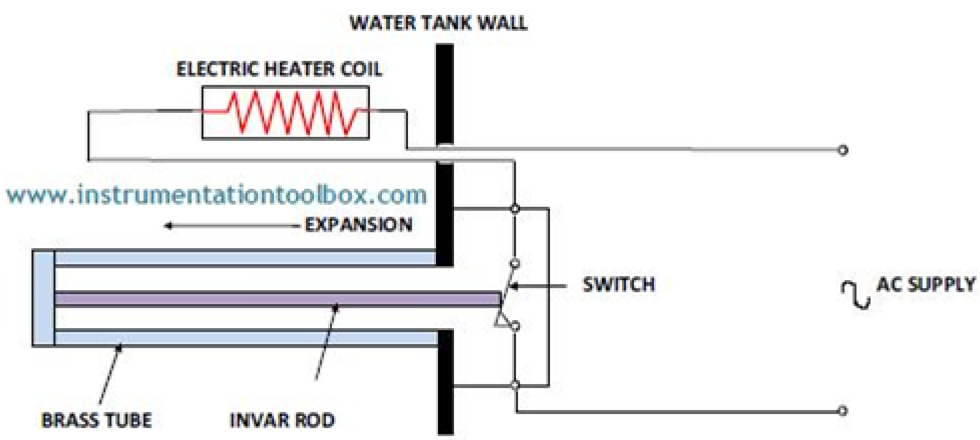
PCT2075 Digital Temperature Sensors - NXP Semiconductors | Mouser

Thermal Switches

Thermal switches open and close switches based on the ambient temperature. Also known as temperature switches, these are electromechanical devices that change their position from normally-open (NO) to normally-closed (NC) when a threshold temperature is reached. Thermal switches are used for monitoring and controlling temperature in various industrial processes.



Thermal Switch Diagram



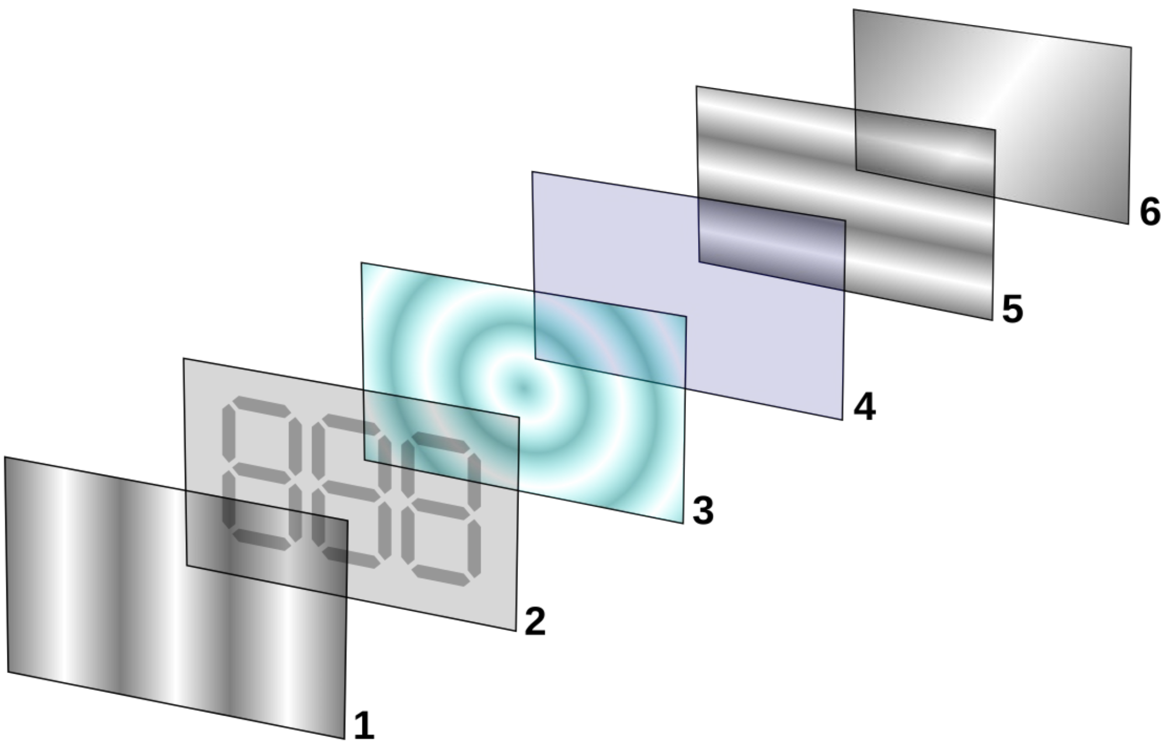
Thermal Switch Circuit

4.14 Electronic Displays

This section briefly discusses various electronic displays: **liquid-crystal displays (LCDs)**, **light-emitting diodes (LEDs)**, and **seven-segment displays**.

Liquid-Crystal Displays (LCDs)

A **liquid-crystal display (LCD)** is a flat-panel display using the light-modulating properties of liquid crystals, combined with polarizers, to display information. Rather than emitting light directly, liquid crystals use a backlight (reflector) to produce images in colour.

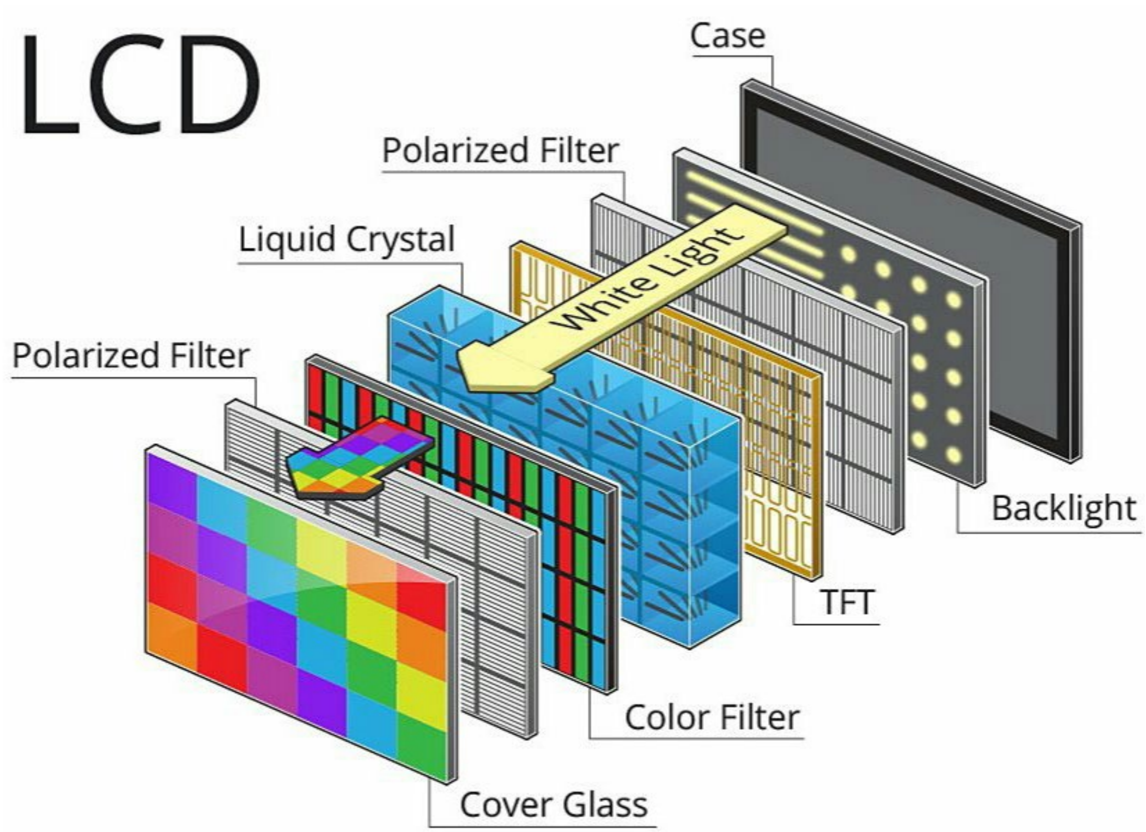


LCD Construction

"Layers of a reflective twisted nematic liquid crystal display:
1. Polarizing filter film with a vertical axis to polarize light as it enters.
2. Glass substrate with ITO electrodes. The shapes of these electrodes will determine the shapes that will appear when the LCD is switched ON. Vertical ridges etched on the surface are smooth.
3. Twisted nematic liquid crystal. It normally rotates the light's polarization by 90°. But if the surrounding electrodes are charged, the light's polarization won't be rotated.
4. Glass substrate with common electrode film (ITO) with horizontal ridges to line up with the horizontal filter.
5. Polarizing filter film with a horizontal axis. Light whose polarization was rotated by the liquid crystal will pass through, but light that wasn't rotated will be blocked.
6. Reflective surface to send light back to viewer. (In a backlit LCD, this layer is replaced or complemented with a light source.)"

Liquid-crystal display - Wikipedia

LCDs were first developed in the 1960s. Now LCDs are used in varied applications such as LCD televisions, computer monitors, instrument panels, aircraft cockpit displays, plus signage. Small LCD screens are implemented in digital cameras, watches, calculators, and smartphones. LCD screens have largely replaced the cathode-ray tube (CRT), since LCDs are lighter, smaller, and more energy-efficient. LCDs do not suffer from screen burn-in like CRTs. LCDs are susceptible to image persistence (or image retention, wherein previous display information can still be seen).



LCD Construction in Technicolour

[Why is LCD (Liquid Crystal Display) device so named? - My Q/A Corner]

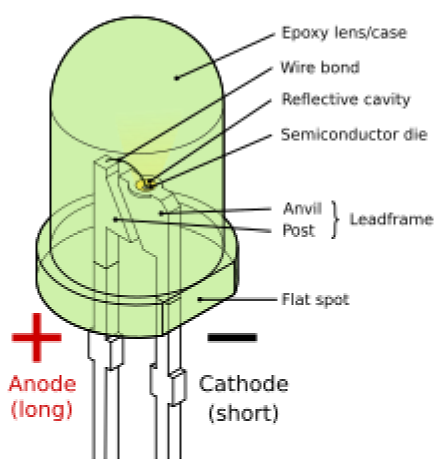
Light-Emitting Diodes (LEDs)

A **light-emitting diode (LED)** is a semiconductor diode that glows with visible light when current flows through it in the correct direction, caused by an applied voltage. The working principle is electroluminescence (optical/electrical emission of light from a material via an electric current or electric field). Electrons in the semiconductor recombine with electron holes, releasing energy as photons. The colour of the resulting light depends on the semiconductor band gap energy. Apparently invented in Russia in the 1920s, practical LED components appeared in 1962. These first emitted infrared (IR) light, such as that still used in remote control devices.

LEDs appear in various applications, including replacing incandescent lightbulbs, Christmas light strings, aviation lighting, fairy lighting, strip lighting, automobile headlights, advertising, traffic signals, camera flashes, horticultural lighting, and medical devices.

LEDs have the following advantages compared to incandescent light sources: lower power requirement, lower heat emissions and thus lower temperatures, longer lifetime, stronger designs, smaller sizes, and faster switching.

Coloured LEDs



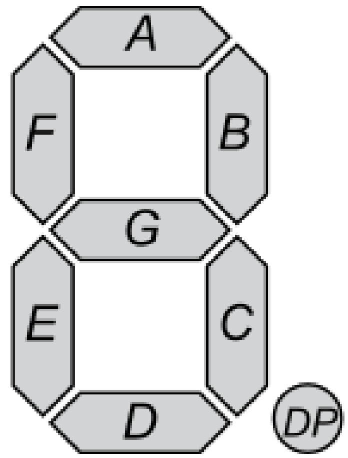
LED Construction

[Light-emitting diode - Wikipedia]

Seven-Segment Displays

A **seven-segment display** is an electronic device for displaying decimal digits. It has also been used to display hexadecimal digits, with the addition of A – F. Seven-segment displays have been widely applied in digital clocks, electronic meters, calculators, and other electronic devices requiring numerical displays. The digit '8' uses all seven straight-line segments. Each segment is basically a rectangle, slanted to aid readability, with triangular end-caps. One data byte is sufficient to represent all possible states.

Seven-Segment Display showing 8



Segment Names

A seven-segment display was patented in 1903 (C. Kinsley, U.S. Patent 1,126,641). LED seven-segment displays for electronics appeared in the 1970s. Nowadays LCDs have largely replaced LEDs in seven-segment displays.

Seven-Segment Display for Petrol Prices

4-Digit Seven-Segment Display

[Seven-segment display - Wikipedia]

5. Microcontrollers

This chapter is about microprocessors and microcontrollers. First, computer number representation systems are presented, with examples. A general introduction to microprocessors is next presented, including Moore's Law and the radical increases in transistors count, decrease in transistor physical size, and decrease in memory cost. This is followed by microprocessor architectures, the Raspberry Pi Pico microcontroller, and an introduction to Micropython programming. This chapter concludes with Programmable Logic Controllers (PLCs); these important industrial automation controllers are not featured in ME 3550 or 4550 laboratory.

5.1 Number Systems

There are various ways to represent a number in mathematics and in a computer.

- Binary base 2 digits 0 or 1
- Octal base 8 not common today, used in early computing, digits 0-7
- Decimal base 10 digits 0-9
- Hexadecimal base 16 more common than octal today, digits 0-9 and A-F (10-15)

Binary number representation uses digits 0-1 and base 2. This is the main number representation method in computers including microprocessors.

Octal number representation uses digits 0-7 and base 8. Octal was used in early computing simply to save word size compared to binary number representation. Octal is generally not used these days.

Decimal number representation uses digits 0-9 and base 10. This is the familiar number system from maths education. This popularity arose because most humans have ten fingers.

Hexadecimal number representation uses digits 0-9 and A-F for values 10-15. It uses base 16. Each hexadecimal digit represents 4 binary digits, i.e. one nibble. Hence 2 hexadecimal digits represent one byte of 8 binary digits.

The table below shows the value of **Hexadecimal** digits vs. **Decimal** digits.

Hex	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Dec	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Example: Number Representation for 234_{10} (base 10).

System	Representation
Binary	11101010_2
Octal	352_8
Decimal	234_{10}
Hexadecimal	EA_{16}

Binary

$$1\times2^7+1\times2^6+1\times2^5+0\times2^4+1\times2^3+0\times2^2+1\times2^1+0\times2^0=128+64+32+8+2=234_{10}$$

Octal

$$3\times8^2+5\times8^1+2\times8^0=192+40+2=234_{10}$$

Decimal

$$2\times10^2+3\times10^1+4\times10^0=200+30+4=234_{10}$$

Hexadecimal

$$\mathrm{E}\times16^1+\mathrm{A}\times16^0=14\times16^1+10\times16^0=224+10=234_{10}$$

Often number systems are explicitly identified by a trailing subscript indicating the base; from the above example:

$$11101010_2 \quad = \quad 352_8 \quad = \quad 234_{10} \quad = \quad EA_{16}$$

Example concludes.

Binary Representation

A **bit** (binary digit) is a single 0 or 1.

A **byte** is 8 binary digits. Based on powers of 2, the maximum number for a single byte is $2^8 = 256$.

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
128	64	32	16	8	4	2	1	sum is 255 (0 to 255 yields 256)

A **nibble** is 4 binary digits, i.e. half a byte. The maximum number for a single nibble is $2^4 = 16$.

2^3	2^2	2^1	2^0	
8	4	2	1	sum is 15 (0 to 15 yields 16)

Binary Addition

This is performed just like the familiar base 10 addition, but in base 2 with only 0s and 1s. Carrying works in a similar manner. Example:

$$\begin{array}{r} 124 \\ +63 \\ \hline 187 \end{array} \qquad \begin{array}{r} 01111100 \\ +00111111 \\ \hline 10111011 \end{array} \qquad 10111011_2 = 187_{10}$$

base 10 (decimal) base 2 (binary)

More binary addition examples:

$$\begin{array}{r} 0110 \\ +0111 \\ \hline 1101 \end{array} \qquad \begin{array}{r} 00010110 \\ +00010111 \\ \hline 00101101 \end{array} \qquad \begin{array}{r} 00110101 \\ 00100110 \\ +00100111 \\ \hline 10000010 \end{array}$$

Binary addition forms the basis for other maths operations as well (e.g. subtraction, multiplication, and division).

To represent a negative number in **Binary**, using the **Signed Magnitude** method, append a leading bit to the left of the original number. If this number is to be positive, the leading appended bit must be 0, but if this number is to be negative, the leading appended bit must be 1.

Binary Subtraction

Computers and microprocessors are not capable of subtracting one binary number from another. To do the subtraction $C_2 = B_2 – A_2$, perform the following steps (the **minuend** is the first number B_2 and the **subtrahend** is the second number A_2; the **difference** is the answer, C_2). The principle to binary subtraction is adding a negative number.

0. Step 0. If the **minuend** and **subtrahend** are of a different numbers of digits, pad the shorter one with zeros to the left.
1. Step 1. Find the **1's complement** of the **subtrahend** A_2.
2. Step 2. Add the Step 1 result to the **minuend** B_2.
3. Step 3. If there is a carryover digit, discard it, then add '1' to the remaining Step 2 result. This is called the **2's complement** of the Step 2 result. *answer*
4. Step 4. If there is no carryover digit, take the **1's complement** of the Step 2 result. Then attach a minus sign in front. *answer*

The **1's complement** of any binary number is simply found by reversing each digit of the original number (all 1s to 0s and all 0s to 1s).

Example:

The **1's complement** of 01011010_2 is 10100101_2.

Example concludes.

Example 1:

$$\begin{array}{r} 124 \\ -63 \\ \hline 61 \end{array}$$

base 10 (decimal)

$B_2 = 1111100$, minuend $\quad A_2 = 111111$, subtrahend

base 2 (binary)

0. Step 0. $A_2 = 0111111$.

1. Step 1. The **1's complement** of A_2 is 1000000.

2. Step 2. Add the Step 1 result to the **minuend** B_2.

$$\begin{array}{r} 1111100 \\ +1000000 \\ \hline \underline{1}0111100 \end{array}$$

the carryover digit '1' has an underbar

3. Step 3. If there is a carryover digit, discard it, then add '1' to the remaining Step 2 result.

$$\begin{array}{r} 0111100 \\ +0000001 \\ \hline 0111101 \end{array}$$

and $0111101_2 = 61_{10}$

This is the answer.

4. Step 4. Does NOT apply to this example.

Example 2:

$$\begin{array}{r} 63 \\ -124 \\ \hline -61 \end{array}$$

base 10 (decimal)

$B_2 = 111111$, minuend $\quad A_2 = 1111100$, subtrahend

base 2 (binary)

0. Step 0. $B_2 = 0111111$.

1. Step 1. The **1's complement** of A_2 is 0000011.

2. Step 2. Add the Step 1 result to the **minuend** B_2.

$$\begin{array}{r} 0111111 \\ +0000011 \\ \hline 1000010 \end{array}$$

there is NO carryover digit '1'

3. Step 3. Does NOT apply to this example.

4. Step 4. If there is no carryover digit, take the **1's complement** of the Step 2 result. Then attach a minus sign in front.

$0111101 \rightarrow -0111101$ and $-0111101_2 = -61_{10}$

This is the answer.

Conversion between Number Systems

How does one convert a number in one base to the same number in another base? There are procedures one can learn via trusty-old Google. However, in this section we will present number-system conversions based on math and logic.

Decimal-to-Binary conversion example: convert 217_{10} to its equivalent number in **Binary**.

First, construct a table representing a byte, including the associated powers of 2. Leave a blanque row for the binary digits (0 or 1).

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1

Now attack the most significant digit first, i.e. leading to the largest number. 128 fits into 217, so put a 1 in that bit.

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
1							

Perform the difference 217 – 128 = 89. 64 fits into 89, so put a 1 in that bit.

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
1	**1**						

Perform the difference 89 – 64 = 25. 32 does not fit into 25, so put a 0 in that bit. 16 fits into 25, so put a 1 in that bit.

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
1	**1**	**0**	**1**				

Perform the difference 25 – 16 = 9. 8 fits into 9, so put a 1 in that bit.

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
1	**1**	**0**	**1**	**1**			

Perform the difference 9 – 8 = 1. 4 does not fit into 1, so put a 0 in that bit. 2 does not fit into 1, so put a 0 in that bit. To finish, put a 1 in the 2^0 bit.

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
128	64	32	16	8	4	2	1
1	**1**	**0**	**1**	**1**	**0**	**0**	**1**

Example concludes.

Decimal-to-Octal and Decimal-to-Hexadecimal conversions

These two conversions can be accomplished in the same manner as the **Decimal-to-Binary conversion** presented above. Simply replace the **Binary** byte table with the left table below for **Octal** and the right table below for **Hexadecimal**.

8^3	8^2	8^1	8^0
512	64	8	1

16^3	16^2	16^1	16^0
4096	256	16	1

Note now many more digits are in play, not just 1 and 0 (0 – 7 for **Octal** and 0 – 9 & A – F for **Hexadecimal**). Therefore, for each most significant location, you have to figure out how many digits fit in (not just 1). An example is presented later in this section under **Hexadecimal-to-Octal conversion**.

Binary-to-Decimal, Octal-to-Decimal, Hexadecimal-to-Decimal conversion examples

These conversions are rather easier to perform, and the first example in this section has already shown this. Additional examples are now presented here.

a. convert 10110111_2 to its equivalent number in **Decimal**.

$$1\times2^7+0\times2^6+1\times2^5+1\times2^4+0\times2^3+1\times2^2+1\times2^1+1\times2^0=128+32+16+4+2+1=183_{10}$$

b. convert 321_8 to its equivalent number in **Decimal**.

$$3\times8^2+2\times8^1+1\times8^0=192+16+1=209_{10}$$

c. convert ABC_{16} to its equivalent number in **Decimal**.

$$A\times16^2+B\times16^1+C\times16^0=10\times16^2+11\times16^1+12\times16^0=2560+176+12=2748_{10}$$

Number System conversions not involving Decimal

What about number system conversions without **Decimal** in the mix? The table below gives all these possibilities.

Binary-to-Octal	**Octal-to-Binary**	**Hexadecimal-to-Binary**
Binary-to-Hexadecimal	**Octal-to-Hexadecimal**	**Hexadecimal-to-Octal**

Since I am a **Decimal**-centric person (ain't we all?), I choose to convert first to **Decimal** and then from **Decimal** to the desired base result.

Binary-Decimal-Octal	**Octal-Decimal-Binary**	**Hexadecimal-Decimal-Binary**
Binary-Decimal-Hexadecimal	**Octal-Decimal-Hexadecimal**	**Hexadecimal-Decimal-Octal**

Out of these six possibilities, we now present one example only. The other five are performed in a similar manner.

Hexadecimal-to-Octal conversion example: convert $9D_{16}$ to its equivalent number in **Octal**.

Let us choose to perform the **Hexadecimal-to-Decimal-to-Octal conversion**

1. **Hexadecimal-to-Decimal conversion**: convert $9D_{16}$ to its equivalent number in **Decimal**.

$$9D_{16} = 9 \times 16^1 + D \times 16^0 = 9 \times 16^1 + 13 \times 16^0 = 144 + 13 = 157_{10}$$

2. **Decimal-to-Octal conversion**: convert 157_{10} to its equivalent number in **Octal**.

First, construct an **Octal** table – we will need three columns since 157_{10} is between 64_{10} and 512_{10}. Leave a blanque row for the Octal digits (0 through 7).

8^2	8^1	8^0
64	8	1

Now attack the most significant digit first, i.e. leading to the largest number. 64 fits twice into 157, so put a 2 in that digit.

8^2	8^1	8^0
64	8	1
2		

Perform the difference 157 – 2(64) = 29. 8 fits thrice into 29, so put a 3 in that digit.

8^2	8^1	8^0
64	8	1
2	**3**	

Perform the difference 29 – 3(8) = 5. To finish, put a 5 in the one's digit.

8^2	8^1	8^0
64	8	1
2	**3**	**5**

$9D_{16} = 235_8$ answer

Example concludes.

Byte Sizes

size	# bytes	power of 10
bit (b)	0.125	–0.9
nibble (N)	0.5	–0.3
byte (B)	1	0
kilobyte (KB)	1,000	3
megabyte (MB)	1,000,000	6
gigabyte (GB)	1,000,000,000	9
terabyte (TB)	1,000,000,000,000	12
petabyte (PB)	1,000,000,000,000,000	15
exabyte (EB)	1,000,000,000,000,000,000	18
zettabyte (ZB)	1,000,000,000,000,000,000,000	21
yottabyte (YB)	1,000,000,000,000,000,000,000,000	24

What is a Byte?

Word-size Computing History

A higher number of bits in a computer word allows for larger numbers, more complex calculations and larger amounts of data to be processed. A higher number of bits also leads to better precision in computations.

# bits	# bytes	max number	Year	System
8	1	256	mid-1960s	IBM 360 mainframe
16	2	65536	late 1970s	PCs – Intel x86
32	4	4.29E+09	1985	Intel 80386
64	8	1.84E+19	2003	Intel Pentium 4

The Evolution of 32-bit Computing: From Past to Present - Ask.com

Integer vs. Floating-Point Computer Number Representation

An **integer** is a number that can be written without a fractional component. Integers have no decimal point. Integers are the infinite set of negative and positive natural numbers, plus zero. Integers are a subset of rational numbers, which are a subset of real numbers. The English word “integer” comes from the Latin meaning “whole” or “untouched”.

In contrast, a **floating-point number** is a data type in computer programming that stores a number (usually in base 2 binary, but also base 10 decimal can be used) with a floating (moveable) decimal point. A computer stores a floating-point number in three parts, the sign, the exponent, and the significand (the significant digits without the decimal point). The exponent sets the decimal point and hence the overall number. For example, for the number:

$$5.6789_{10} = 56789 \times 10^{-4}$$

the sign is 1 (positive – 0 if negative), the base-ten exponent is –4, and the significand is 56789. In general the significand can be a ratio of two integers, rounding where necessary due to storage limitations (see precision below).

Floating-point numbers can be used for very large things such as the distance between galaxies, or very small things like the distance between protons in an atom. The 1985 standard IEEE 754 is the accepted convention for floating-point arithmetic. The speed of different computers is often compared with FLOPS, the number of floating-point operations per second.

Computers store and perform computations with integers differently than floating-point numbers. There are also fixed-point numbers and calculations, such as in banking software, where there are always two digits after the decimal in base 10 (for pennies).

Older CPUs did not have the hardware support for floating-point operations, necessitating these computations in software, which was inefficient. Those chips used a co-processor for floating-point computation. Modern CPUs allow floating-point computations and hence do not need a co-processor.

Numerical precision is defined as the measure of detail in which a number can be expressed, i.e. the number of digits that are used to express a value. Higher precision is preferable to lower precision, especially in ensuing computations. However, higher precision requires more resources, memory, and time. The difference between the single-precision and double-precision representation of floating-point numbers is described in the table below, assuming each are represented in binary, as is the case in computers.

Single-Precision vs. Double-Precision, binary representation

precision	base	# sign bits	# exponent bits	# significand bits	# bits
single	2	1	8	23	32
double	2	1	11	52	64

I find that most modern students have totally lost touch with the concept of significant figures (see the following page), possibly due to the wide availability and use of calculators and computers. Further, they assume the computer represents numbers perfectly (it cannot!) and that all of the 16 digits to the right of the decimal point are meaningful and should be reported (they are not and should not be!).

Significant Figures

Significant Figures (abbreviated sig figs or s.f.) are important in any branch of science and engineering involving numerical calculations. It is my observation that with the advent and mainstream use of digital calculators and computers, many students, faculty, and engineers have lost sight of the important concept of significant figures. Please pay attention to significant figures in all homeworks and laboratory experiments this semester – indeed, in all of your engineering work into the future.

The amount of significant figures in any number is the amount of digits with meaning. This is related to the precision of a number or measurement, which is the degree of closeness a number is reported to. In calculations involving multiple numbers, the amount of significant figures in the result must be equal to that of the least significant number that enters the calculations.

In this context, precision is NOT the same as accuracy, which is how close a number is known to its true value.

One cannot report more digits in a measurement than is warranted by the measuring device. For instance, in a protractor marked to the nearest 1 degree, I was taught that you can report results to the nearest ½ degree, but the result is significant only to the nearest 1 degree.

Significant Figures Rules

- all non-zero digits are significant
  15 has two sig figs and 5.4321 has five sig figs

- leading zeros or trailing zeros are not significant unless otherwise stated
  0.15 has two sig figs, 0.00015 also has two sig figs, and 200 has one sig fig

  exceptions:
  200. has three sig figs, $2\underline{0}0$ has two sig figs, and $20\underline{0}$ has three sig figs

- trailing zeros after the decimal point are significant
  0.000150 has three sig figs and 0.00015000 has five sig figs

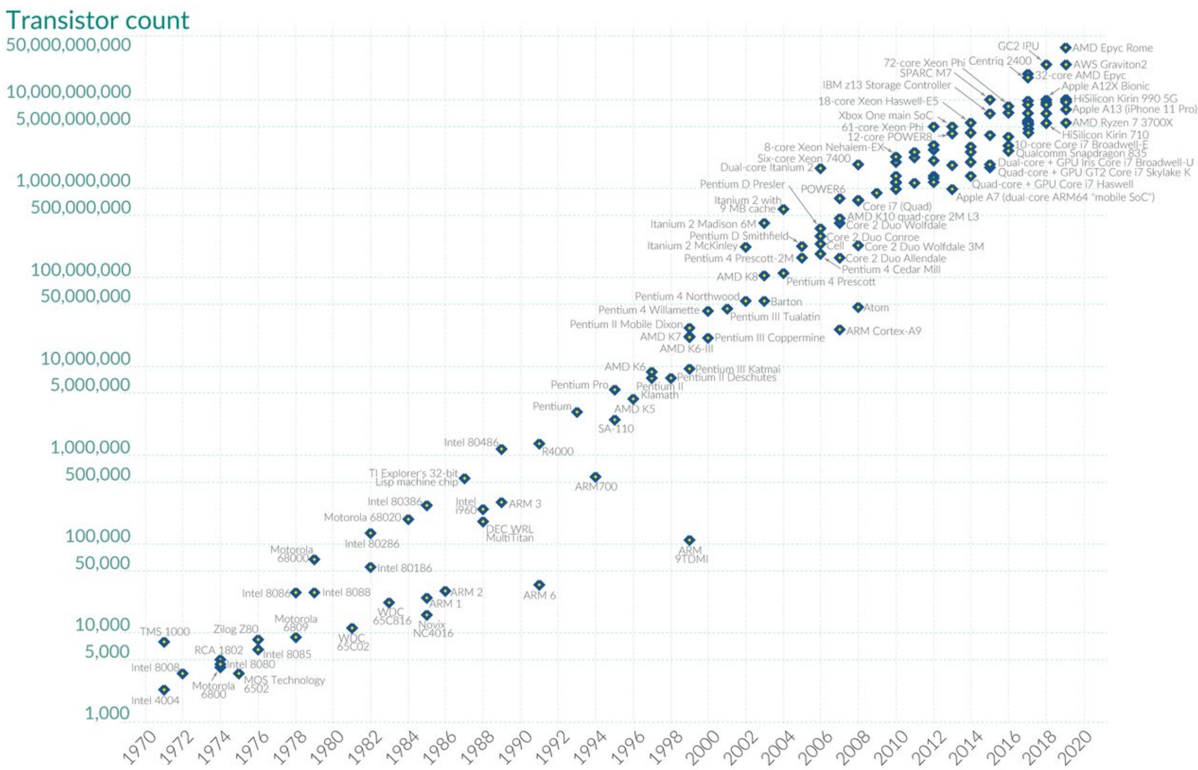
- intermediate zeros are significant
  101 has three sig figs and 202.408 has six sig figs

- scientific notation has the same rules
  $1.5\text{x}10^{-4}$ has two sig figs and $1.5000\text{x}10^{-4}$ has five sig figs

5.2 Microprocessors Introduction

A microprocessor is an integrated circuit providing all the functions of a computer CPU (central processing unit). Microprocessors date to the 1960s and 1970s. In mechatronics we are interested in microprocessors used to control external electromechanical devices such as motors and to read external sensors such as thermocouples; hence, these are called microcontrollers.

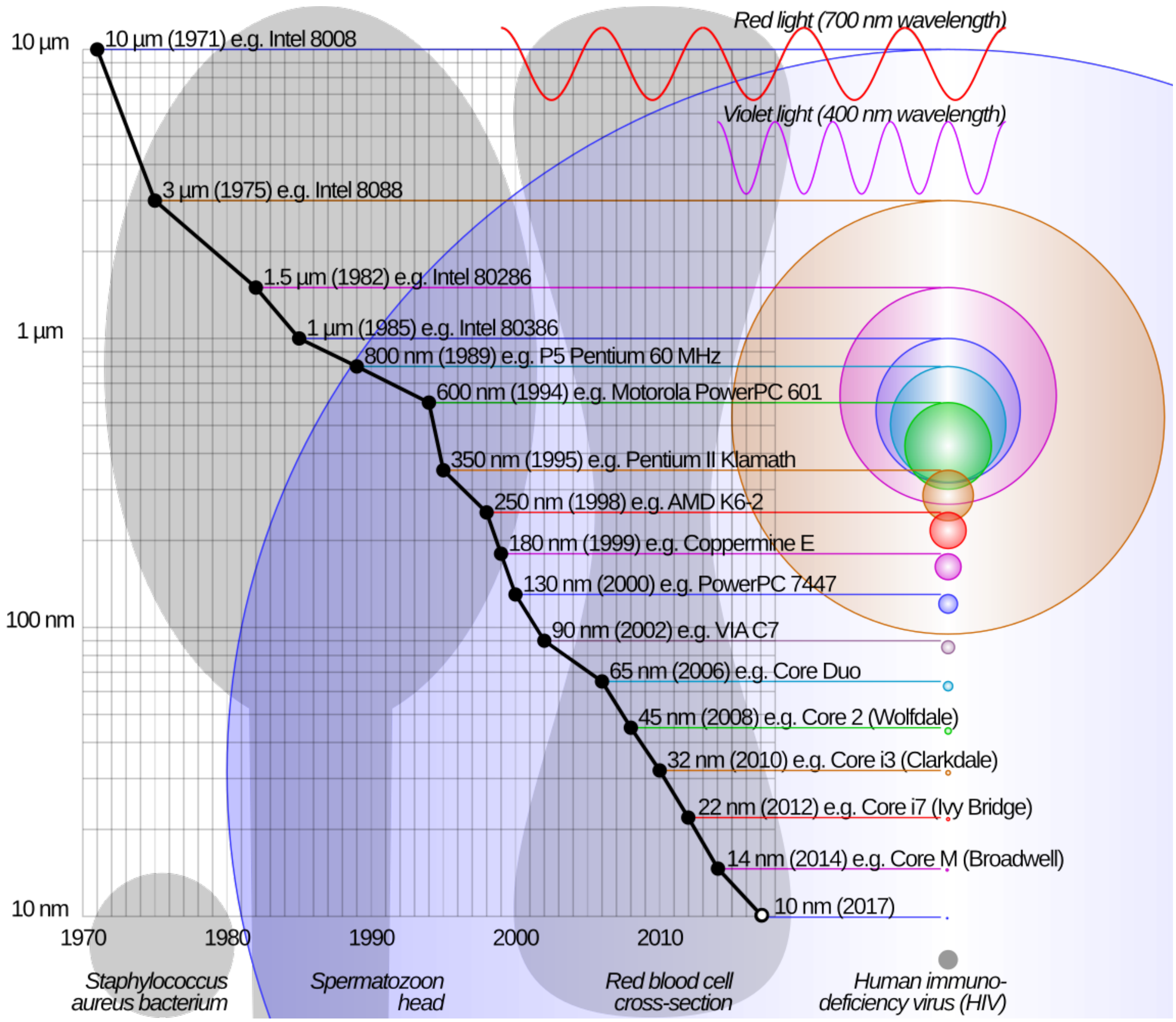
Microprocessors are small computers that are common in applications, relatively cheap, with great power and high functionality. Example applications are industrial automated controls, robotics, mobile phones, microwave ovens, programmable televisions, video game consoles, bathroom scales, etc.



Moore's Law

Moore's Law: (1965) the number of transistors on microchips (density) doubles every two years. The corollary is that cost and size decrease and processing speed and efficiency increase.

(Doubling is not linear, there is an extreme exponential increase represented by the log scale on the abscissa above.)

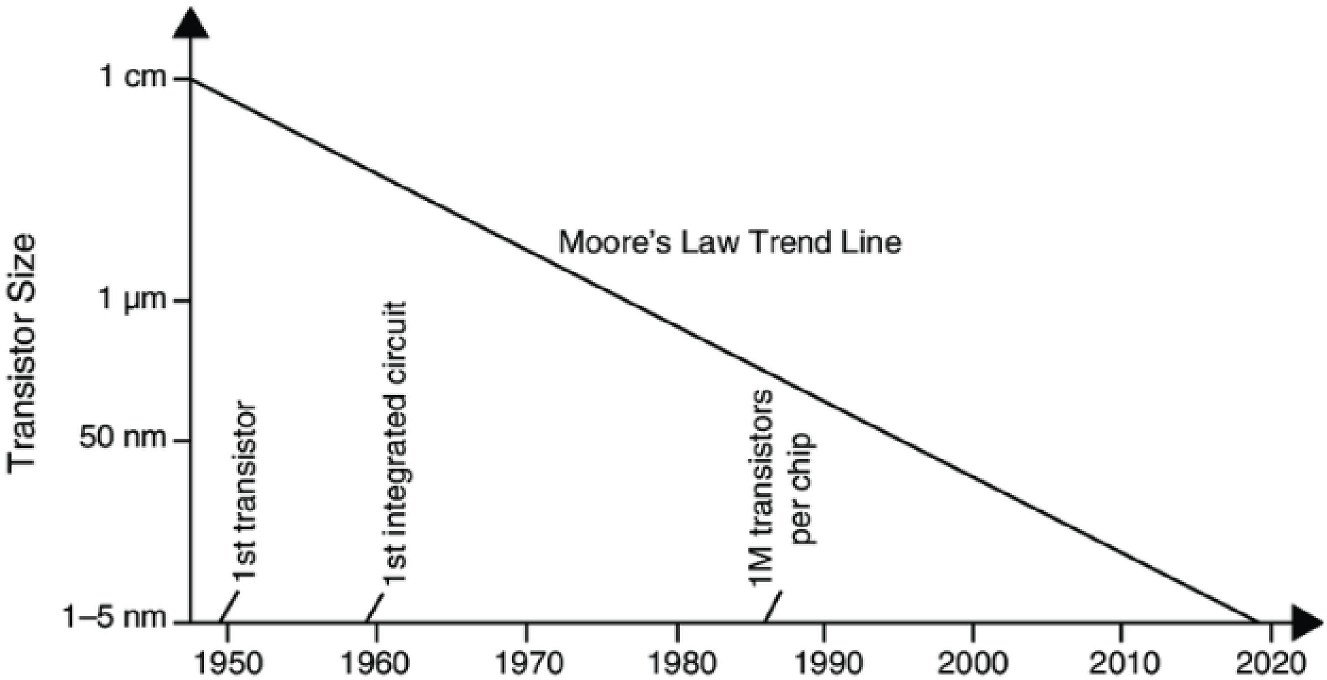


Microprocessor Miniaturization over the Years

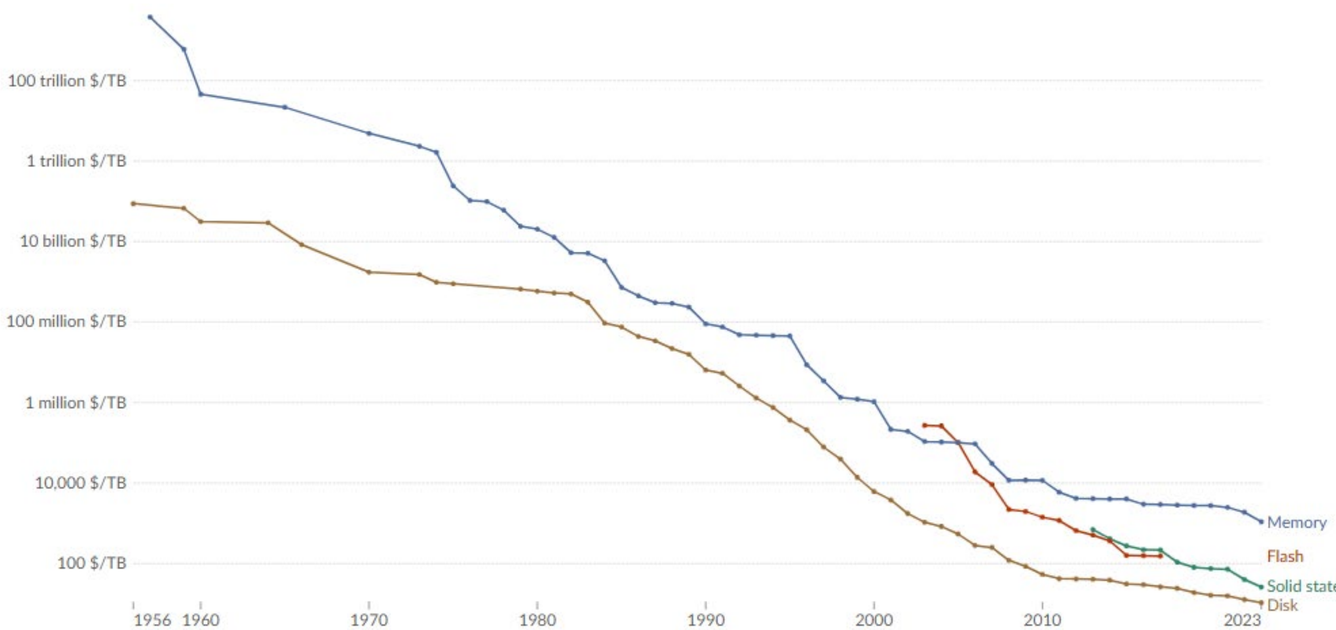
Comparison semiconductor process nodes - Microprocessor chronology - Wikipedia

The figure above shows the progress of microprocessor miniaturization over the years. Again, please note the log scale on the vertical axis above. This vertical axis plots semiconductor manufacturing process nodes size.

The micrometer size (µm) for microprocessors was accomplished in the mid-1980s. It appears the “micro” name stuck, rather than following the ensuing size trends. Shall we rename ‘microprocessors’ to ‘nanoprocessors’?

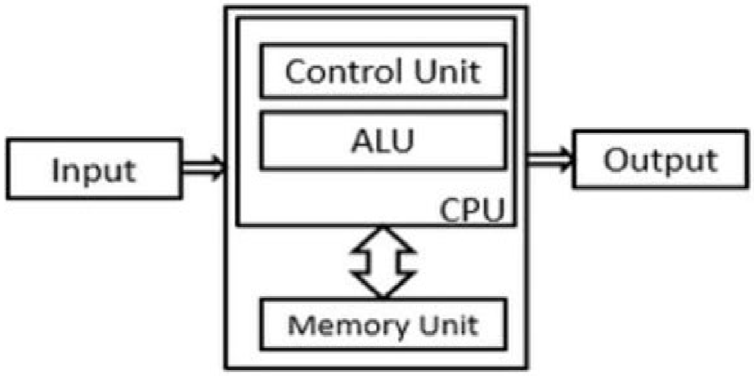


Decreasing Transistor Size over the Decades

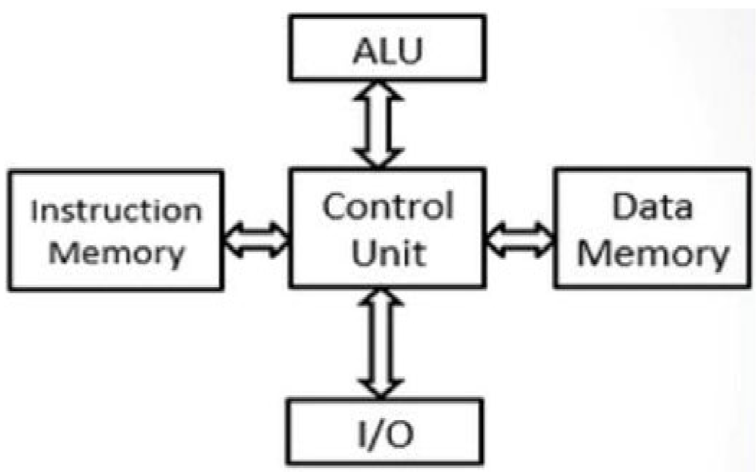


Cost of Computer Memory over the Decades

5.3 Microcontroller Architectures

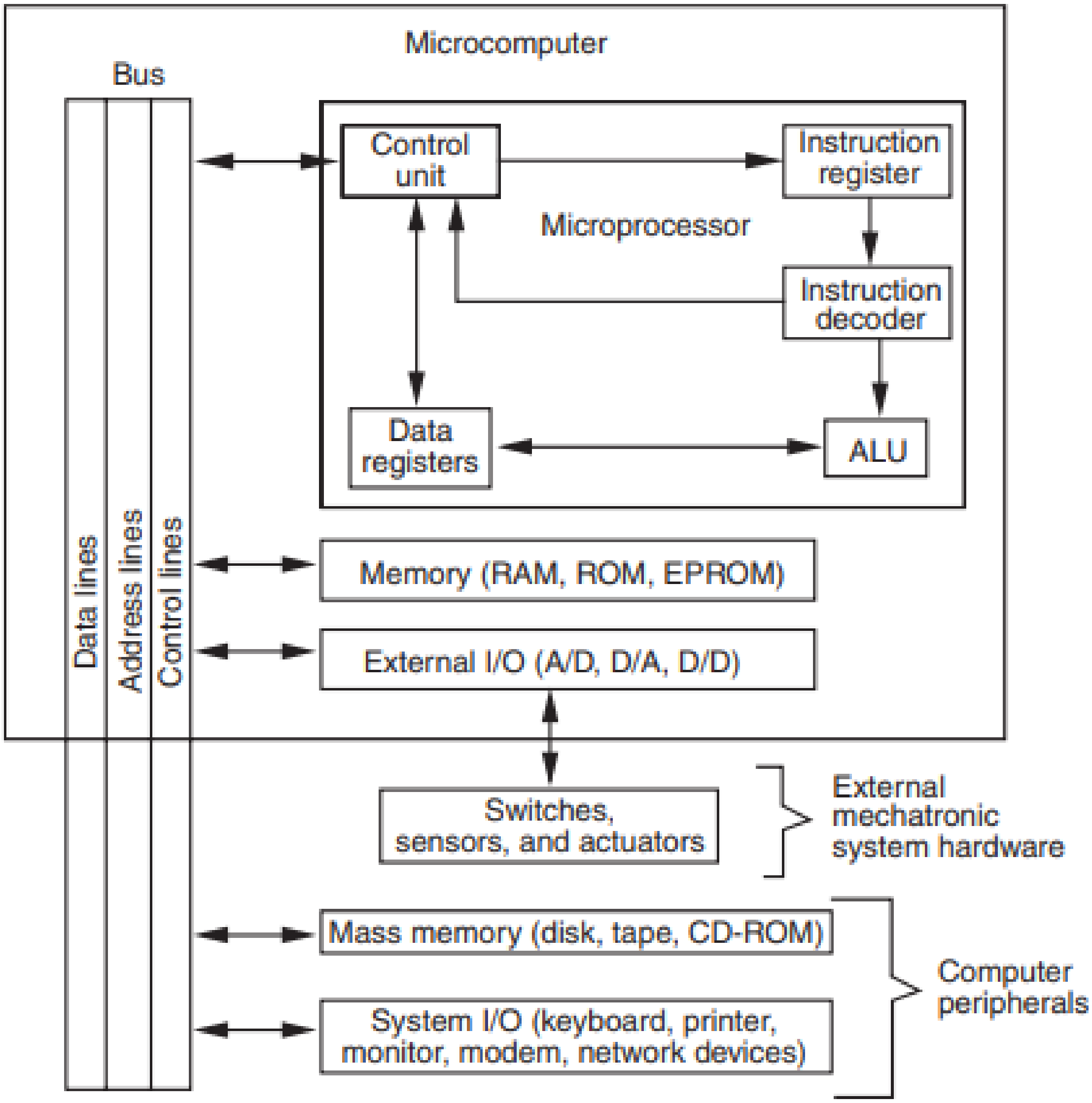


Von Neumann Model Microcontroller Architecture



Harvard Model Microcontroller Architecture

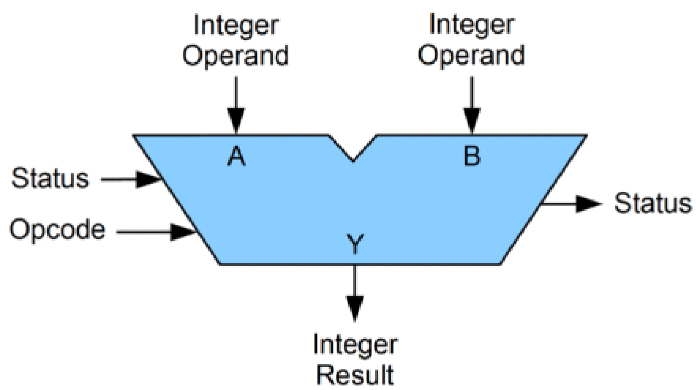
Raspberry Pi Picos in the ME 3550 and 4550 laboratory use the Harvard Microcontroller Architecture Model (as most modern computers do). This helps avoid the risk of allocating code to the wrong memory location.



Basic Microcontroller Structure, Harvard Model

The terms **microprocessor** and **microcontroller** are often used interchangeably. Actually, the microprocessor is the CPU basis of the computer. When adding capability to control external devices and read external sensors, this greater unit is now called a microcontroller.

The ALU is a computer's arithmetic logic unit, performing binary maths operations. The inputs and outputs are binary words (multiple bit data packages such as bytes). The data registers are the working memory for these operations. A bus allows the ALU to communicate with other devices and the CPU. I2C (I squared C) is a common bus standard for communication with external sensors (like inertial measurement units, IMUs).



Arithmetic Logic Unit (ALU)

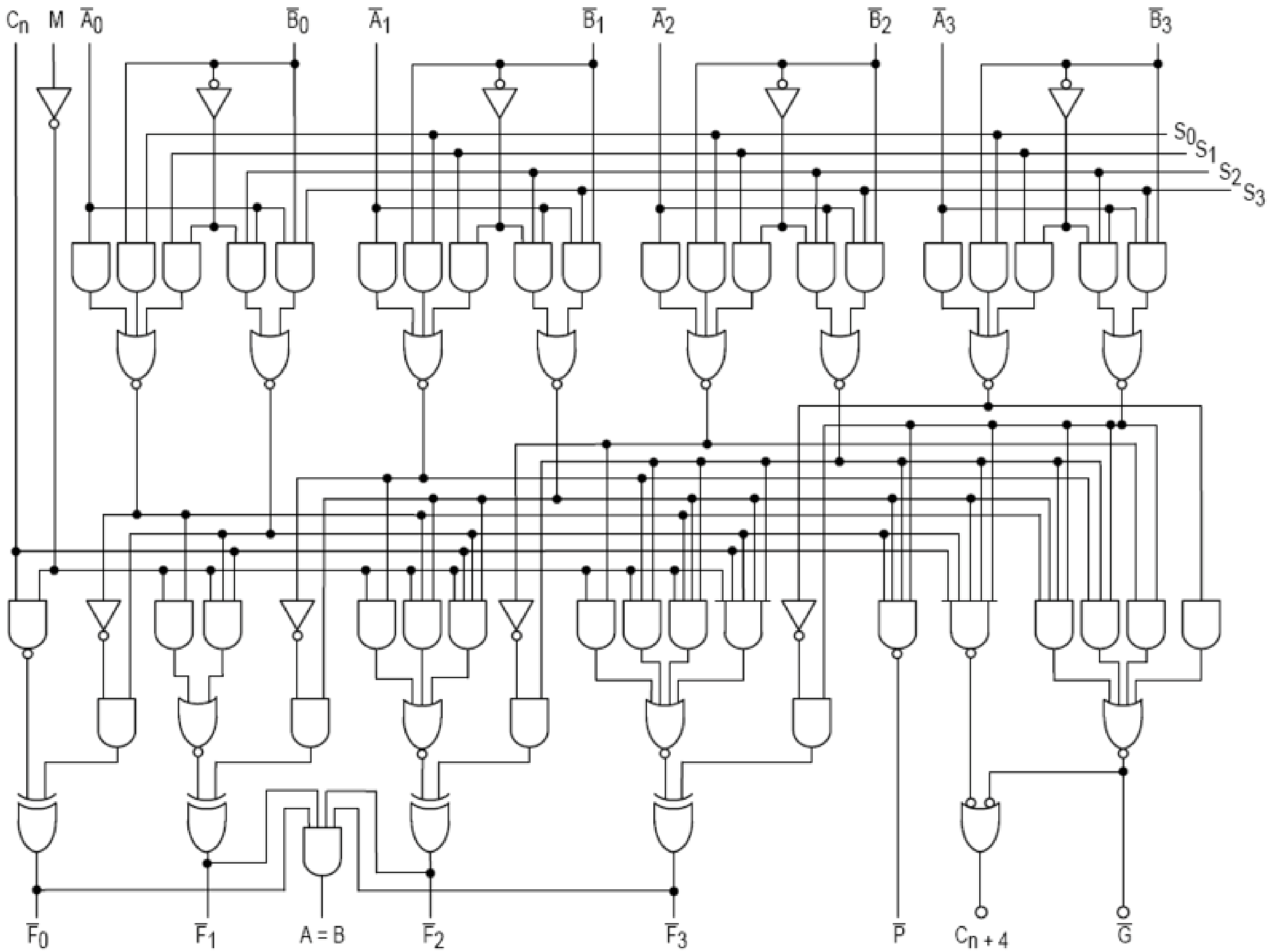
The ALU is used to perform binary arithmetic. The inputs and outputs are binary words (multiple bit packages; e.g. one byte in, one byte out).

Bitwise ALU operations:

- AND
- OR
- XOR (NOT gate)
- Ones Complement

Non-Bitwise ALU operations:

- Twos Complement
- Addition
- Subtraction
- Increment (add 1)
- Decrement (subtract 1)



4-bit ALU

We see this example 4-bit ALU is based on logic gates and transistors.

Memory

- Non-Volatile
  - ROM
  - Flash
  - EPROM
  - EEPROM

- Volatile – RAM

When the microprocessor power is shut off, non-volatile memory remains, but volatile memory is lost. The ROM (read-only memory) is fixed one-time by programming. UV rays erased EPROMs (erasable programmable read-only memory) when the user/programmer removed a protective tape. EEPROMs (electronically-erasable programmable read-only memory) don't require this tape.

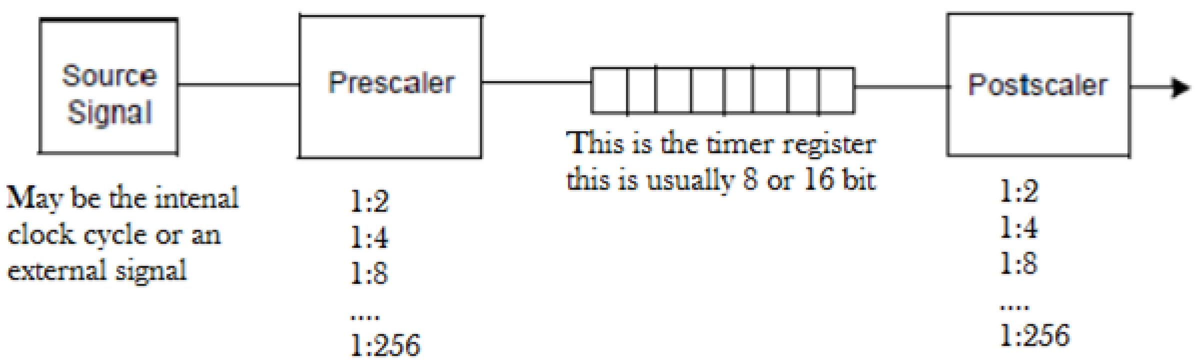
Upon power-up, a chunk of code is typically run. Then another set of code runs on an infinite loop. Thus, **interrupts** are used to interrupt whatever the code is doing at the moment and directs the microprocessor elsewhere.

Interrupts

- Software
- Hardware
- Interrupt Service Routines (ISR)

Without interrupts, a precision timing device (such as a sensor for motor shaft speed), will not be able to measure accurately. This requires a hardware interrupt.

Timers



Microcontroller Timer Diagram

- **Source Signal** internal or external clock. Clock could be same as the processor clock, sometimes different.

- A clock cycle of 100 MHz is too high – the **Prescaler** makes it easier to track clock cycles (for a sensor with a 100 msec delay; 100 MHz is a significantly-different time scale (0.001 msec)).

- The **Register** is counting (incrementing) with 8 or 16 bits depending on the specific processor.

- The **Postscaler** converts back to the required magnitude of time scale.

- General purpose timers; also real-time clock (24-hour scale) for system time; for human interface.

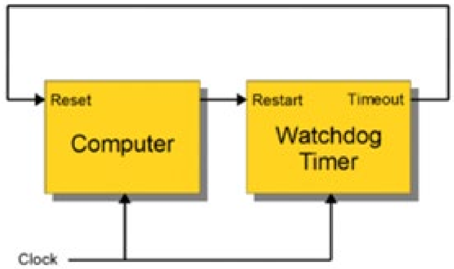
- Button – CMOS (Complementary Metal-Oxide Semiconductor) real-time clock.

- Time starts on power-up, never stop counting until register fills, then resets.

- Delay block to keep to clock cycle (processor does nothing until next time cycle).

- System clock can allow processor to do another task while waiting.

Watchdog Timer



Watchdog Timer (in hardware)

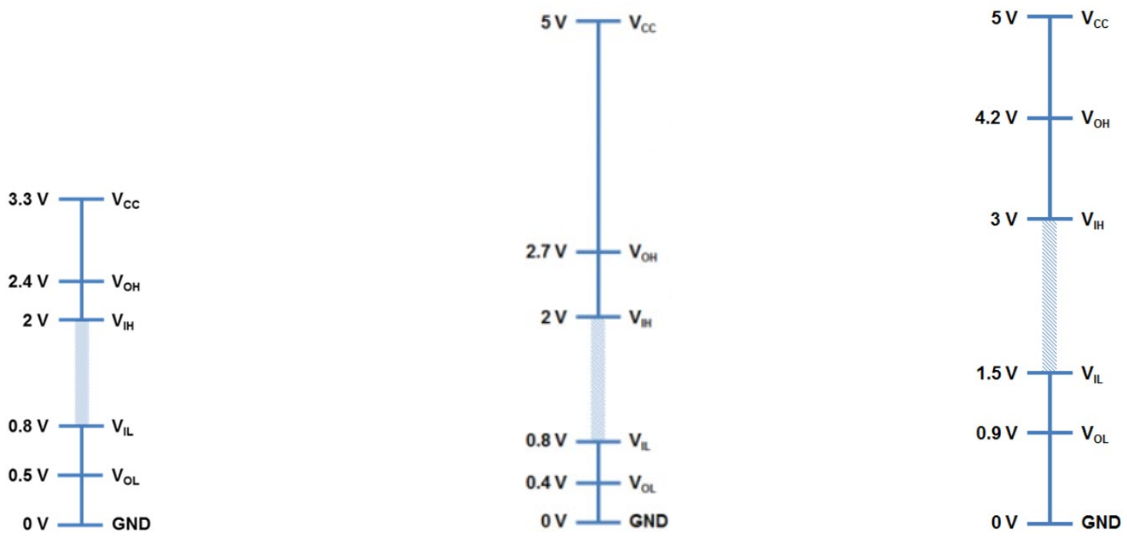
The watchdog timer is implemented in hardware. It ensures recovery in case of a time issue during microprocessor executions.

General-Purpose I/O (GPIO)

- Digital I/O
- Analog I/O
- Peripheral communication (buses)

Buses (I2C and SPI) – use for serial communication within and external to a microcontroller.

Logic Voltage Levels



3.3 V **Standard 5V TTL** **ATMega328**

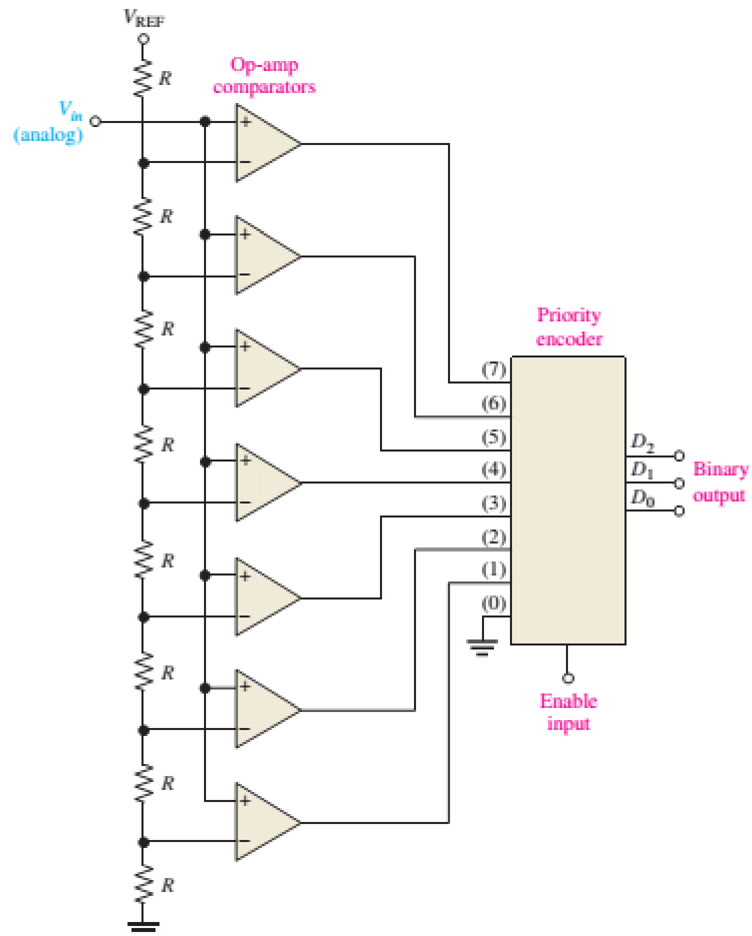
Logic Voltage Levels

High/Low (1/0) Voltages

Convention	High (V)	Low (V)
3.3 V	2.4 – 3.3	0.0 – 0.5
5 V TTL	2.7 – 5.0	0.0 – 0.4
ATMega328	4.2 – 5.0	0.0 – 0.9

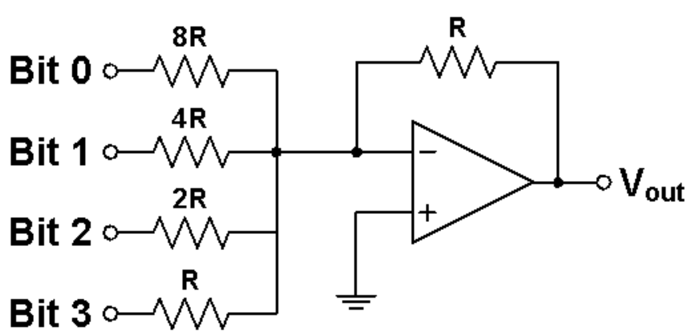
TTL (Time-To-Live) is the amount of time that a data packet is set to exist in a network before being discarded by a router.

The ATMega328 is a standard Arduino microcontroller.



Analog-to-Digital Converter (ADC)

An ADC converts analog voltages from real-world devices to binary representation in a microprocessor. Most are 8-bit (no lower). The more bits, the higher the ADC resolution.



Digital-to-Analog Converter (DAC)

A DAC converts from binary representation in a microprocessor to analog voltages for real-world devices. Most are 8-bit (no lower). There is a summing op-amp configuration; V_{out} is the sum of all bit voltages. There is a shift register for serial communications – both ADC and DAC.

5.4 Raspberry Pi Pico

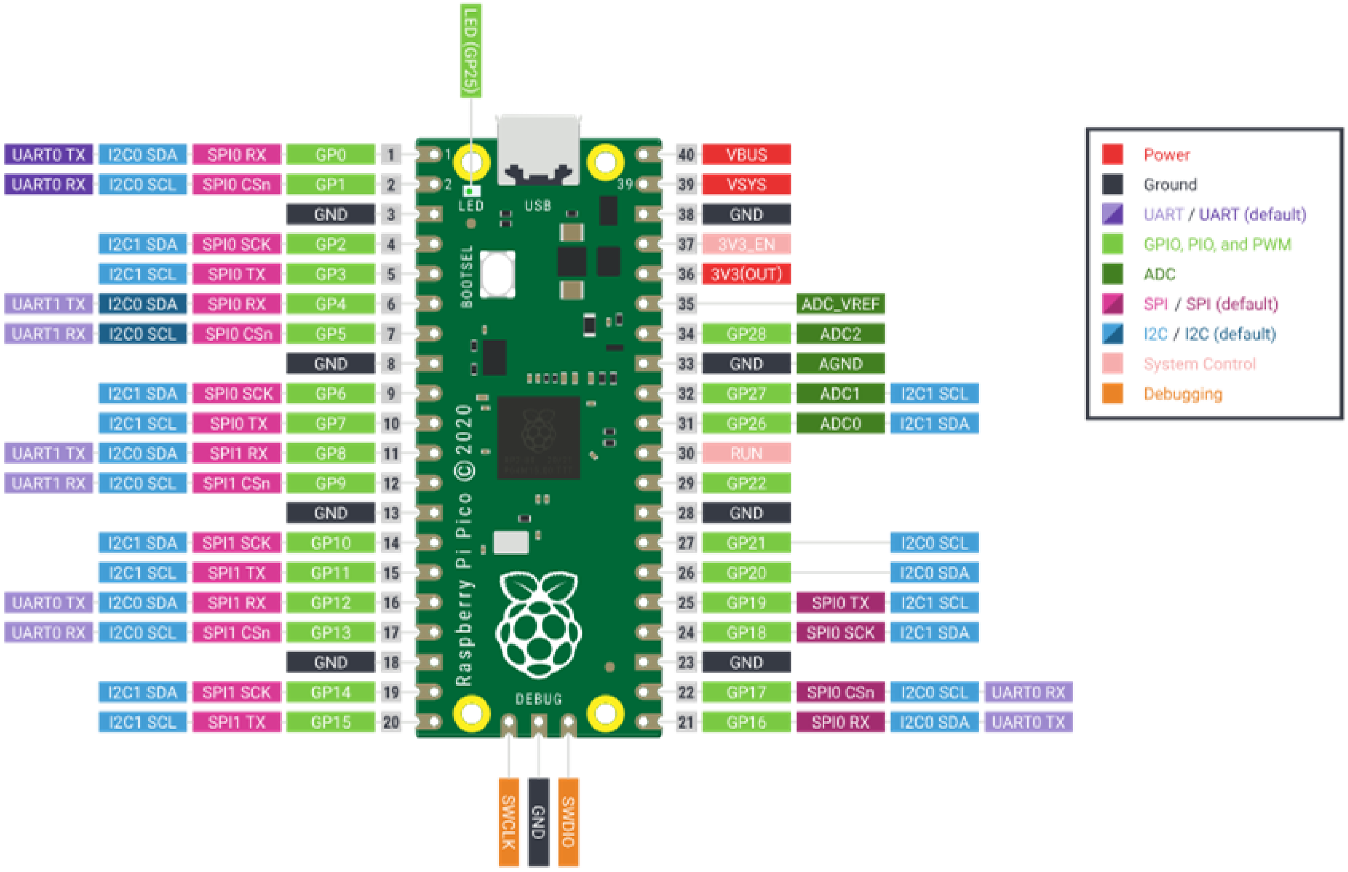
The Raspberry Pi Pico microcontroller is the development board now used in ME 3550 laboratory. It is an entire computer, not just a microprocessor. The Pi Pico has a 32-bit architecture.

"The Raspberry Pi Pico is a low-cost, high-performance microcontroller board with flexible digital interfaces. Key features include:

- RP2040 microcontroller chip designed by Raspberry Pi in the United Kingdom
- Dual-core Arm Cortex M0+ processor, flexible clock running up to 133 MHz
- 264kB of SRAM (**S**tatic **R**andom-**A**ccess **M**emory), and 2MB of on-board flash memory
- USB 1.1 with device and host support
- Low-power sleep and dormant modes
- Drag-and-drop programming using mass storage over USB (**U**niversal **S**erial **B**us)
- 26 × multi-function GPIO (**G**eneral-**P**urpose **I**nput / **O**utput) pins
- 2 × SPI, 2 × I2C, 2 × UART, 3 × 12-bit ADC, 16 × controllable PWM (**P**ulse-**W**idth **M**odulation) channels
- Accurate clock and timer on-chip
- Temperature sensor
- Accelerated floating-point libraries on-chip
- 8 × PIO (**P**rogrammable **I**nput / **O**utput) state machines for custom peripheral support

The Raspberry Pi Pico comes as a castellated module which allows soldering direct to carrier boards"

Pico-series Microcontrollers - Raspberry Pi Documentation



Raspberry Pi Pico Pinout Diagram

The Raspberry Pi Pico has a dual-core processor which can run two operations simultaneously. There are two separate independent processors sharing just some memory. To transfer information, a FIFO (first-in-first-out) buffer is used. There are 26 hardware interrupts.

Memory

- 16 kB ROM (**R**ead-**O**nly **M**emory)
  - non-volatile, configured at factory
  - converts user machine code to make Pi Pico work
  - equivalent of BIOS on desktop computer (executes on power-up)
- 246 kB SRAM (**S**tatic **R**andom-**A**ccess **M**emory)
  - volatile static RAM
  - disappears with power off
- 2 MB Flash
  - non-volatile, stored on board
  - maintains memory with power off
  - stores user program

Static RAM uses a series of latches. Compared to dynamic RAM, static RAM is faster to access, but slower to write.

Dynamic RAM uses capacitors. Small capacitors lose their charge even without a discharge circuit. Therefore, a constantly-refreshing charge is required for dynamic RAM.

GPIO (General-Purpose Input/Output)

- 2 × SPI (**S**erial **P**eripheral **I**nterface)
- 2 × UART (**U**niversal **A**synchronous **R**eceiver-**T**ransmitter)
- 2 × I2C (**I**nter-**I**ntegrated **C**ircuit): two-wire serial interface, *we say 'I squared C'*
- 8 × two-channel controllable PWM (**P**ulse-**W**idth **M**odulation) output (16 MHz)
- 2 × external clock inputs
- 4 × general-purpose clock outputs
- 4 × 12-bit (0 – 4095) ADC (**A**nalog-to-**D**igital **C**onverter) inputs (2^{12} = 4096)

Power

- 3.3 V
- 5 V input pin V bus
- 3.3 V logic level

ADC (**A**nalog-to-**D**igital **C**onverter)

- Voltage resolution: 12-bit, 3.3 V / 4096 counts = 0.8 mV
- ADC value: Input / Voltage Resolution

  Example: 2 V / 0.008 V = 2500 (e.g. 2 V input on ADC pin)

Pins – see previous Raspberry Pi Pico Pinout Diagram in TechniColour.

- Many ground pins are available, in black GND.
- The VBUS (**V**oltage **BUS**) is a USB (**U**niversal **S**erial **B**us) wire supplying nominal 5V input, regulated to the required system voltage of 3.3 V VSYS. There is also a 3.3 V output pin 3V3(OUT).
- A level of 3.3 V out is used for some sensors; 5 V is required by other sensors.
- As seen in forest green, ADC (**A**nalog-to-**D**igital **C**onverter) has five pins, ADC0, ADC1, ADC2, plus ADC reference voltage 3.3 V ADC_VREF, and ground AGND.
- I2C and SPI are only allotted two buses apiece.
- Digital I/O and Analog I/O are available on any lime-green GPIO pin GPi.
- For PWM (**P**ulse-**W**idth **M**odulation) use any lime-green pin GPi.
- The GP25 pin activates the LED (**L**ight-**E**mitting **D**iode).

The Raspberry Pi Pico W has wireless capability; this is not yet available in our ME 3550 / 4550 laboratory.

5.5 Coding

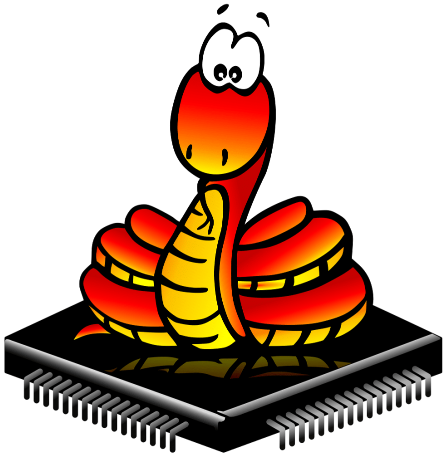
Programming

- The IDE (**I**ntegrated **D**evelopment **E**nvironment) automates a lot of the programming process.
- With the Pi Pico unplugged, hold boot select button, then plug into USB port, release the button, then the Pi Pico appears as a flash drive to the host computer.
- This will be covered in one of our ME 3550 Laboratory experiments.
- The programming language is MicroPython (a hardware implementation of Python for microprocessors).
- A **`.uf2`** file is downloaded to flash memory (this is in machine code, converted to run user programs on the Pi Pico).

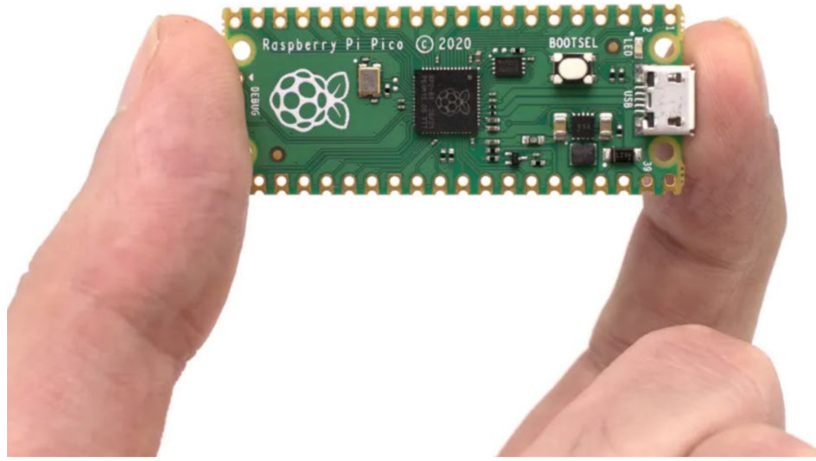
Software

- Microprocessors run Machine Code (1s and 0s), which is not human-readable.
- Assembly code using an assembler is a low-level manipulation-style code. Assembly code is human-readable, as long as the human being is an expert. Assembly code is the fastest, most efficient human-readable code.
- High-Level Languages are used for human-based programming.
    - BASIC
    - C/C++
    - Micropython – in the ME 3550 / 4550 laboratory

In the recent past, the C++ programming language was used with the Arduino microcontroller in ME 3550 laboratory. The change to the Raspberry Pi Pico was made since the Arduino is based on outdated 1990s technology: 8-bit processor, 16 Mhz clock rate, and integer-only number representation and maths operations. In contrast, the Raspberry Pi Pico is a more modern choice (Pentium II-like) with a dual-core 32-bit processor, 133 Mhz clock rate, and floating-point number representation and maths operations. The Pi Pico is still small and economical like the Arduino.



MicroPython Icon



Raspberry Pi Pico

Now in ME 3550 we use the MicroPython programming language and the Raspberry Pi Pico microcontroller. MicroPython is a software implementation of a programming language largely compatible with the Python 3 programming language. It is written in the C programming language and is optimized to run on microcontrollers. MicroPython contains a small subset of the Python standard library. MicroPython runs on microcontrollers with a C compiler and interpreter. Specifically, we use MS Visual Studio for MicroPython Development.

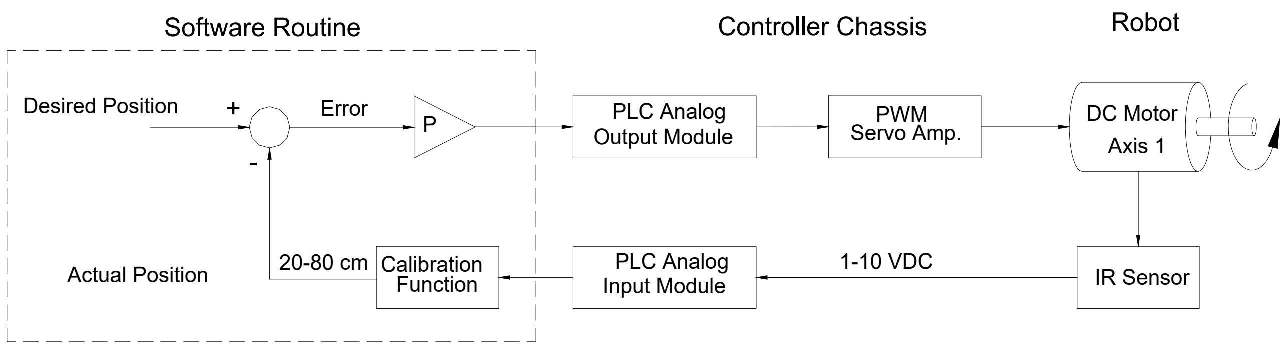
5.6 Programmable Logic Controllers (PLCs)

A Programmable Logic Controller (PLC) is a specialized computer used widely in industrial automation applications. PLCs allow for precise numerical control of machines and processes, increasing industrial efficiency, reliability, and safety. PLCs are small, modular, solid-state computers customized for specific tasks. PLCs have largely replaced mechanical relays, drum sequencers, and cam timers in industrial applications. PLCs are programmed using ladder logic, relying on principles from Section 4.5 Logic Circuits. PLCs can handle a large number of digital and analog I/O.

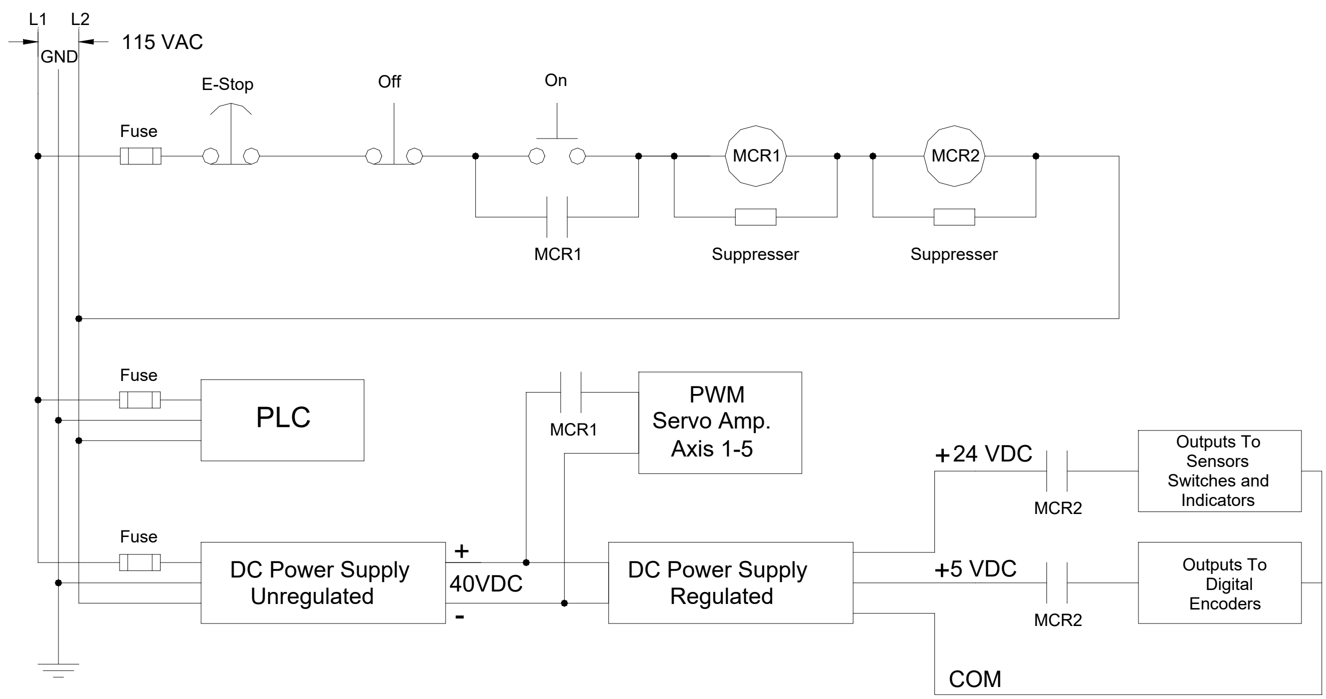
The first PLC, the Modicon 084, was invented by Dick Morley for GM in 1968 to replace hard-wired relay logic systems. Applications include manufacturing, assembly lines, automation, robotics, washing machines, and roller coasters.

Programmable logic controller - Wikipedia

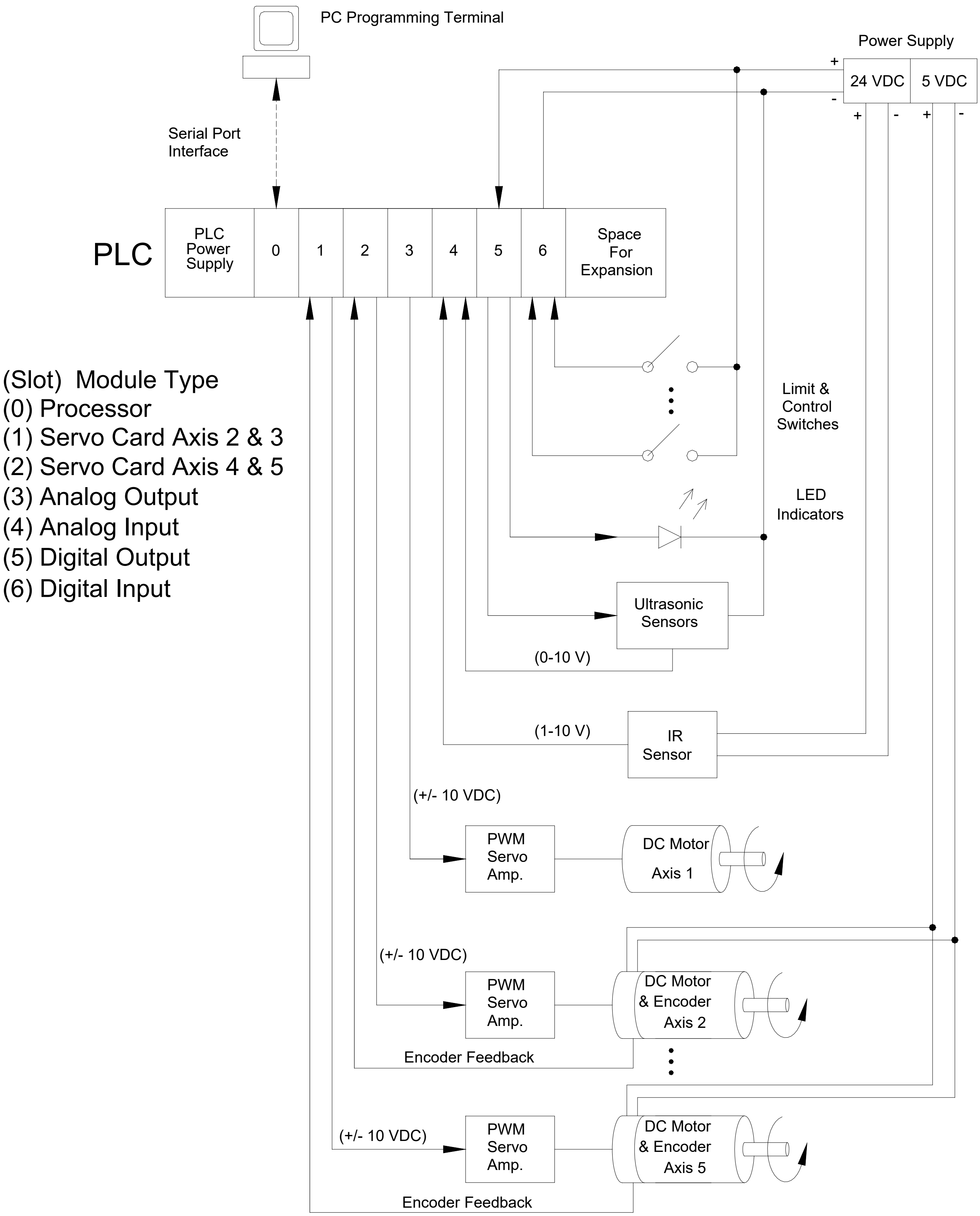
PLC Controller Diagrams



Closed-Loop Position Control of Axis 1

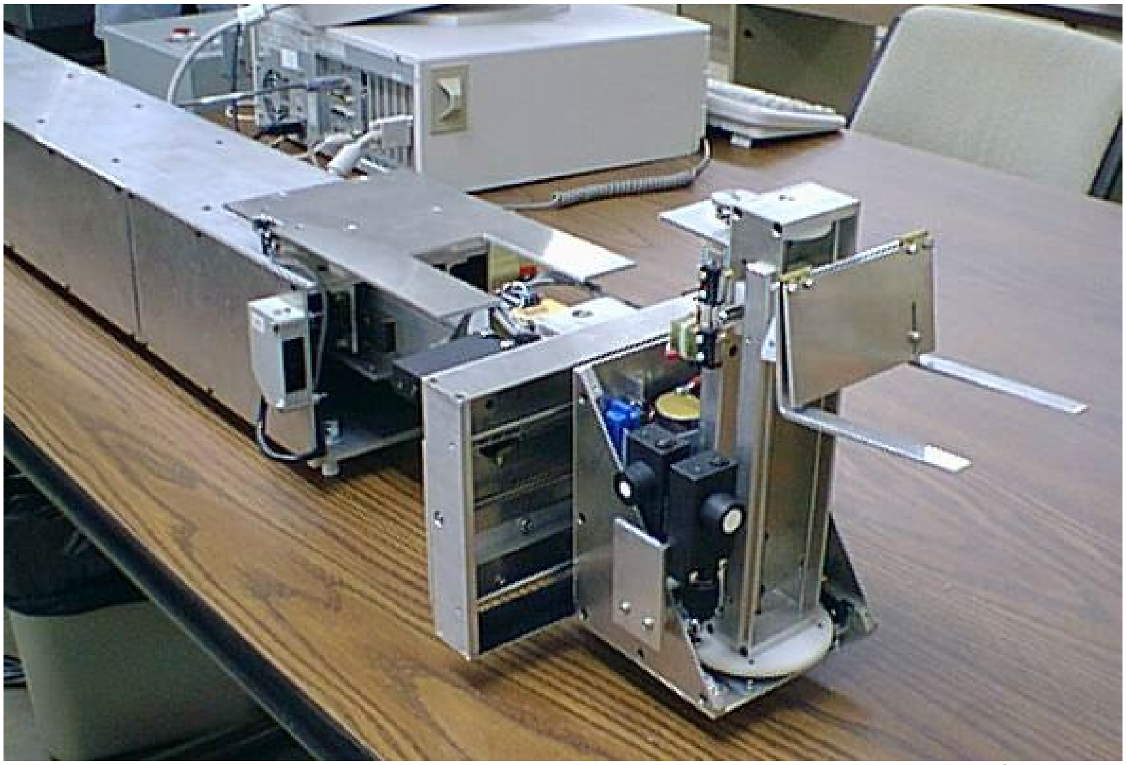


Schematic of Controller Layout and Power Distribution



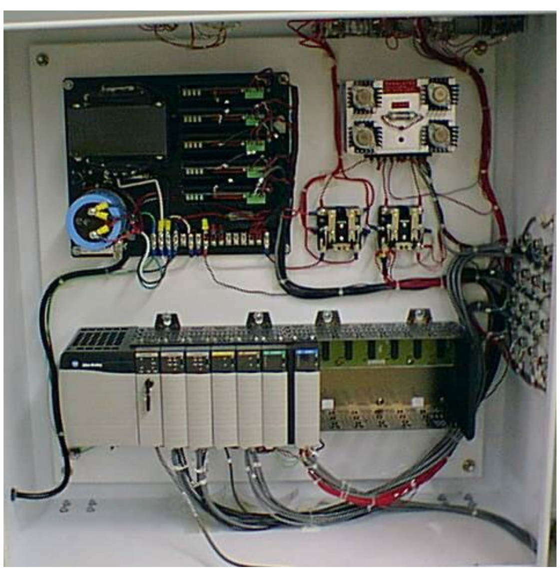
PLC Control System Hardware: Ladder Logic

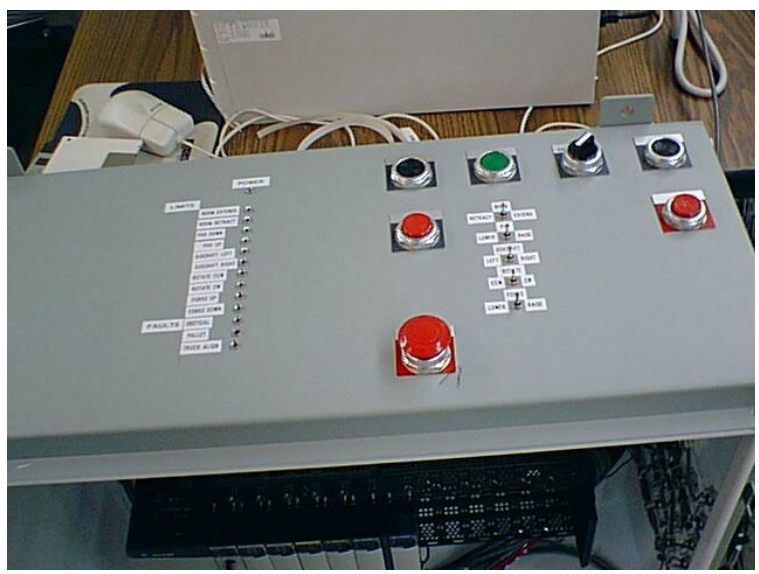
Controlled via PLC in Dr. Bob's Robotics Lab



John Ottersbach, MSME, 1999

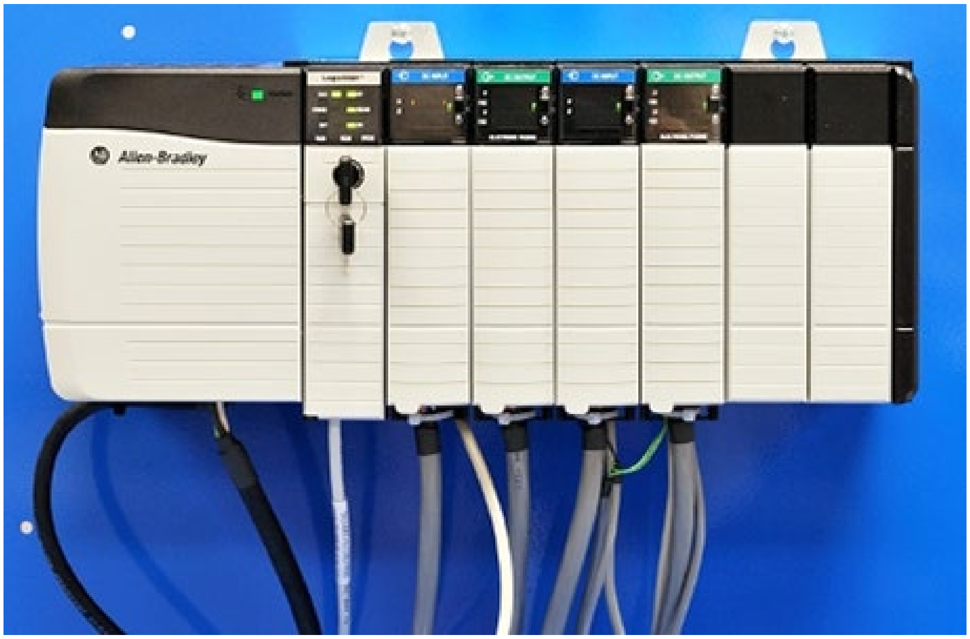
Stewart-Glapat Pallet Handling Device (PHD)



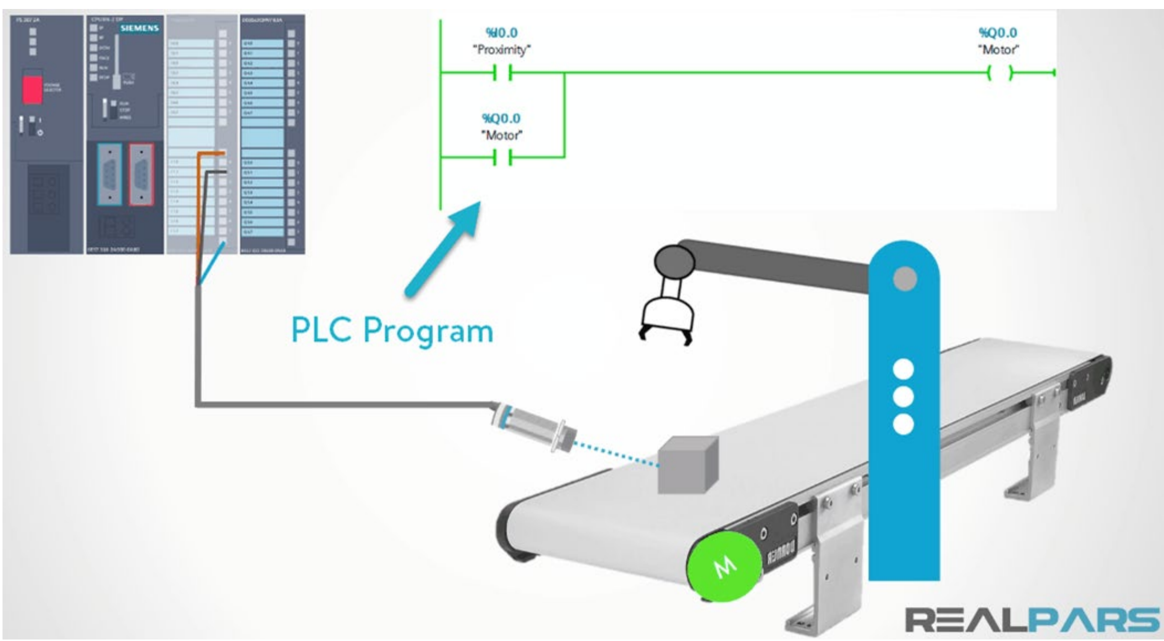


PLC Cabinet

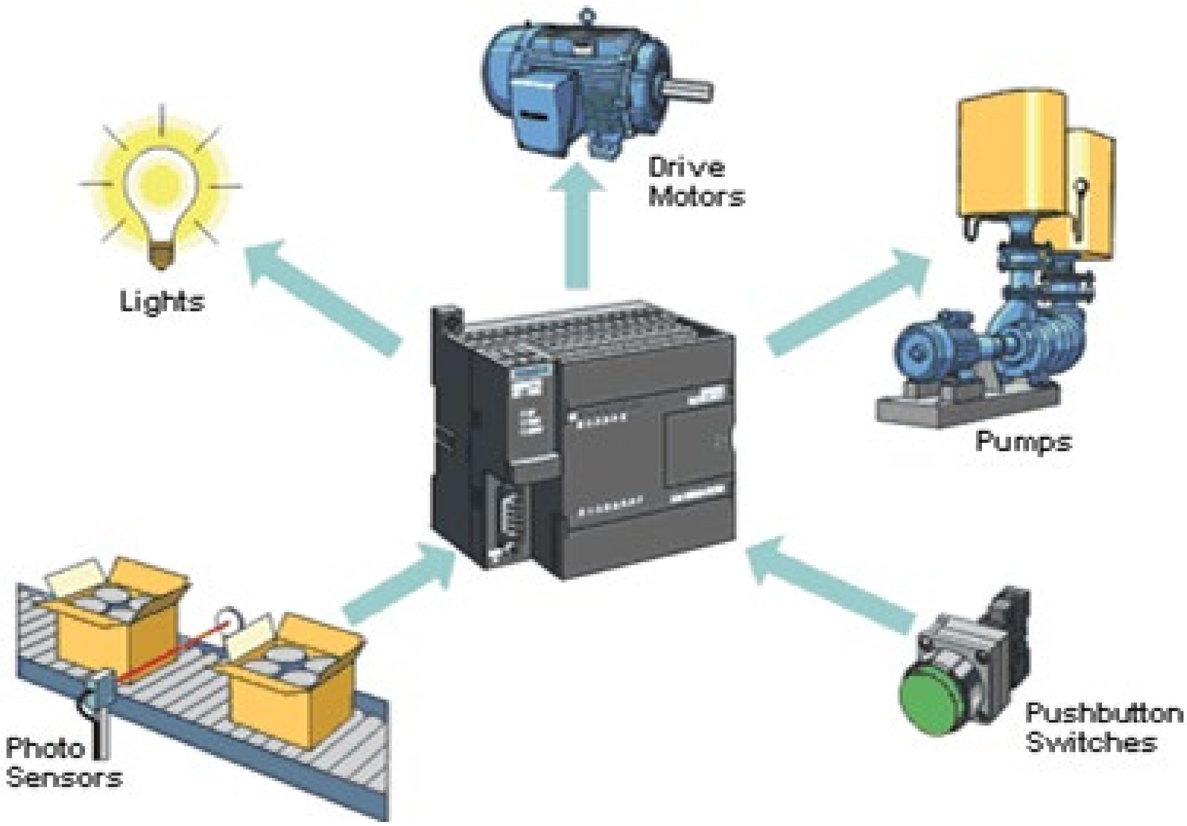
PLC Interface



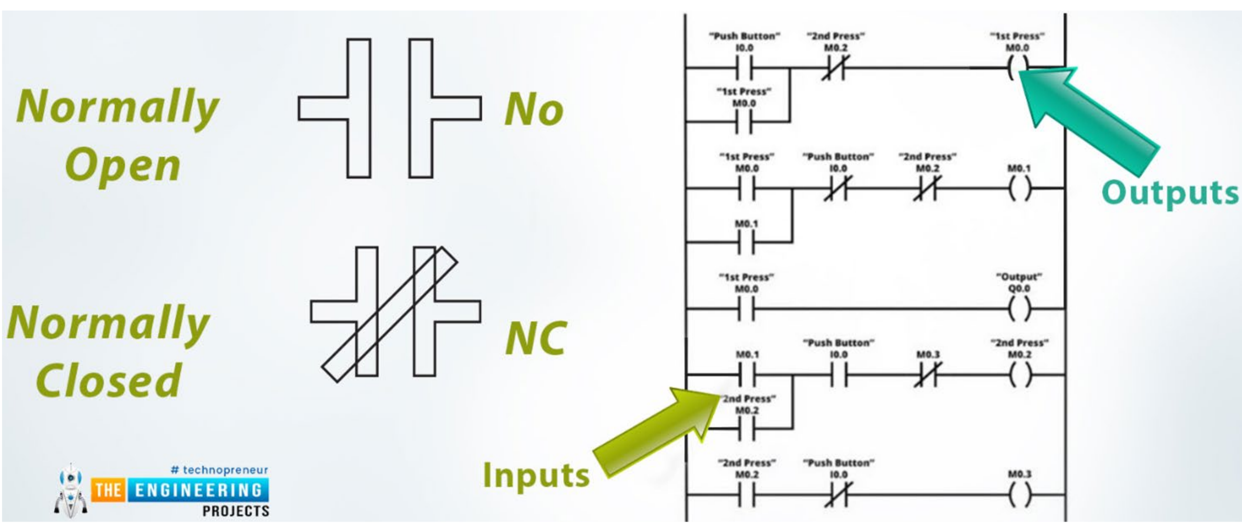
Allen-Bradley Commercial PLC



PLC in a Robotic Workcell



PLC in a Manufacturing Application



PLC Ladder Logic Example

6. DC Electric Motors

This chapter focuses on DC electric motors. The operating principles of DC electric motors are presented, along with brushed DC motors, brushless DC motors, stepper motors, and linear motors. Pulse-width modulation is presented for speed control of DC motors.

Benjamin Franklin experimented with the first electric motors (electrostatic devices) in the 1740s. Understanding of Coulomb's Law and the invention of electrochemical batteries aided electric motor development historically.

An electric motor is a machine that converts electrical energy into (usually rotational) mechanical energy. The basic operation principle is an interaction between the motor's magnetic field and current flowing through the motor windings (coils). The result is torque on the motor's shaft, causing rotation.

In DC motors, **the output torque is linearly proportional to the input current** and **the output shaft angular speed is linearly proportional to the input voltage**.

Electric generators work on the same principles, but in reverse. An external input power source (such as flowing wind or water) rotates the shaft and this produces electrical energy output.

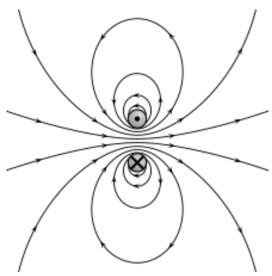
Electric motors can be powered either by DC or AC power sources. ME 3550 and 4550 will focus strictly on DC motors.

Electric motor applications are diverse and widespread. These applications include industrial fans, blowers, and pumps, machine tools, home appliances, power tools, vehicles, and disk drives. Small motors are used in electric watches, and large motors are used in ship propulsion, pipeline compression, and pumped storage (up to 100 MW power).

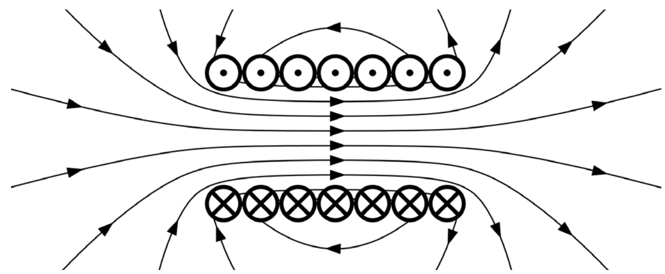
Electric motor - Wikipedia

Electromagnetism

- An electrical current creates a magnetic field.
- Maxwell's equations relate the electrical current to the magnetic field.
- The strength of the magnetic field is related to the level of current and the number of wire turns.



Single-Turn Electromagnet



Multiple-Windings Electromagnet

Rotational Motion using Electromagnetism

Types of DC Electric Motors
- Brushed motors
- Brushless permanent magnet motors
- Stepper motors
- Servomotors

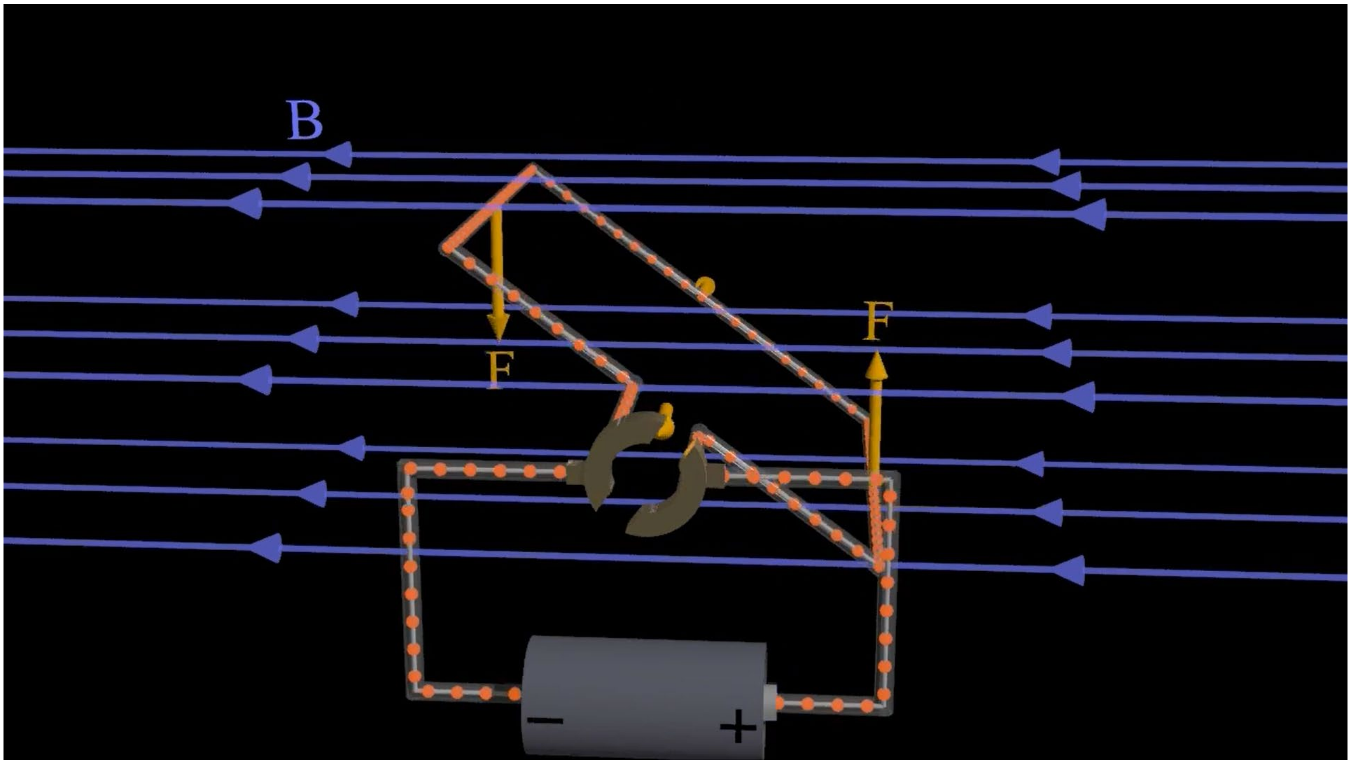
Types of AC Electric Motors
- Synchronous motors
- Asynchronous motors
- Induction motors

It is easier to implement control systems using DC, rather than AC, motors. Therefore, DC motors are generally used to control motor shaft angular position, speed, or torque. It was the invention of AC electric motors that made possible the triumph of Tesla's AC vs. Edison's DC current delivery.

DC motors are used exclusively in ME 3550 and ME 4550 labs.

Electric Motor Operation Principle

The output torque in a DC electric motor is created by magnetic forces exerted on the electric current carried by the rotating part (rotor) of the motor. In the figure below, conventional current is shown flowing through the rotor, and the magnetic field **B** is represented by the blue lines.



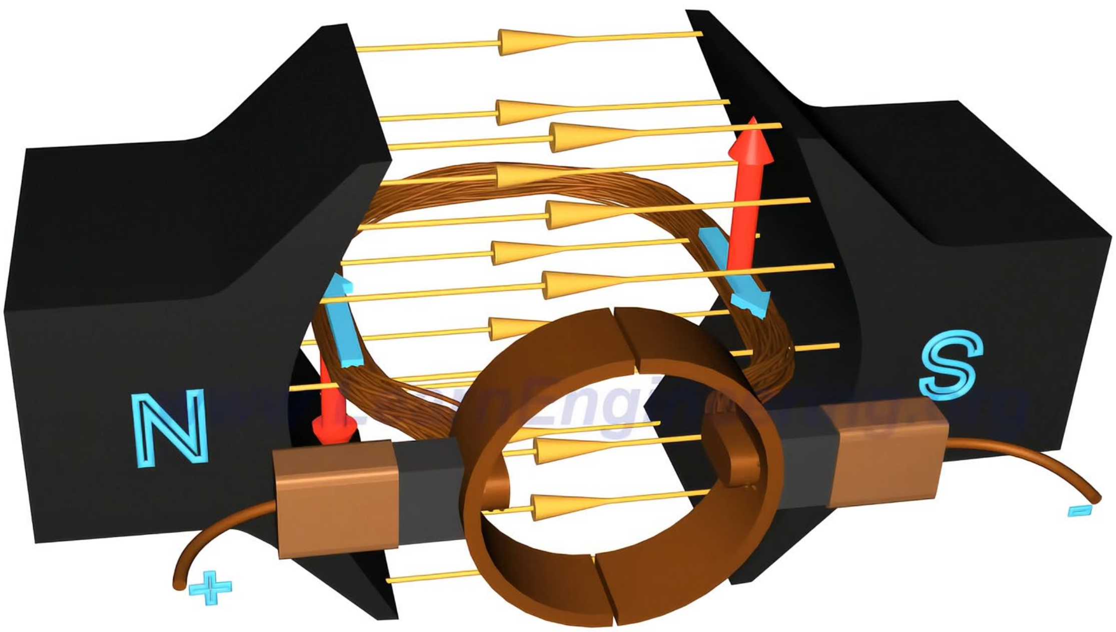
Forces from Current in a Magnetic Field

www.youtube.com/watch?v=aMH7pdn-qr4

The unbalanced vertical forces generate torque at the motor shaft (not shown). The associated axial (horizontal) forces are balanced and do not contribute to the output torque.

The DC power source (battery) shown produces a constant voltage that can only drive the current in one direction. To avoid dead spots when the rotor is vertical, the current must be able to change direction. This is achieved by the commutator (the split copper disc), wherein the DC current switches direction every half rotation (in this single coil example).

As shown in the figure below, the stator (with two north/south pole permanent magnets) provides a constant magnetic field (the linear gold vectors). The rotor (armature) is a coil of copper wire. The armature is connected to a DC power source through the pair of commutator rings. When current flows through the armature coil, an electromagnetic force is induced upon in according to Lorentz' Law. This force couple (the red vectors) causes a torque, which rotates the rotor.



Torque from Electromagnetic Force Couple

www.youtube.com/watch?v=LAtPHANEfQo

According to Lorentz' Law, the motor output force $\vec{F}$ acting on the armature is proportional to the current and magnetic field:

$$\vec{F} = i\vec{L} \times \vec{B}$$

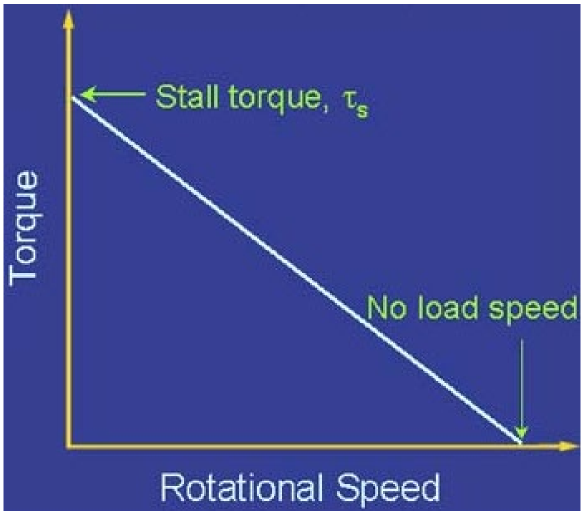
where $\vec{L}$ is the vector (whose magnitude is the wire length, and whose direction is along the wire in the direction of current i), and $\vec{B}$ is the magnetic field.

As the coil rotates, the commutator rings connect with the power source of opposite polarity such that on the left side, electricity will always flow "away", while on the right side electricity will always flow "toward". This means that the torque will always be in the same direction throughout the rotational motion.

However, when this single coil is vertical, there is no torque instantaneously. Armature inertia in motion can overcome this dead spot, but a single-coil DC motor will run rough and irregularly. To fix this, multiple coils are used, generally a minimum of three, spaced symmetrically with respect to the motor shaft.

Electric Motor Torque vs. Speed Curves

For standard DC electric motors, the shaft torque τ vs. angular shaft speed ω is a linear inverse relationship as shown in the plot below.



DC Motor Torque / Speed Curve

D.C. Motor Torque/Speed Curve Tutorial:::Understanding Motor Characteristics

This linear relationship is not theoretical, but holds good experimentally for the entire useful range of a DC motor. Let τ_S be the stall torque (maximum torque, at zero speed), and let ω_{NL} be the no-load speed (maximum speed, at zero torque). Then the motor torque for any speed can be simply calculated from this straight line:

$$\tau(\omega) = \tau_S - \frac{\tau_S}{\omega_{NL}}\omega = \tau_S\left[1 - \frac{\omega}{\omega_{NL}}\right]$$

The motor torque τ equation above will take the units of the torque supplied by the engineer (e.g. Nm). Torque-speed curves are often plotted in units of Nm vs. rpm. This is fine since the units of ω cancel in the above equation for $\tau(\omega)$.

The DC electric motor power P, a function of ω, is the product of τ and ω:

$$P(\omega) = \tau\omega = \tau_S\omega\left[1 - \frac{\omega}{\omega_{NL}}\right]$$

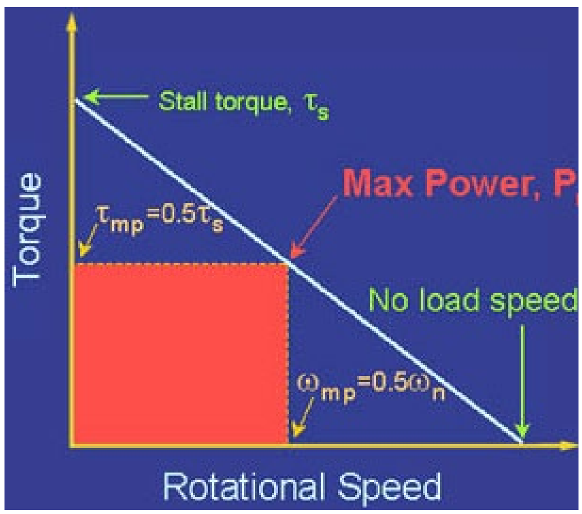
The SI units of power are Watts (Joules/sec, i.e. Nm/sec). In order for the power equation to yield correct SI units of Watts, it is **essential** that the ω units are rad/sec in the $P(\omega)$ equation above. After the power calculation, the units of ω may be converted to rpm for plotting, if so desired.

The power P for any τ and ω combination is the area of a rectangle from the origin to the inverse relationship line on the torque/speed plot above (show some cases). From the equation above we see the motor power is a flipped parabola (opening down) as a function of motor speed ω.

Due to the linear inverse relationship between torque τ and speed ω, the maximum power occurs for half of the stall torque τ_S, which is also for half of the no-load speed ω_{NL}:

$$P_{MAX} = \frac{\tau_S}{2}\frac{\omega_{NL}}{2} = \frac{1}{4}\tau_S\omega_{NL}$$

Since high power requires high torque and high speed, and these two terms are inversely related, the maximum power occurs in the middle of both variables, striking a compromise between these competing factors.

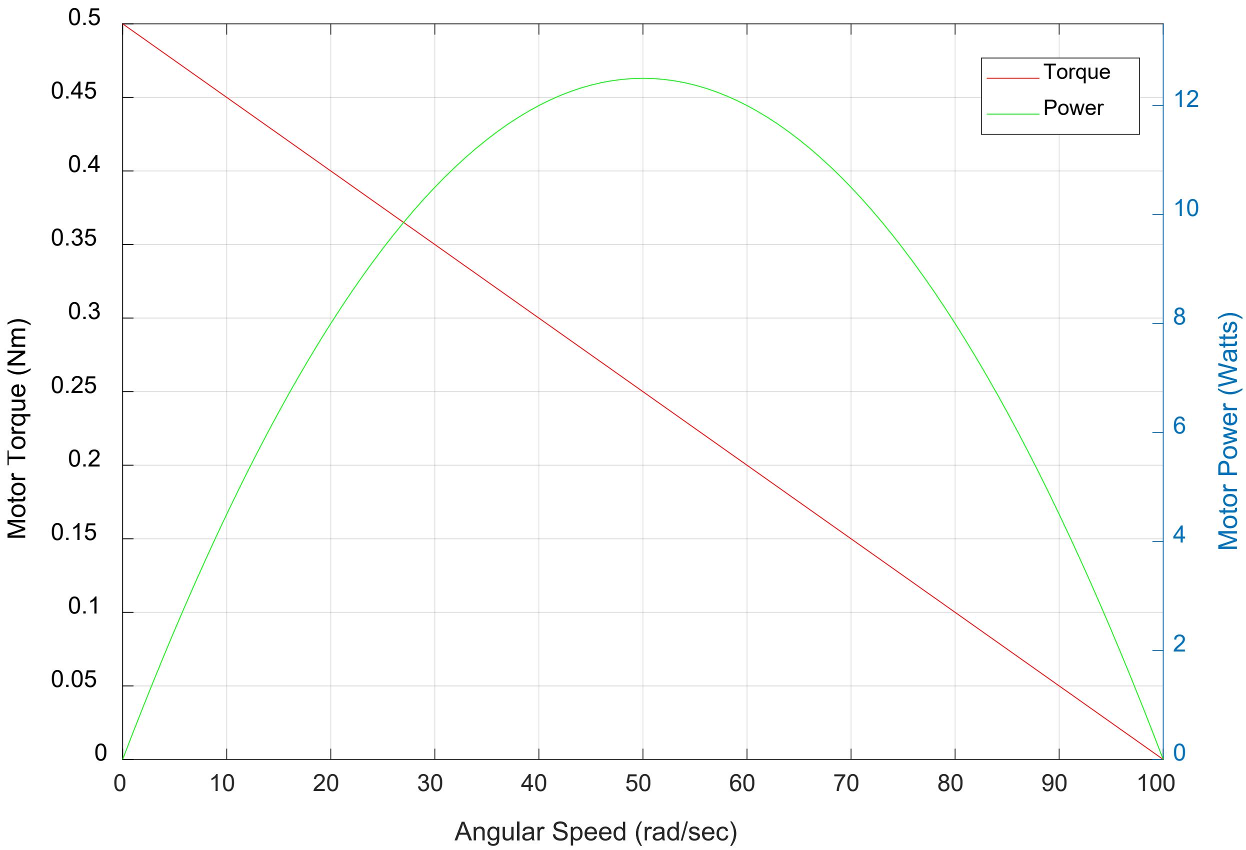


DC Motor Torque / Speed Curve with Max Power

D.C. Motor Torque/Speed Curve Tutorial:::Understanding Motor Characteristics

Example Torque vs. Speed and Power vs. Speed Curves

For a certain DC electric motor we are given stall torque $\tau_S = 0.5$ Nm and no-load angular speed $\omega_{NL} = 100$ rad/sec (954.9 rpm). Based on the equations presented above, plot the associated Torque vs. Speed and Power vs. Speed Curves.



Example DC Motor Torque and Power vs. Speed Curves

The maximum power is:

$$P_{MAX} = \frac{1}{4}\tau_S\omega_{NL} = \frac{0.5(100)}{4} = 12.5 \text{ Watts}$$

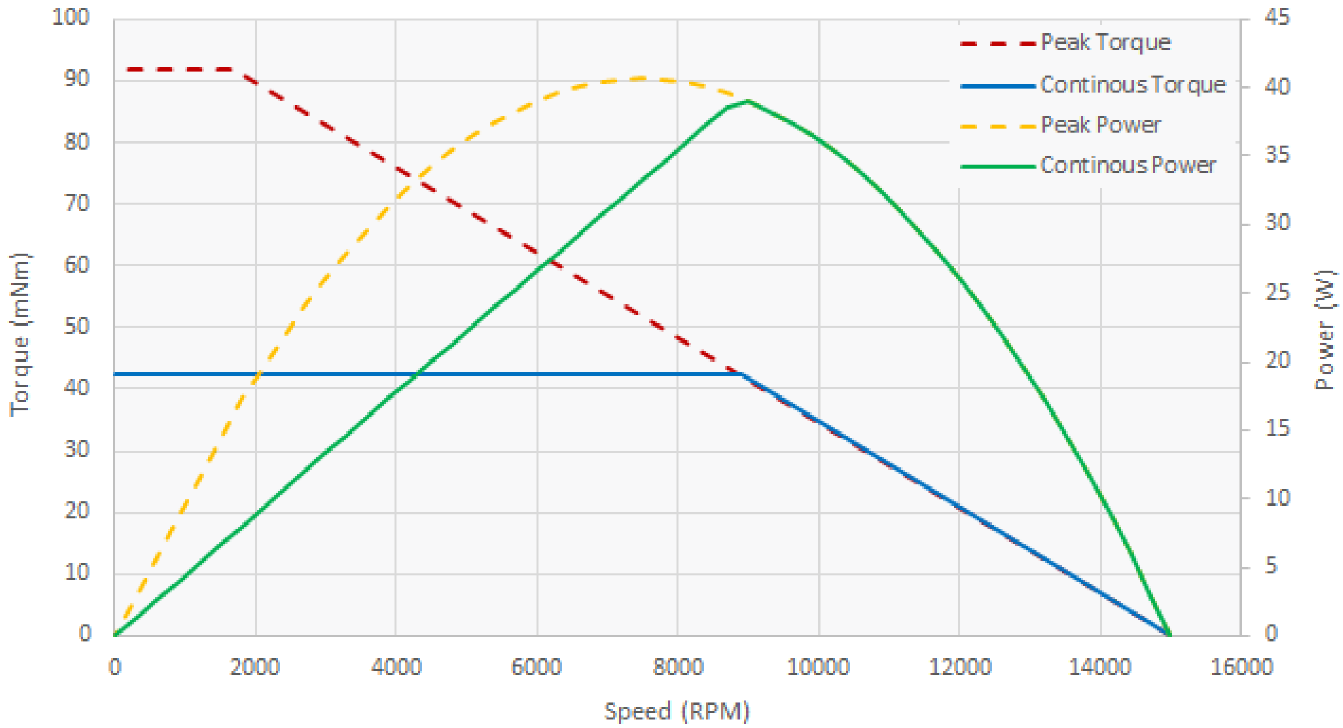
at:

$$\tau = \frac{\tau_S}{2} = 0.25 \text{ Nm} \qquad \omega = \frac{\omega_{NL}}{2} = 50 \text{ rad/sec}$$

as seen in the plot above.

Example concludes.

Here are similar torque and power vs. speed curves, from Dr. Wilhelm, incorporating the concepts of peak vs. continuous motor torque τ and power P as a function of motor speed ω.



Peak and Continuous Torque and Power vs. Speed Curves

6.1 Brushed DC Motors

Standard DC motors use wound coils of wire to create a magnetic field within permanent magnets. Brushed DC motors have four main components:

1. The Stator

The stationary stator provides a continuous magnetic field, from permanent magnets (or electromagnets in larger DC motors).

2. The Rotor

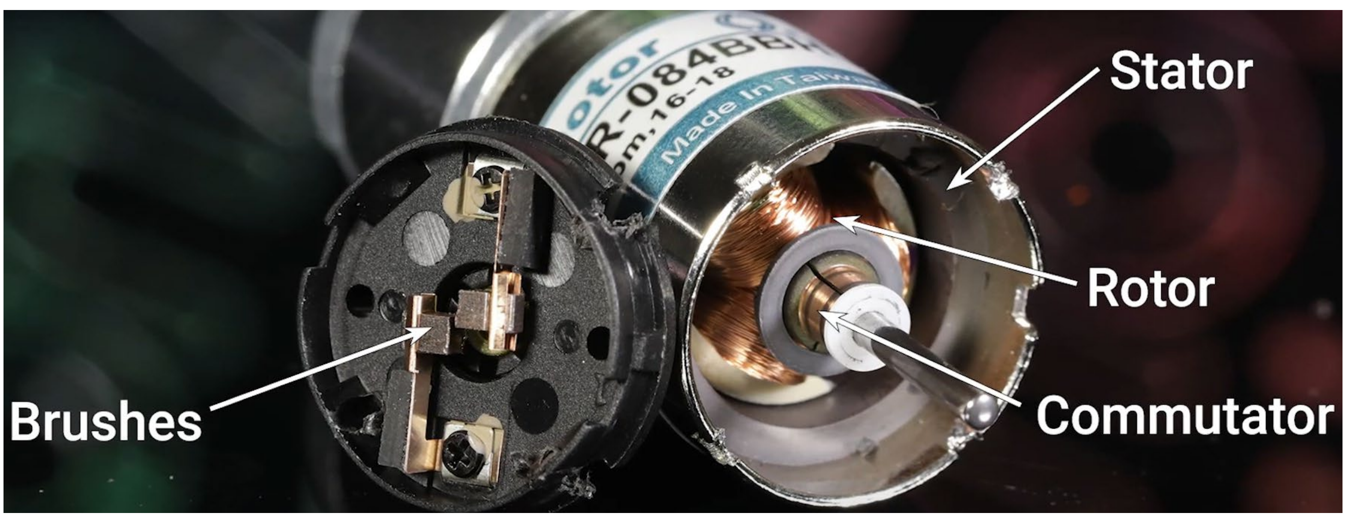
The rotor (aka motor shaft, aka armature) rotates to provide output angular velocity and torque. The aforementioned coils are part of the rotor. The motor is made of iron.

3. The Commutator

The commutator is connected to the brushes. The more rotor coils and commutators included in a brushed DC motor design, the smoother the resulting output rotation and torque will be. Generally three coils (separated symmetrically by 120 deg) is the minimum number in standard brushed DC motors.

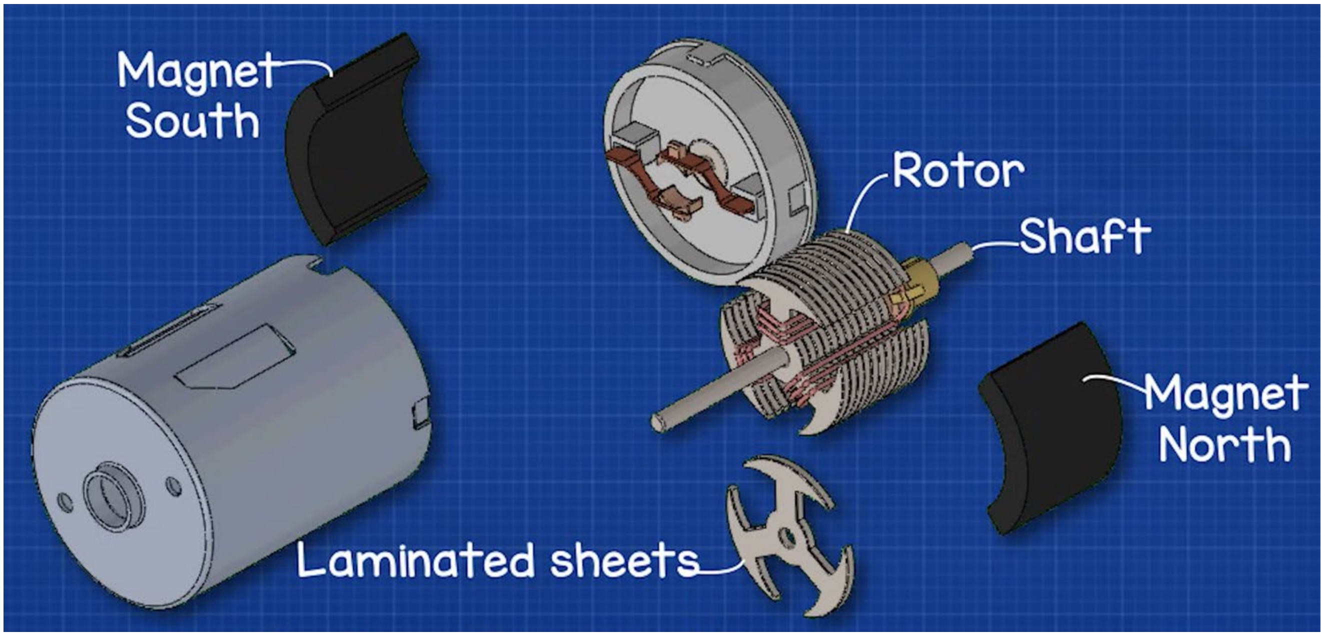
4. The Brushes

The brushes are connected to the DC power supply. The brushes physically rub against the commutator, providing a path for the DC electricity to flow. Due to this friction, contact, and electric arcing, the brushes on this type of DC motor will wear out over time and usage.



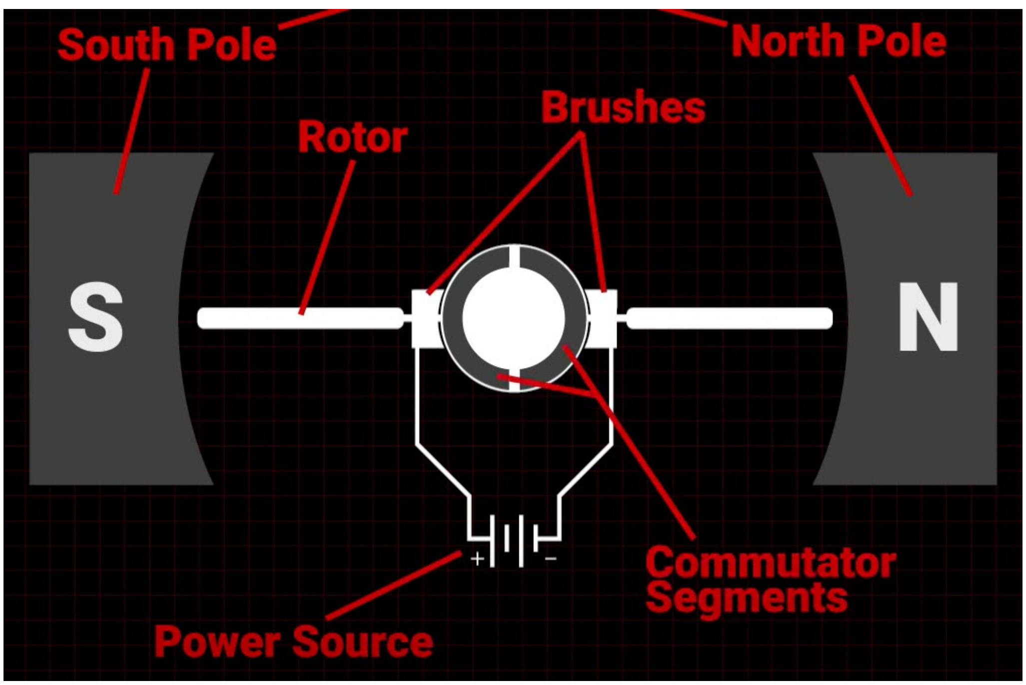
Parts of a Brushed DC Motor

Bing Videos



Parts of a Brushed DC Motor II

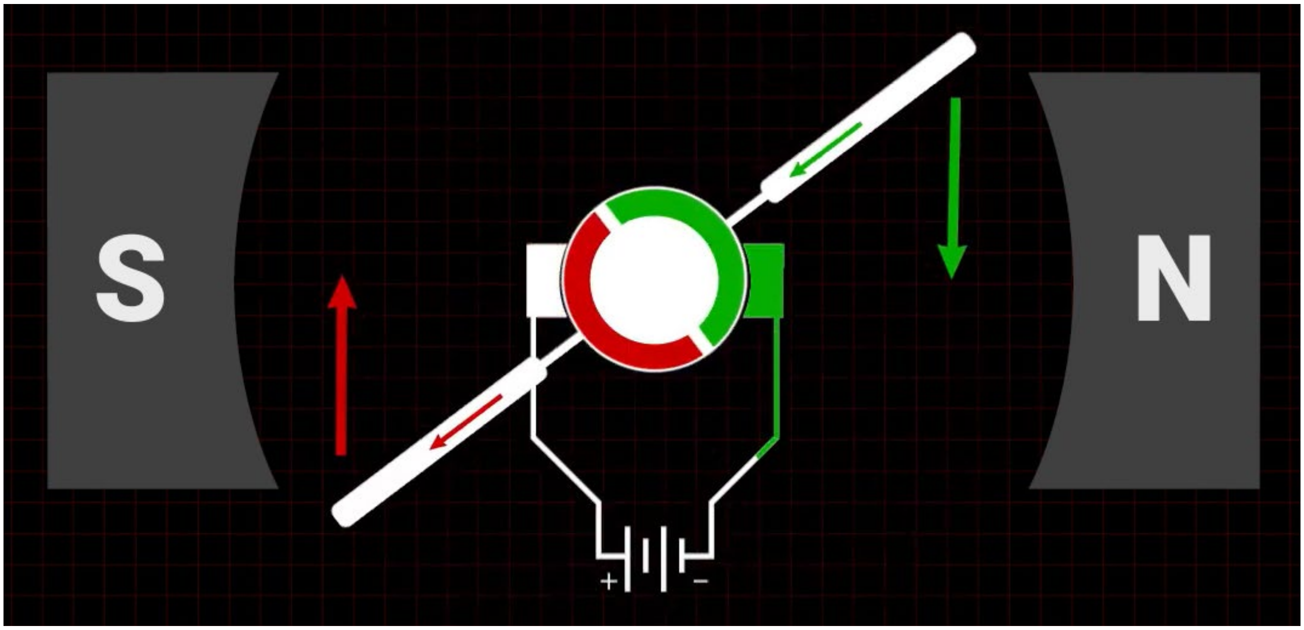
www.youtube.com/watch?v=1AaUK6pT_cE



Brushed DC Motor Components

Bing Videos

The rotor is energized by a DC power source, creating a polarized electromagnet that is acting against the force of the polarized permanent magnets. As current passes through the windings of the rotor (passed to the commutator via the brushes), this creates a magnetic force that generates torque, which causes the rotor to rotate.

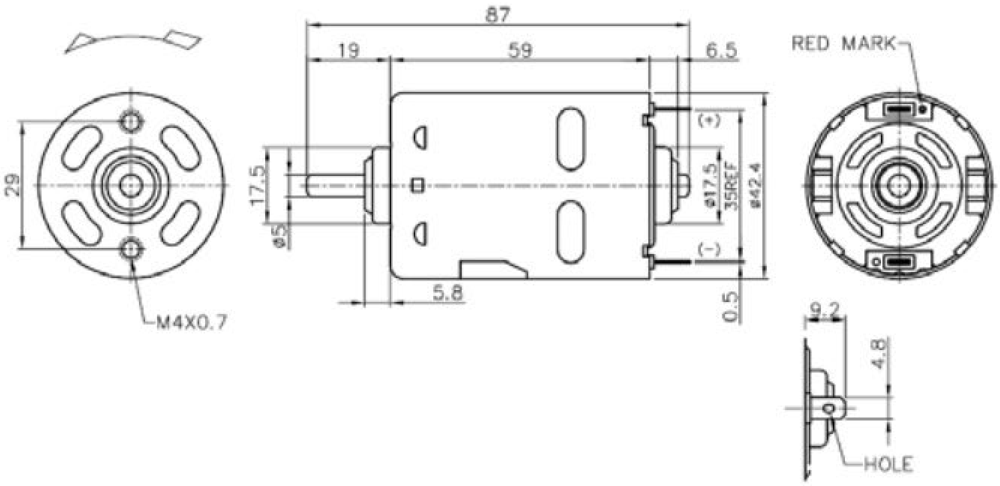


Brushed DC Motor Operation Principle

Bing Videos

Brushed DC motors have physical brushes used to commutate the motor shaft, causing it to spin.

- Brushes – Rotating to non-rotating
- Minimum of 2 poles
- Relies on a fixed permanent magnet (i.e. the stator)
- The windings rotate (i.e. the rotor)
- There is a push / pull using magnets
- 2 wires for + / – voltage input.

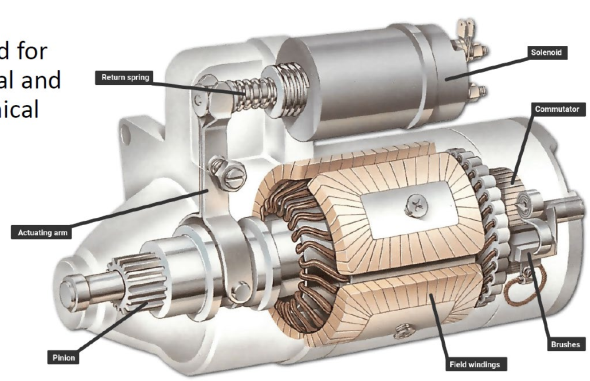


755 Series Brushed DC Motor / 42.4 mm dia / 14 – 144 W

Example Brushed DC Motor Data Sheet

Characteristic	Value
Part name	755-8512F-C
Diameter	42.2 mm
Length	59.0 mm
Nominal voltage	9.6 V
Nominal speed	14,000 rpm
Nominal torque	98.2 mN-m
Nominal current	19.8 A
No load speed	16,700 rpm
No load current	2.8 A
Stall torque	635.3 mN-m
Starting current	112.6 A
Power output	144 W
Efficiency	76%
Operating temperature range	[–10 +60]

Solenoid for electrical and mechanical needs



Automobile Starter and Solenoid

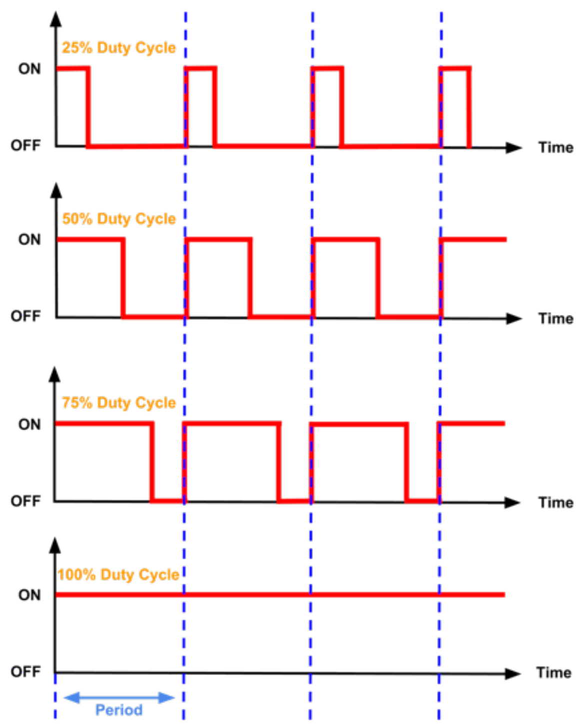
Brushed DC Motor Speed Control

Despite not being as sexy as their brushless counterparts, brushed DC motors are the most widely used motor due to simplicity. Applications range from toys, small electrical appliances to industrial automation machines and robotics. In most applications, variable speed control is required since motors are not generally run at a constant speed. Controlling motor speed is easy since the speed of a brushed DC motor is linearly proportional to the applied voltage (also, the torque output of a brushed DC motor is linearly proportional to the input current). Therefore, a means of varying the voltage of the power supply is required.

community.robotshop.com/blog/show/fundamental-of-pwm-speed-control-for-brushed-dc-motor-1

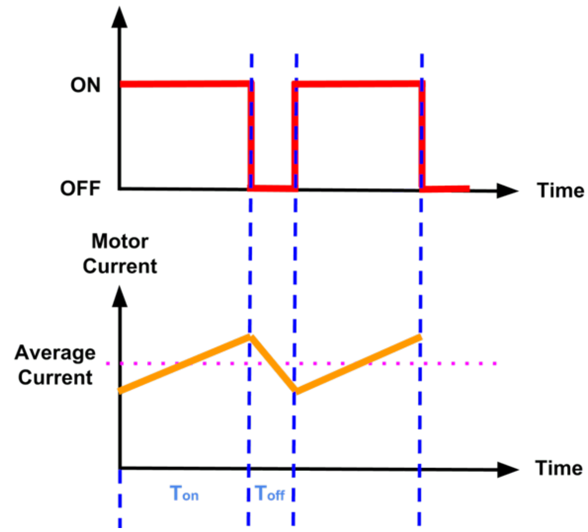
The method of varying the constant input voltage from a power supply for motor speed control is called Pulse-Width Modulation (PWM). Essentially it involves switching the motor on and off at a high frequency (e.g. 20 kHz). This can be accomplished by using Bipolar Junction Transistors (BJTs) or Metal-Oxide-Semiconductor Field-Effect-Transistor (MOSFETs) as electronic switches capable of operating up to hundreds of thousands of cycles per second (Hz) using a microcontroller.

To control motor speed using PWM, the ratio of motor on- and off-time is varied; this is called the duty cycle.



PWM Duty Cycles

When a motor is switched on/off at high frequency, it is like being driven from a constant DC voltage source, due to mechanical inertia and the coil inductance. The motor shaft inertia prevents rapid speed changes and the inductor will not allow instantaneous current changes.



PWM Characteristics

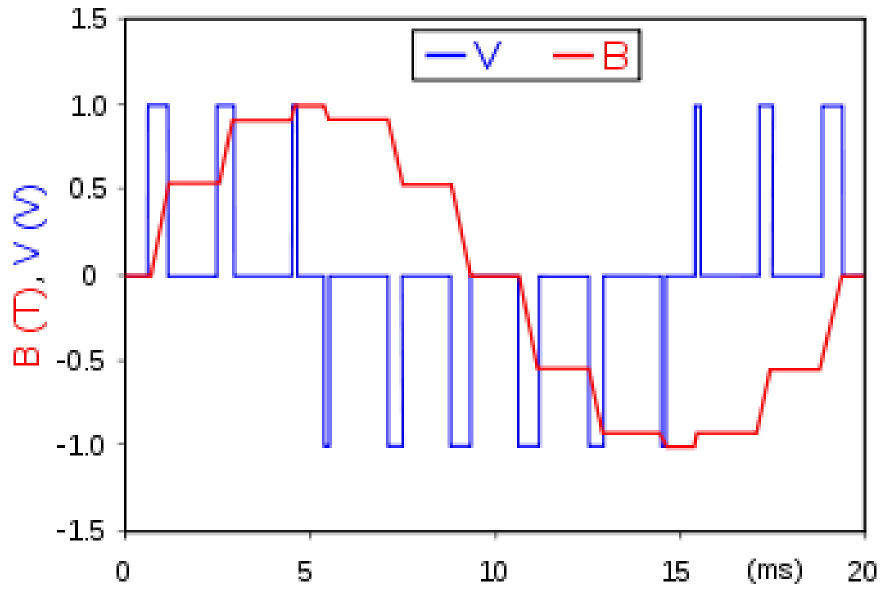
There is no known optimum PWM frequency. Generally between 16 – 20 kHz is acceptable (this is beyond the human audible range and is easy for MOSFETs to provide). The Arduino microcontroller default PWM frequency is 490 Hz, which is very low but usable. The Raspberry Pi Pico microcontroller default PWM frequency is much higher, 125 MHz, which is certainly preferable for smooth motor speed control.

Low PWM Frequency:

- Jerky motor rotation
- Lower motor efficiency
- Noisy if within human audible range
- Lower MOSFET switching loss
- Cheaper components

Higher PWM Frequency:

- Smoother motor rotation
- Higher motor efficiency
- Quiet (>16 kHz)
- Higher MOSFET switching loss
- Higher-cost components

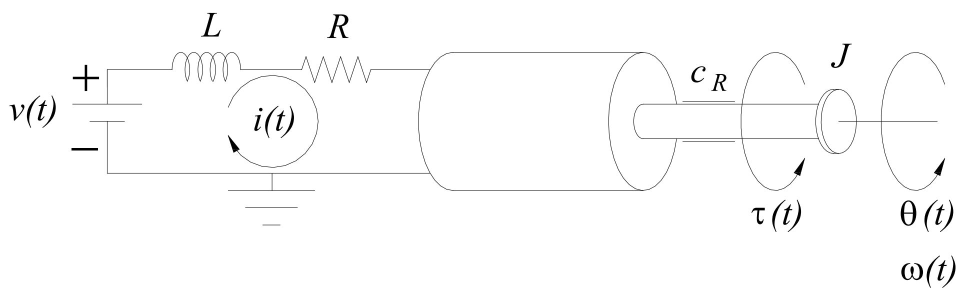


Pulse-Width Modulation (PWM)

The PWM method converts digital computer commands to analog voltage signals to control the rotational speed of electric motors. An H-Bridge device controls the rotational direction but not speed of brushed DC electric motors. PWM allows speed control. V is the digital (discrete) input and B is the analog (continuous) output.

DC Motor Model

The figure below shows a simple diagram for deriving the model of a DC motor, which is a rotational electromechanical system. On the circuit side, $v(t)$ is the input armature voltage, L is the inductance constant, R is the resistance constant, and $i(t)$ is the armature circuit current. On the rotational mechanical side, J is the lumped rotational inertia of the motor shaft and load, c_R is the rotational viscous damping coefficient, and the output is angular displacement $\theta(t)$, whose time derivative is angular velocity $\omega(t)$.



(Williams and Lawrence, 2007)

There are 3 levels for the open-loop transfer function.

input: voltage $v(t)$	output: current $i(t)$
input: current $i(t)$	output: torque $\tau(t)$
input: torque $\tau(t)$	output: angle $\theta(t)$

This model is interesting because the current $i(t)$ and torque $\tau(t)$ play dual roles, i.e. first output and then input to the next block. Of course, this is engineering convenience; in the real world, all of this occurs simultaneously. From an earlier derivation in Section 2.6, the *RL* series circuit model is a first-order ordinary differential equation (ODE):

$$L\frac{di(t)}{dt} + Ri(t) = v(t)$$

where we have ignored the motor back emf voltage. Usually the time constant for the electrical system τ_{RL} is much **smaller** than the time constant for the rotational mechanical system τ_{Jc}, which means that the electrical system current $i(t)$ rises much **faster** than the mechanical displacement $\theta(t)$. Therefore, we can ignore the circuit dynamics ($L \approx 0$), so the electrical circuit model simplifies to $Ri(t) = v(t)$, which is simply Ohm's Law.

$$\tau_{RL} = \frac{L}{R} \qquad \tau_{Jc} = \frac{J}{c_R}$$

In a DC servomotor, the generated motor torque is proportional to the circuit current, a linear proportional relationship that holds good for nearly the entire range of operation of the motor:

$$\tau(t) = K_T i(t)$$

K_T is the motor torque constant, which is generally stamped on the motor housing, available from the motor manufacturer, or measurable by experiment.

The rotational mechanical system dynamic model is derived from a free-body diagram of the rotating motor shaft, using Euler's rotational dynamics law $\sum M = J\ddot{\theta}(t)$:

$$J\ddot{\theta}(t) + c_R\dot{\theta}(t) = \tau(t)$$

Substituting the electrical models into the rotational mechanical system dynamic model yields:

$$J\ddot{\theta}(t) + c_R\dot{\theta}(t) = \tau(t) = K_T i(t) = \frac{K_T}{R}v(t) \qquad G_\theta(s) = \frac{\Theta(s)}{V(s)} = \frac{K_T/R}{s(Js + c_R)}$$

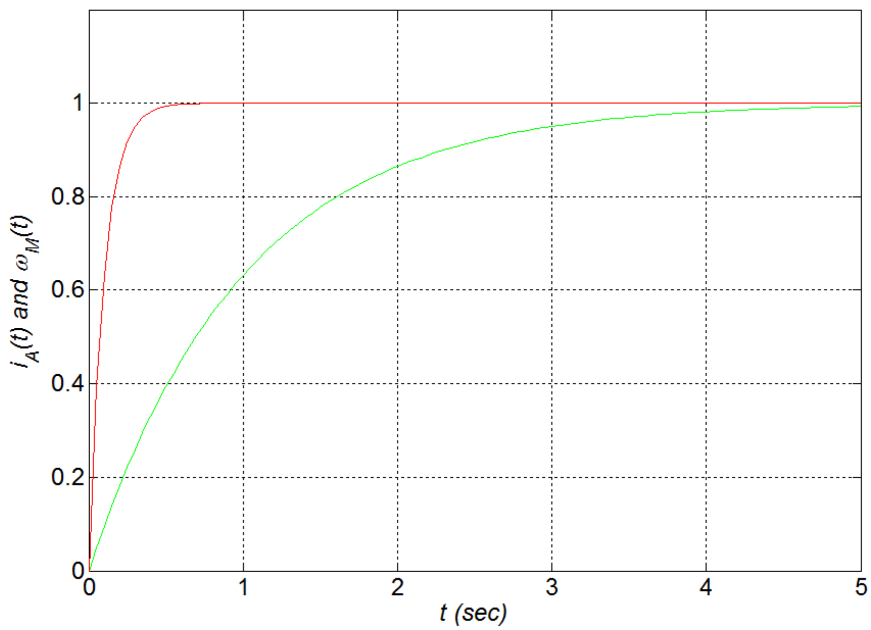
This is a linear, lumped-parameter, constant-coefficient, second-order ordinary differential equation (ODE). The same model written for angular velocity $\omega(t)$ output is a first-order model:

$$J\dot{\omega}(t) + c_R\omega(t) = \frac{K_T}{R}v(t) \qquad G_\omega(s) = \frac{\Omega(s)}{V(s)} = \frac{K_T/R}{Js + c_R}$$

Electrical vs. Mechanical Rise Time

Usually the electrical system time constant L/R is small relative to the mechanical system time constant J/c_R. This means that when the input voltage $v(t)$ is applied to the armature circuit, the armature current $i(t)$ rises much faster than the motor shaft angular velocity $\omega(t)$ does (see the figure below) when $i(t)$ is applied to generate motor torque $\tau(t)$. Here are the component first-order transfer functions and time constants for the armature circuit and rotational mechanical system dynamics.

$$G_1(s) = \frac{I(s)}{V(s) - V_B(s)} = \frac{1}{Ls + R} \qquad \frac{L}{R} = 0.1 \qquad\qquad G_3(s) = \frac{\Omega(s)}{\mathrm{T}(s)} = \frac{1}{Js + c_R} \qquad \frac{J}{c_R} \cong 1$$



Electrical vs. Mechanical Rise Time

armature current motor shaft angular velocity

Therefore, the electromechanical system open-loop block diagram transfer function could be simplified as follows: set the armature circuit time constant $\frac{L}{R}$ to zero relative to the mechanical system time constant $\frac{J_E}{c_E}$ since the mechanical system dominates the time response.

6.2 Brushless DC Motors

Compared to brushed DC motors, brushless DC motors replace the physical commutation brushes with electronic control, causing the motor shaft to spin.

Advantages and Disadvantages of Brushed vs. Brushless DC Motors

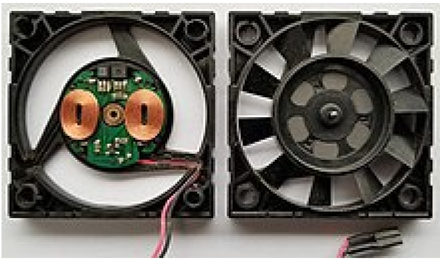
Characteristic	Brushed	Brushless
Lifetime	short (brushes wear out)	long (no brushes)
Speed and acceleration	medium	high
Efficiency	medium	high
Electrical noise	noisy (brush arcing)	quiet
Acoustic noise and ripple	poor	medium (trapezoidal) / good (sine)
Cost	lowest	medium due to added electronics

Due to decreasing cost and better performance, use of brushless DC motors is increasing. In automobiles, most motors running whenever the car is running (e.g. pumps and fans) are now brushless motors due to their increased reliability. The higher cost of the motor and electronics is balanced by a lower rate of failures and decreased maintenance over the lifetime of the automobile.

Motors that are operated infrequently (e.g. power seats and power windows) still largely use brushed DC motors. Since total runtime over the life of the car is small, it is unlikely that these motors will fail over the automobile lifetime.

Brushless DC motors are moving into applications traditionally filled by brushed DC motors. For example, seat adjustment motors in luxury cars have adopted brushless motors due to less acoustic noise.

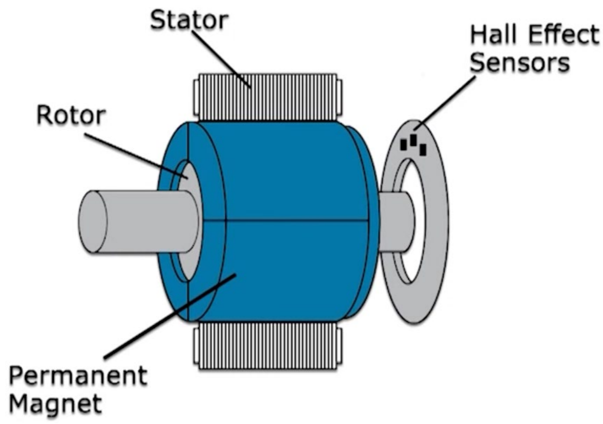
Brushless Vs Brushed DC Motors: When and Why to Choose One Over the Other | Article | MPS



Example Brushless DC Motor (computer fan)

Brushless DC electric motor - Wikipedia

Compared to brushless DC motors, brushed DC motors have simplified circuit wiring and controls is simply on/off (bang-bang). Brushed motors are also less expensive.



Brushless DC Motor Components

www.youtube.com/watch?v=Y7nQI2xM2as

Brushed DC motors are less efficient than brushless. Continuous creating and breaking of inductive circuits by the commutator causes electromagnetic noise. Brushes and commutators are known to wear out due to being in constant physical contact.

Brushless DC motors have a longer lifespan since there are no brushes to wear out. They also require less maintenance since there are no brushes to replace. Brushless DC motors have higher efficiency (85 – 90%) compared to brushed DC motors (75 – 80%). This is because brushed DC motors lose some energy in waste heat due to friction of the brushes in contact.

Brushless DC motors have a higher initial cost than brushed DC motors since brushless motors require a commutating device (such as an encoder) and a controller drive.

6.3 Stepper Motors

A stepper motor is a brushless DC motor that rotates in a series of small discrete angular steps. These motors operate in an open-loop controls fashion, without the need for angular feedback to perform adequately. The stepping angular position may be rapidly driven in either direction to achieve continuous rotation. Alternatively, the motor may be commanded to hold a given step angular position.

Stepper motors effectively have multiple toothed electromagnets arranged as a stator around a central rotor (a gear-shaped piece of iron). A stepper motor converts a train of input square wave pulses to a precisely-defined increment in the shaft angular position. Each pulse rotates the shaft through a fixed angle.

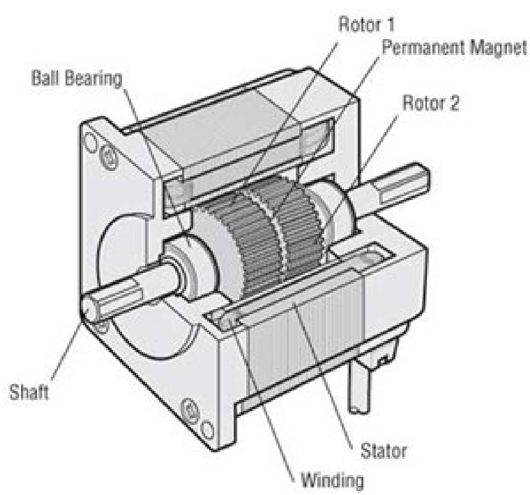
Stepper motors are deployed in computer disk drives, scanners, printers, plotters, slot machines, CD drives, lighting, camera lenses, CNC machines, and 3D printers.

Stepper Motor Advantages

- Low cost
- High torque at startup and low speeds
- Ruggedness, simplicity of construction, relatively long life
- Open-loop control system
- Low maintenance and high reliability
- Less likely to stall or slip than other electric motors
- Precise positioning and repeatability of movement, having an accuracy of 3–5% per step (and non-cumulative error from step to step).
- Good response to starting / stopping / reversing

Stepper Motor Disadvantages

- Unwanted resonance effect at low speeds
- Decreasing torque with increasing speed



Stepper Motor Diagram

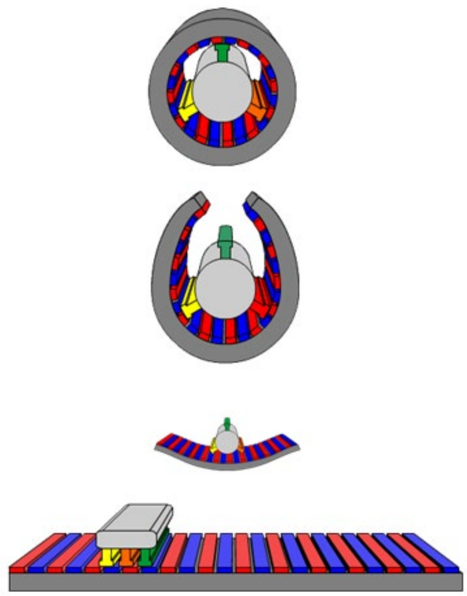
Example Stepper Motor

Stepper motor - Wikipedia

6.4 Linear Motors

Linear motors are widely applied in the packaging industry (in automated conveyors) and CNC (computer numerical-control) machines.

A linear motor is a DC electric motor with voltage input and translational motion output. Here linear refers to translational motion, rather than the linear mathematical relationship between current and torque, and voltage and speed (though that also applies here). A linear motor has its stator and rotor "unrolled" as pictured below. Another way to look at it is that the motor radius has gone to infinity, causing flat translational motion.



Stator and Rotor Unrolled

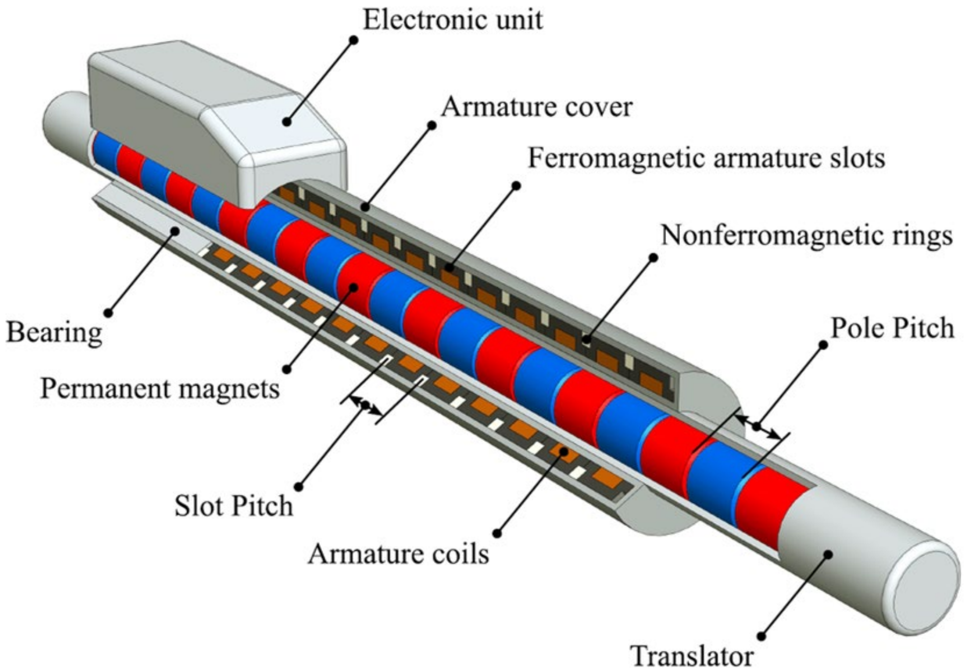
Linear motor - Wikipedia

Unlike rotational DC electric motors, linear electric motors necessarily have limits of travel. Many linear electric motors are designed as Lorentz-type actuators, wherein the linear motor output force $\vec{F}$ is proportional to the current and magnetic field:

$$\vec{F} = i\vec{L} \times \vec{B}$$

where $\vec{L}$ is the vector (whose magnitude is the wire length, and whose direction is along the wire in the direction of current i), and $\vec{B}$ is the magnetic field.

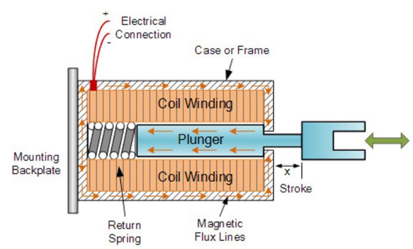
Linear motors are designed in two categories, low-acceleration and high-acceleration. Low-acceleration linear motors are used in maglev trains. High-acceleration linear motors have short travel, and can be used in a coil gun. Like rotational DC electric motors, linear DC electric motors may be either brushed or brushless, with similar advantages and disadvantages discussed earlier for the rotational electric motors.



Linear Motor Diagram

Solenoid

A solenoid is a linear electromechanical switch. An electromagnet produces translational mechanical motion output in a limited range.



Solenoid Product **Solenoid Operation**

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Appendices

These appendices presents (mostly electrical) mathematical models for various mechatronically-important circuits and devices.

The model in each case is the algebraic or integro-differential equation representing each component. The transfer function $G(s)$ is a dynamic representation for the same component, but in the Laplace frequency s domain. Transfer functions are the basic building blocks of classical controls.

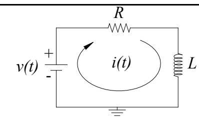
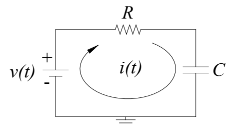
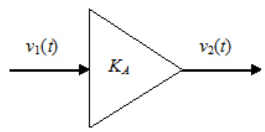
Presented are various important zeroth-order (no differentials), first-order, and second-order systems.

A.1 Zeroth-Order System Examples

Name	**Model**	***G*(*s*)**
accelerometer	$(k/m)x(t)=\omega^2 x_{IN}(t)$	$\frac{X(s)}{X_{IN}(s)}=\frac{\omega^2}{k/m}$
motor torque	$\tau(t)=K_T i(t)$	$\frac{T(s)}{I(s)}=K_T$
back emf	$v_B(t)=K_B\omega_M(t)$	$\frac{V_B(s)}{\Omega_M(s)}=K_B$

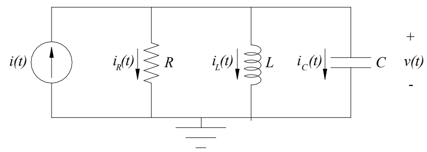
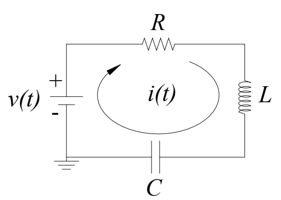
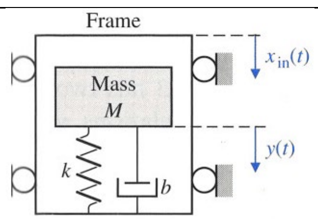
Name	**Model**	***G*(*s*)**
capacitor ($q(t)$ is charge)	$q(t)=Cv(t)$	$\frac{Q(s)}{V(s)}=C \quad \frac{V(s)}{Q(s)}=\frac{1}{C}$
resistor	$v(t)=Ri(t)$	$\frac{V(s)}{I(s)}=R \quad \frac{I(s)}{V(s)}=\frac{1}{R}$
inductor ($\phi(t)$ is flux)	$\phi(t)=Li(t)$	$\frac{\Phi(s)}{I(s)}=L \quad \frac{I(s)}{\Phi(s)}=\frac{1}{L}$
potentiometer	$v_1(t)R_2=v_2(t)(R_1+R_2)$	$\frac{V_2(s)}{V_1(s)}=\frac{R_2}{R_1+R_2}$
tachometer	$v(t)=K_t\omega(t)$	$\frac{V(s)}{\Omega(s)}=K_t$
DC amplifier, $\tau=0$	$v_2(t)=K_A v_1(t)$	$\frac{V_2(s)}{V_1(s)}=K_A$
series / parallel resistors	$v(t)=(R_1+R_2)i(t)$ $i(t)=\left[\frac{1}{R_1}+\frac{1}{R_2}\right]v(t)$	$\frac{V(s)}{I(s)}=R_1+R_2$ $\frac{V(s)}{I(s)}=\frac{R_1R_2}{R_1+R_2}$

A.2 First-Order System Examples

Name	Diagram	Model	$G(s)$	τ
LR series electrical circuit		$L\dfrac{di(t)}{dt}+Ri(t)=v(t)$	$\dfrac{I(s)}{V(s)}=\dfrac{1}{Ls+R}$	$\dfrac{L}{R}$
RC series electrical circuit		$R\dot{q}(t)+\dfrac{1}{C}q(t)=v(t)$ $Ri(t)+\dfrac{1}{C}\int i(t)dt=v(t)$	$\dfrac{Q(s)}{V(s)}=\dfrac{C}{RCs+1}$ $\dfrac{I(s)}{V(s)}=\dfrac{Cs}{RCs+1}$	RC RC
DC amplifier with time constant		$\tau\dot{v}_2(t)+v_2(t)=K_A v_1(t)$	$\dfrac{V_2(s)}{V_1(s)}=\dfrac{K_A}{\tau s+1}$	τ

Name	Model	$G(s)$
capacitor	$i(t)=C\dfrac{dv(t)}{dt}\qquad v(t)=\dfrac{1}{C}\int i(t)dt$	$\dfrac{I(s)}{V(s)}=Cs\qquad\dfrac{V(s)}{I(s)}=\dfrac{1}{Cs}$
resistor	$q(t)=\dfrac{1}{R}\int v(t)dt\qquad v(t)=R\dfrac{dq(t)}{dt}$	$\dfrac{Q(s)}{V(s)}=\dfrac{1}{Rs}\qquad\dfrac{V(s)}{Q(s)}=Rs$
inductor	$i(t)=\dfrac{1}{L}\int v(t)dt\qquad v(t)=L\dfrac{di(t)}{dt}$	$\dfrac{I(s)}{V(s)}=\dfrac{1}{Ls}\qquad\dfrac{V(s)}{I(s)}=Ls$
generic sensor	$\tau\dot{y}_{SENS}(t)+y_{SENS}(t)=ky(t)$	$H(s)=\dfrac{Y_{SENS}(s)}{Y(s)}=\dfrac{k}{\tau s+1}$ k gain $\quad\tau$ time constant

A.3 Second-Order System Examples

#	Diagram	Model	*G*(*s*)
1		$C\frac{dv(t)}{dt}+\frac{v(t)}{R}+\frac{1}{L}\int v(t)dt=i(t)$ $C\ddot{\phi}(t)+\frac{1}{R}\dot{\phi}(t)+\frac{1}{L}\phi(t)=i(t)$	$\frac{V(s)}{I(s)}=\frac{RLs}{CRLs^2+Ls+R}$ $\frac{\Phi(s)}{I(s)}=\frac{RL}{CRLs^2+Ls+R}$
2		$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)dt=v(t)$ $L\ddot{q}(t)+R\dot{q}(t)+\frac{1}{C}q(t)=v(t)$	$\frac{I(s)}{V(s)}=\frac{Cs}{LCs^2+RCs+1}$ $\frac{Q(s)}{V(s)}=\frac{C}{LCs^2+RCs+1}$
3		$m\ddot{x}(t)+b\dot{x}(t)+kx(t)=-m\ddot{x}_{IN}(t)$	$\frac{X(s)}{X_{IN}(s)}=\frac{-s^2}{s^2+(b/m)s+(k/m)}$

Row # Key

1. Parallel DC *RLC* current-driven circuit
2. Series DC *RLC* voltage-driven circuit
3. accelerometer